Research Article

A Semianalytical Approach to the Solution of Time-Fractional Navier-Stokes Equation

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In this manuscript, a semianalytical solution of the time-fractional Navier-Stokes equation under Caputo fractional derivatives using Optimal Homotopy Asymptotic Method (OHAM) is proposed. The above-mentioned technique produces an accurate approximation of the desired solutions and hence is known as the semianalytical approach. The main advantage of OHAM is that it does not require any small perturbations, linearization, or discretization and many reductions of the computations. Here, the proposed approach's reliability and efficiency are demonstrated by two applications of one-dimensional motion of a viscous fluid in a tube governed by the flow field by converting them to time-fractional Navier-Stokes equations in cylindrical coordinates using fractional derivatives in the sense of Caputo. For the first problem, OHAM provides the exact solution, and for the second problem, it performs a highly accurate numerical approximation of the solution compare with the exact solution. The presented simulation results of OHAM comparison with analytical and numerical approaches reveal that the method is an efficient technique to simulate the solution of time-fractional types of Navier-Stokes equation.

1. Introduction

Partial differential equations (PDEs) are utilized to mathematically formulate and thus help solve physical and other problems involving functions of several variables, such as the propagation of sound or heat, electrostatics, fluid flow, elasticity, and electrodynamics. In fluid mechanics, the Navier-Stokes equation is a PDE that illustrates incompressible fluids' flow. This equation is a generalization of the equation developed to illustrate the flow of frictionless and incompressible fluids by Euler in the eighteenth century. In 1821, Navier added the viscosity (friction) element to make viscous fluids more realistic and complex. The British physicist and mathematician Stokes improved it during the middle of the nineteenth century, though complete solutions were achieved only for simple two-dimensional flows [1]. That is why the equation is called the Navier-Stokes equation. The

mathematical model of the above-mentioned equation is given by:

$$\frac{\partial \mathcal{U}(s,t)}{\partial t} + (\mathcal{U}(s,t) \cdot \nabla)\mathcal{U}(s,t) = -\frac{1}{\rho}\nabla \mathbf{p} + \vartheta\nabla^2\mathcal{U}(s,t), \quad (1)$$

where $\mathcal{U}$ is the velocity, t is the time, ρ is the density, s is the spatial variable, ϑ is the kinematics viscosity, p is the pressure, and ∇ denotes the gradient differential operator.

Fractional differential equations have proven to be a powerful tool for modeling real-world problems in the literature. It was noticed that time-fractional derivatives usually appear as infinitesimal generators of the time evolution when choosing a long-time scaling limit. Several essential phenomena in physics and polymer technology [2], electrical circuits [3], electrochemistry [4], electrodynamics

of complex medium [5], control theory [6], thermodynamics [7], viscoelasticity [8], aerodynamics [9], capacitor theory [10], biology [11], blood flow [12], and fitting of experimental data [13], are well described by the aforesaid equations. Equation (1) can be converted to time-fractional derivative of order $\alpha \in (0, 1]$, as given by:

$$\frac{\partial^\alpha \mathscr{U}(s,t)}{\partial t^\alpha} + (\mathscr{U}(s,t) \cdot \nabla)\mathscr{U}(s,t) = -\frac{1}{\rho}\nabla\mathbf{p} + \vartheta\nabla^2\mathscr{U}(s,t). \quad (2)$$

Here, $\partial^\alpha/\partial t^\alpha$ is the Caputo fractional derivative. As the equation mentioned above is nonlinear, there is no known general method for resolving it. There are very few cases where it is possible to achieve the exact solution of Equation (2), making some predictions about the fluid's state and a simple arrangement considered for the flow pattern; for detail, see [14].

Analytical and numerical techniques are extensively utilized to solve nonlinear differential equations modeling physical phenomena. This is because the exact solutions of the above-mentioned equations are challenging to achieve. In recent decades, a new variety of perturbation methods have developed, which is loosely based on Poincare's homotopy applied in topology. Liao [15] introduced the Homotopy Analysis Method (HAM) in 1992. In 1998, He [16] followed Liao's work and developed the Homotopy Perturbation Method (HPM). Both methods have been successfully implemented to the problems, which exist in engineering and science fields. For example, Ganji and Rafei [17] solved nonlinear Hirota-Satsuma coupled Korteweg-De Vries equation by HPM. Lu and Liu [18] solved the Korteweg-De Vries-Burgers equation via the help of HAM. Siddiqui et al. [19] utilized HPM and examined the irregular 2D flow of a viscous magnetohydrodynamics fluid within two parallel plates.

In [20], Marinca and Herisanu proposed a technique called OHAM. The benefit of the above-mentioned technique is in the built-in convergence criteria alike to HAM but extraflexible. The researchers have successfully implemented this approach to solving essential science problems and have also explained its reliability and effectiveness, for example, the dynamics of an electrical machine exhibiting nonlinear vibration [21], the oscillations of a particle that moves on a rotating parabola [22], the explicit solutions for some oscillators with discontinuities and a fractional power restoring force [23], and nonlinear equations arising in heat transfer [20], in an application to the steady flow of a fourth-grade fluid [24]. The above-mentioned technique is the HAM's modification, which is based on minimizing the residual error. In OHAM, the adjustment and control of the convergence region are provided conveniently.

In [14], Momani and Odibat considered unsteady one-dimensional motion of a viscous fluid in a tube. The equations of motion which govern the flow field in the tube are the Navier-Stokes equations in cylindrical coordinates. They converted Equation (2) to the operator form as:

$$\frac{\partial^\alpha \mathscr{U}(s,t)}{\partial t^\alpha} = P + \vartheta\left(\frac{\partial^2 \mathscr{U}(s,t)}{\partial s^2} + \frac{1}{s}\frac{\partial \mathscr{U}(s,t)}{\partial s}\right), \quad (3)$$

where $0 < \alpha \leq 1$ is the fractional order derivative and $P = -\partial\mathbf{p}/\rho\partial z$. For $\alpha = 1$, we can get the standard Navier-Stokes equation. For the analytical solution, they have utilized the Adomian decomposition method. Lately, several powerful analytical techniques have been utilized to achieve the solution of Equation (3), such as the modified Laplace decomposition method [25], the q-homotopy analysis transform scheme [26], the new homotopy perturbation transform method [27], the iterative Elzaki transform method [28], the natural homotopy perturbation method [29], Elzaki transform with homotopy perturbation technique [30], and He's homotopy perturbation and variational iteration methods [31]. The aforementioned methods are called analytical, and no one has used the semianalytical technique for the solution of Equation (3) in the previous studies. Therefore, the objective of this manuscript is to present the semianalytical solution of Equation (3) by a semianalytical approach called OHAM.

The rest of the manuscript is structured as follows: in Section 2, we recall some definitions and properties of noninteger order operators. Section 3 is devoted to the basic formulation of OHAM. In Section 4, we apply the above-mentioned technique to time-fractional Navier-Stokes type of equations and discuss the method reliability through tables and plots. Section 5 is devoted to the conclusion.

2. Definitions and Properties

This section deals with some definitions and properties that are used in the manuscript.

Definition 1. A real function $\psi(t)$, $t > 0$ is supposed to be in the space C_κ $(\kappa > 0)$ if it can express as $\psi(t) = t^p\psi_1(t)$ for certain $p > \kappa$ where $\psi_1(t) \in C[0,\infty)$, and it is supposed to be in the space $C_\kappa^m \Leftrightarrow \psi^m \in C_\kappa$, $m \in \mathbb{N}$.

Definition 2 (see [32]). The $\alpha \geq 0$ order integral operator for a function $\psi \in C_\kappa$, $\kappa \geq -1$ in the Riemann-Liouville sense is defined as:

$$\mathbf{I}^\alpha\psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi(\theta)}{(t-\theta)^{1-\alpha}}d\theta. \quad (4)$$

Let $\psi \in C_\kappa$, $\kappa \geq -1$, $\alpha, \gamma \geq 0$, and $\mu > -1$, then we have the properties [32] given by:

$$\begin{aligned}
\mathbf{I}^\alpha\mathbf{I}^\gamma\psi(t) &= \mathbf{I}^{\alpha+\gamma}\psi(t), \\
\mathbf{I}^\alpha\mathbf{I}^\gamma\psi(t) &= \mathbf{I}^\gamma\mathbf{I}^\alpha\psi(t), \\
\mathbf{I}^\alpha t^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}.
\end{aligned} \quad (5)$$

Definition 3 (see [32]). The $\alpha > 0$ order Caputo derivative operator for a function $\psi \in C_{-1}^m$, $m \in \mathbb{N}$ is defined as:

$$^c\mathbf{D}^\alpha\psi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\psi^{(n)}(\theta)}{(t-\theta)^{\alpha-n+1}}d\theta, \quad t > 0, \quad (6)$$

where $[\alpha]$ is the integer part of α and $n = [\alpha] + 1$. For ψ, $\phi \in C_\kappa^m, \kappa \geq -1$, we have the properties [32] given by:

$$
\begin{aligned}
{}^c\mathbf{D}^\alpha(a\psi(t) + b\phi(t)) &= a\,{}^c\mathbf{D}^\alpha\psi(t) + b\,{}^c\mathbf{D}^\alpha\phi(t), \quad a, b \in \mathrm{IR}, \\
{}^c\mathbf{D}^\alpha\mathbf{I}^\alpha\psi(t) &= \psi(t), \\
\mathbf{I}^\alpha[{}^c\mathbf{D}^\alpha]\psi(t) &= \psi(t) - \sum_{j=0}^{k-1} \psi^{(j)}(0)\frac{x_j}{j!}.
\end{aligned}
\tag{7}
$$

3. Basic Formulation of the OHAM

We formulate OHAM for PDEs with boundary condition, in the steps given by [33]:

$$
\mathscr{L}(\mathscr{U}(s,t)) + \mathscr{N}(\mathscr{U}(s,t)) + \hbar(s,t) = 0, \mathscr{B}\left(\mathscr{U}, \frac{\partial\mathscr{U}}{\partial t}\right) = 0, \quad s \in \Upsilon,
\tag{8}
$$

where $\mathscr{L}$ and $\mathscr{N}$ are the linear and nonlinear operators, respectively. $\mathscr{U}(s,t)$ is an unknown function, the boundary operator denoted by $\mathscr{B}$, s and t denote spatial and time variables, respectively, $\hbar(s,t)$ is known function, and Υ is the domain of the problem.

By OHAM, we construct the homotopy $\omega(s,t;\wp)$: $\Upsilon \times J \longrightarrow R$, where $J = [0,1]$, which satisfies:

$$
\begin{aligned}
(1-\wp)\{\mathscr{L}(\omega(s,t;\wp)) + \hbar(s,t)\} = \mathscr{H}(\wp)\{\mathscr{L}(\omega(s,t;\wp)) \\
+ \mathscr{N}(\omega(s,t;\wp)) + \hbar(s,t)\},
\end{aligned}
\tag{9}
$$

where $\wp \in J$ is an embedding parameter and $\mathscr{H}(\wp)$ is a nonzero auxiliary function for $\wp \neq 0$ and $\mathscr{H}(0) = 0$. Equation (9) is to be the optimal homotopy equation. Obviously,

$$
\begin{aligned}
\wp = 0 &\Rightarrow \mathscr{L}(\omega(s,t;\wp)) + \hbar(s,t) = 0, \\
\wp = 1 &\Rightarrow \mathscr{L}(\omega(s,t;\wp)) + \mathscr{N}(\omega(s,t;\wp)) + \hbar(s,t).
\end{aligned}
\tag{10}
$$

For $\wp = 0$, we can obtain $\omega(s,t;0) = \mathscr{U}_0(s,t)$, and for $\wp = 1$, we can get $\omega(s,t;1) = \mathscr{U}(s,t)$. Therefore, as $\wp$ extend from 0 to 1, then $\omega(s,t;\wp)$ moves from $\mathscr{U}_0(s,t)$ to $\mathscr{U}(s,t)$, where $\mathscr{U}_0(s,t)$ is got from Equation (9) for $\wp = 0$:

$$
\mathscr{L}(\mathscr{U}_0(s,t;\wp)) + \hbar(s,t) = 0, \mathscr{B}\left(\mathscr{U}_0, \frac{\partial\mathscr{U}_0}{\partial t}\right) = 0.
\tag{11}
$$

Now, we take $\mathscr{H}(\wp)$, which is called auxiliary function, in the following form:

$$
\mathscr{H}(\wp) = \wp C_1 + \wp^2 C_2 + \wp^3 C_3 + \cdots + \wp^m C_m.
\tag{12}
$$

For the numerical solution, we utilize Taylor's series about $\wp$ and expand $\omega(s,t;\wp,C_p)$ in the following way:

$$
\omega(s,t;\wp,C_p) = \mathscr{U}_0(s,t) + \sum_{q-1}^{\infty} \mathscr{U}_q(s,t;C_p)\wp^q, \quad p = 1, 2, 3, \cdots.
\tag{13}
$$

Plugging Equation (13) in Equation (9) and equating the coefficient of like powers of $\wp$, we get the problem of zerothorder; given in (11), the problems of the first and second order are given by the Equations (14) and (15), respectively, and the general governing equations for $\mathscr{U}_q(s,t)$ are given in Equation (16):

$$
\mathscr{L}(\mathscr{U}_1(s,t)) = C_1\mathscr{N}_0(\mathscr{U}_1(s,t)), \mathscr{B}\left(\mathscr{U}_1, \frac{\partial\mathscr{U}_1}{\partial t}\right) = 0,
\tag{14}
$$

$$
\mathscr{L}(\mathscr{U}_2(s,t)) - \mathscr{L}(\mathscr{U}_1(s,t)) = C_2\mathscr{N}_0(\mathscr{U}_0(s,t)) + C_1[\mathscr{L}(\mathscr{U}_1(s,t)) + \mathscr{N}_1(\mathscr{U}_0(s,t), \mathscr{U}_1(s,t))], \mathscr{B}\left(\mathscr{U}_2, \frac{\partial\mathscr{U}_2}{\partial t}\right) = 0,
\tag{15}
$$

$$
\vdots
$$

$$
\begin{aligned}
\mathscr{L}(\mathscr{U}_q(s,t)) - \mathscr{L}(\mathscr{U}_{q-1}(s,t)) = C_q\mathscr{N}_0(\mathscr{U}_0(s,t)) + \sum_{p=1}^{q-1} C_p[\mathscr{L}(\mathscr{U}_{q-p}(s,t)) \\
+ \mathscr{N}_{q-p}(\mathscr{U}_0(s,t), \mathscr{U}_1(s,t), \mathscr{U}_2(s,t), \cdots, \mathscr{U}_{q-p}(s,t))], \mathscr{B}\left(\mathscr{U}_q, \frac{\partial\mathscr{U}_q}{\partial t}\right), \quad q = 2, 3, \cdots,
\end{aligned}
\tag{16}
$$

where $\mathcal{N}_{q-p}(\mathcal{U}_0(s,t), \mathcal{U}_1(s,t), \cdots, \mathcal{U}_{q-p}(s,t))$ is the coefficient of $\wp^{q-p}$ in the expansion of $\mathcal{N}(\omega(s,t;\wp))$ about the embedding parameter $\wp$.

$$\mathcal{N}\left(\omega\left(s,t;\wp,C_p\right)\right) = \mathcal{N}_0(\mathcal{U}_0(s,t)) + \sum_{q\geq 1} \mathcal{N}_q\left(\mathcal{U}_0, \mathcal{U}_1, \cdots, \mathcal{U}_q\right)\wp^q. \tag{17}$$

Here, $\mathcal{U}_q$ for $q \geq 0$ is the set of linear equations with linear boundary conditions, which can be solved very easily.

The series in Equation (13) depends on $C_1, C_2, \cdots$. If it is convergent at $\wp = 1$, then:

$$\tilde{\mathcal{U}}\left(s,t;C_p\right) = \mathcal{U}_0(s,t) + \sum_{q=1}^{m} \mathcal{U}_q\left(s,t,C_p\right). \tag{18}$$

Putting Equation (18) in Equation (8), one can obtain the residual expression in the form:

$$R\left(s,t;C_p\right) = \mathcal{L}\left(\tilde{\mathcal{U}}\left(s,t;C_p\right)\right) + \mathcal{N}\left(\tilde{\mathcal{U}}\left(s,t;C_p\right)\right) + \hbar(s,t). \tag{19}$$

If $R(s,t;C_p) = 0$, then $\tilde{\mathcal{U}}(s,t;C_p)$ will be the exact solution. But generally, it is not possible in nonlinear problems.

For calculating the C_p, $p = 1, 2, \cdots, m$, one can utilize the least square technique as given by:

$$\Phi(C_p) = \int_0^t \int_\gamma R^2\left(s,t;C_p\right) ds\,dt, \tag{20}$$

where R is the residual given by Equation (19) and

$$\frac{\partial \Phi}{\partial C_1} = \frac{\partial \Phi}{\partial C_2} = \frac{\partial \Phi}{\partial C_3} = \cdots = \frac{\partial \Phi}{\partial C_m} = 0. \tag{21}$$

The convergence based on $C_1, C_2, C_3, \cdots$ can be identified and minimized optimally by Equation (21).

4. Numerical Examples

In this section, the fractional OHAM is utilized to get the solution of time-fractional Navier-Stokes equations.

Example 1. Suppose a time-fractional Navier-Stokes equation:

$$\frac{\partial^\alpha \mathcal{U}(s,t)}{\partial t^\alpha} = P + \frac{\partial^2 \mathcal{U}(s,t)}{\partial s^2} + \frac{1}{s}\frac{\partial \mathcal{U}(s,t)}{\partial s}, \tag{22}$$

with the initial condition:

$$\mathcal{U}(s,0) = 1 - s^2, \tag{23}$$

where $\partial^\alpha/\partial t^\alpha$ is Caputo fractional derivative and $0 < \alpha \leq 1$. The exact solution of Equation (22) is given by [14]:

$$\mathcal{U}(s,t) = 1 - s^2 + (P-4)t. \tag{24}$$

According to Section 3, we can set up the homotopy in the following way:

$$(1-\wp)\frac{\partial^\alpha \omega(s,t;\wp)}{\partial t^\alpha} = \mathcal{H}\left(\wp,C_p\right)$$
$$\left[\frac{\partial^\alpha \omega(s,t;\wp)}{\partial t^\alpha} - P - \frac{\partial^2 \omega(s,t;\wp)}{\partial s^2} - \frac{1}{s}\frac{\partial \omega(s,t;\wp)}{\partial s}\right], \tag{25}$$

where

$$\omega(s,t;\wp) = \mathcal{U}_0(s,t) + \sum_{q=1}^{\infty} \mathcal{U}_q\left(s,t;C_p\right)\wp^q, \quad p = 1, 2, 3 \cdots, \tag{26}$$

$$\mathcal{H}\left(\wp,C_p\right) = \wp C_1 + \wp^2 C_2 + \wp^3 C_3 + \wp^4 C_4 + \cdots. \tag{27}$$

Plugging Equations (26) and (27) in (25) and equating the coefficient of the same powers of $\wp$, one can get the simpler problems, given as:

Zero-order problem:

$$\frac{\partial^\alpha \mathcal{U}_0(s,t)}{\partial t^\alpha} = 0, \quad \mathcal{U}_0(s,0) = 1 - s^2. \tag{28}$$

First-order problem:

$$\frac{\partial^\alpha \mathcal{U}_1(s,t;C_1)}{\partial t^\alpha} = (1 + C_1)\frac{\partial^\alpha \mathcal{U}_0(s,t)}{\partial t^\alpha} - C_1 P - C_1 \frac{\partial^2 \mathcal{U}_0(s,t)}{\partial s^2}$$
$$- \frac{C_1}{s}\frac{\partial \mathcal{U}_0(s,t)}{\partial s}, \quad \mathcal{U}_1(s,0) = 0. \tag{29}$$

Respective solutions of Equations (28) and (29) after apply fractional integral and initial condition are given:
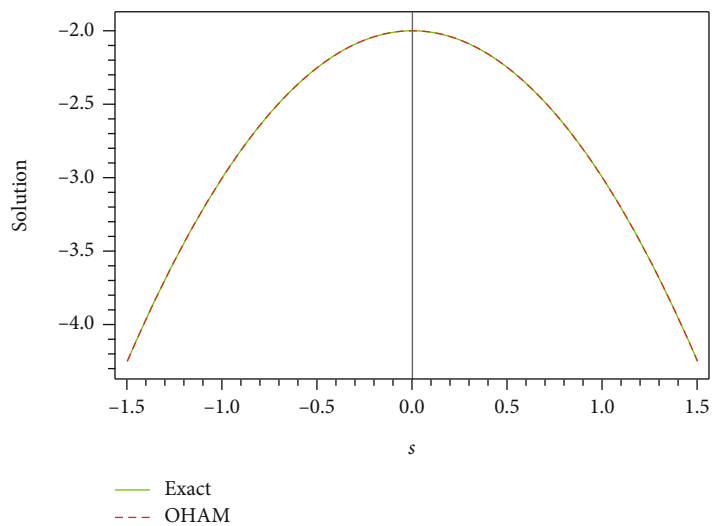
$$\mathcal{U}_0(s,t) = 1 - s^2,$$
$$\mathcal{U}_1(s,t;C_1) = -\frac{C_1(P-4)t^\alpha}{\Gamma(1+\alpha)}. \tag{30}$$
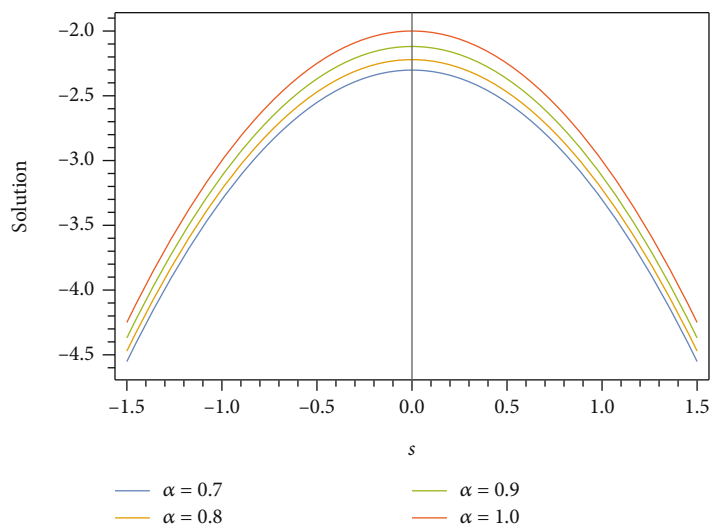
One can get the following expression:

$$\tilde{\mathcal{U}}(s,t) = \mathcal{U}_0(s,t) + \mathcal{U}_1(s,t;C_1) + \mathcal{U}_2(s,t;C_1,C_2) + \cdots$$
$$= 1 - s^2 - \frac{C_1(P-4)t^\alpha}{\Gamma(1+\alpha)}. \tag{31}$$

We used the least square method after finding the residual and then got the auxiliary constant value for $\alpha = 1$; we have $C_1 = -1$. Putting the value of C_1 in Equation (31), we get

$$\tilde{\mathcal{U}}(s,t) = 1 - s^2 + \frac{(P-4)t^\alpha}{\Gamma(1+\alpha)}. \tag{32}$$
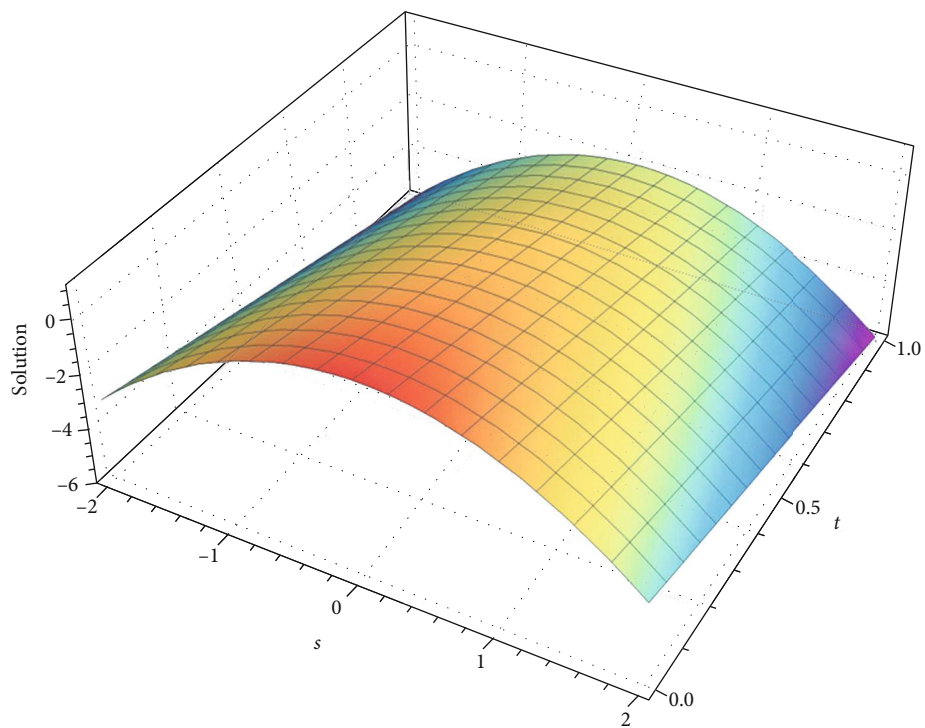
(a) Exact vs. OHAM solution

(b) Solution behavior for different values of α

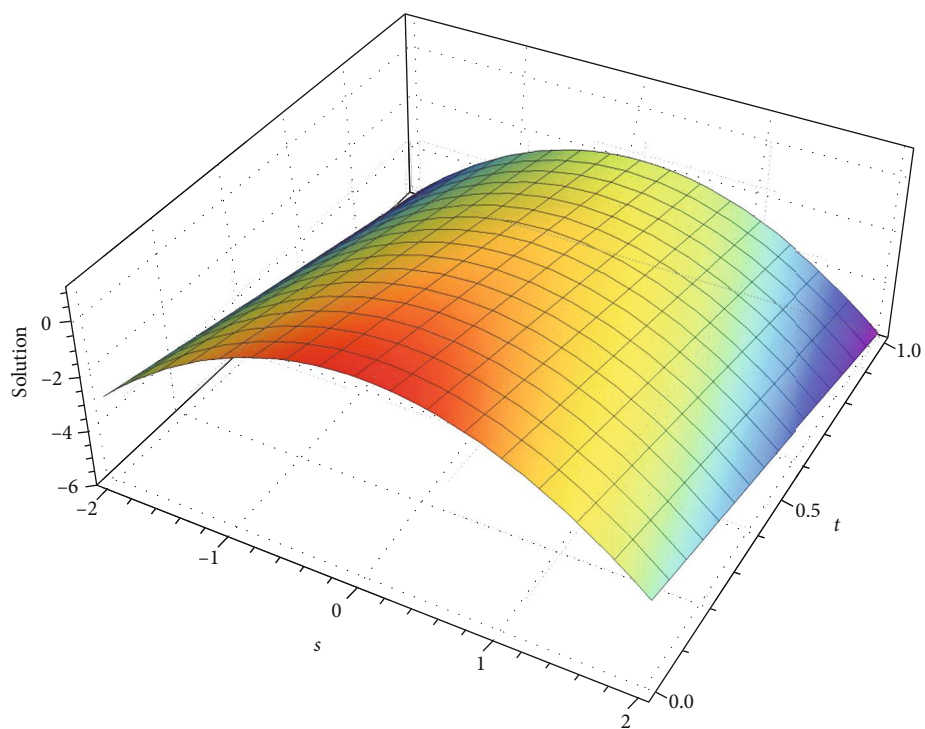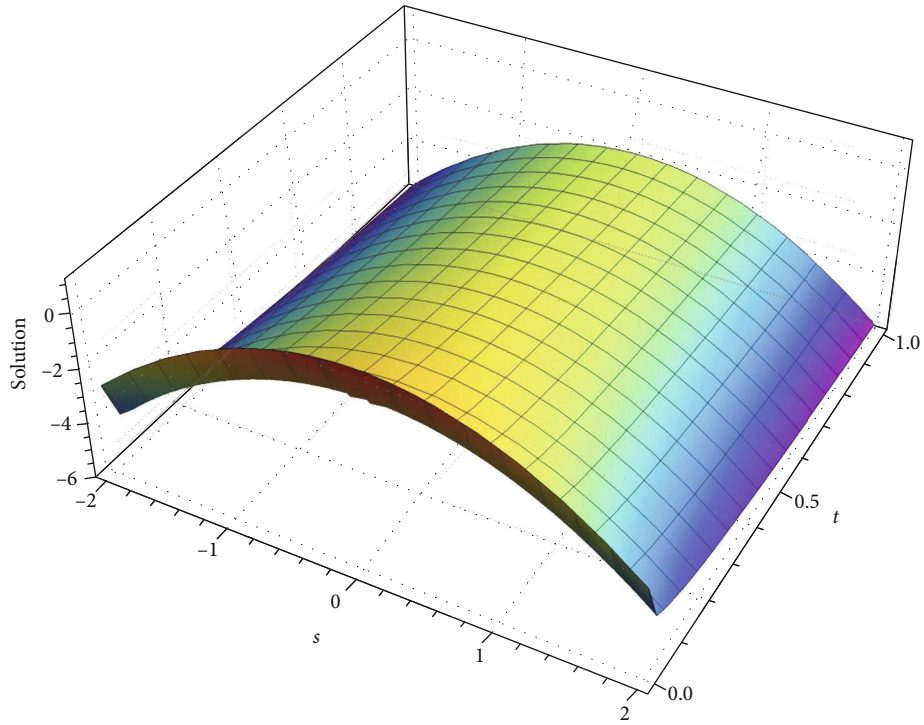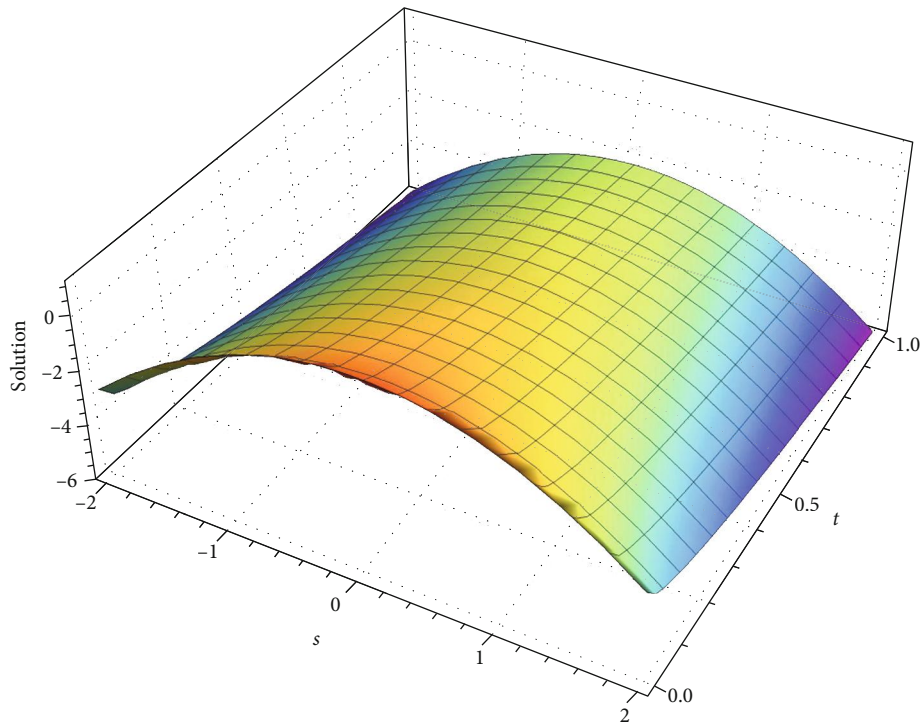FIGURE 1: Continued.

(c) Exact solution

(d) OHAM solution for $\alpha = 1$

Figure 1: Continued.

(e) Solution for $\alpha = 0.2$

(f) Solution for $\alpha = 0.5$

FIGURE 1: Exact solution and OHAM solution behavior of Example 1.

The result (32) is in complete agreement with [14, 25–31]. Figure 1 shows the evaluation results of the semianalytical solution for Example 1 when $P = 1$ and show the dynamics of the obtained solution by OHAM for various noninteger order Brownian motions and for standard motions, i.e., for $\alpha = 1$. It can be seen that the solution acquired via the above-mentioned technique is decreasing very swiftly with the increase in t in Example 1, which is illustrated in Figure 1(b). Figure 1(a) shows the efficiency of the above-mentioned method. Figures 1(e) and 1(f) show the solution behavior for $\alpha = 0.2$ and $\alpha = 0.5$. Besides, we have obtained the exact solution by OHAM.

Example 2. Suppose a time-fractional Navier-Stokes equation:

$$\frac{\partial^{\alpha}\mathcal{U}(s,t)}{\partial t^{\alpha}} = \frac{\partial^2 \mathcal{U}(s,t)}{\partial s^2} + \frac{1}{s}\frac{\partial \mathcal{U}(s,t)}{\partial s}, \tag{33}$$

with the initial condition

$$\mathcal{U}(s,0) = s, \tag{34}$$

where $\partial^{\alpha}/\partial t^{\alpha}$ is Caputo fractional derivative and $0 < \alpha \le 1$. We consider the first four terms of the exact solution of (33) from [14] is given by:

$$\mathcal{U}(s,t) = s + \frac{t}{s} + \frac{t^2}{2s^3} + \frac{9t^3}{6s^5}. \tag{35}$$

According to Section 3, we can set up the homotopy in the following way:

$$(1-\wp)\frac{\partial^{\alpha}\omega(s,t;\wp)}{\partial t^{\alpha}} = \mathcal{H}(\wp,C_p)\left[\frac{\partial^{\alpha}\omega(s,t;\wp)}{\partial t^{\alpha}} - \frac{\partial^2 \omega(s,t;\wp)}{\partial s^2} - \frac{1}{s}\frac{\partial \omega(s,t;\wp)}{\partial s}\right], \tag{36}$$

where

$$\omega(s,t;\wp) = \mathcal{U}_0(s,t) + \sum_{q=1}^{\infty}\mathcal{U}_q(s,t;C_p)\wp^q, \quad p = 1,2,3\cdots \tag{37}$$

$$\mathcal{H}(\wp,C_p) = \wp C_1 + \wp^2 C_2 + \wp^3 C_3 + \wp^4 C_4 + \cdots. \tag{38}$$

Plugging Equations (37) and (38) in (36) and equating the coefficient of the same powers of $\wp$, one can get the simpler problems, given as:

Zero-order problem:

$$\frac{\partial^{\alpha}\mathcal{U}_0(s,t)}{\partial t^{\alpha}} = 0, \quad \mathcal{U}_0(s,0) = s. \tag{39}$$

First-order problem:

$$\frac{\partial^{\alpha}\mathcal{U}_1(s,t;C_1)}{\partial t^{\alpha}} = (1+C_1)\frac{\partial^{\alpha}\mathcal{U}_0(s,t)}{\partial t^{\alpha}} - C_1\frac{\partial^2 \mathcal{U}_0(s,t)}{\partial s^2} \\ - \frac{C_1}{s}\frac{\partial \mathcal{U}_0(s,t)}{\partial s}, \quad \mathcal{U}_1(s,0) = 0. \tag{40}$$

Second-order problem:

$$\frac{\partial^{\alpha}\mathcal{U}_2(s,t;C_1,C_2)}{\partial t^{\alpha}} = (1+C_1)\frac{\partial^{\alpha}\mathcal{U}_1(s,t;C_1)}{\partial t^{\alpha}} - C_1\frac{\partial^2 \mathcal{U}_1(s,t;C_1)}{\partial s^2} \\ - \frac{C_1}{s}\frac{\partial \mathcal{U}_1(s,t;C_1)}{\partial s} + C_2\frac{\partial^{\alpha}\mathcal{U}_0(s,t)}{\partial t^{\alpha}} \\ - C_2\frac{\partial^2 \mathcal{U}_0(s,t)}{\partial s^2} - \frac{C_2}{s}\frac{\partial \mathcal{U}_0(s,t)}{\partial s}, \quad \mathcal{U}_2(s,0) = 0. \tag{41}$$

TABLE 1: C_1, C_2, C_3 for various values of α.

α	C_1	C_2	C_3
0.7	-1.036692817879	-0.001240278601	-4.24687×10^{-5}
0.8	-1.020447062606	-0.000376864638	-7.02931×10^{-6}
0.9	-1.010550021386	-0.000101578844	-9.88500×10^{-7}
0.10	-0.999953117565	-0.000040256679	3.30467×10^{-8}

TABLE 2: Comparison of exact and OHAM solution.

s	Exact	OHAM	Abs error
1.0	2.94735999	2.94736040	$4.04145499 \times 10^{-7}$
1.2	3.05262208	3.05262204	$3.73698401 \times 10^{-8}$
1.4	3.16486784	3.16486694	$8.94947017 \times 10^{-7}$
1.6	3.28483456	3.28483228	$2.27227761 \times 10^{-6}$
1.8	3.41325952	3.41325524	$4.27305321 \times 10^{-6}$
2.0	3.55088000	3.55087299	$7.00096540 \times 10^{-6}$

Third-order problem:

$$\frac{\partial^{\alpha}\mathcal{U}_3(s,t;C_1,C_2,C_3)}{\partial t^{\alpha}} = (1+C_1)\frac{\partial^{\alpha}\mathcal{U}_2(s,t;C_1,C_2)}{\partial t^{\alpha}} - C_1\frac{\partial^2 \mathcal{U}_2(s,t;C_1,C_2)}{\partial s^2} \\ - \frac{C_1}{s}\frac{\partial \mathcal{U}_2(s,t;C_1,C_2)}{\partial s} + C_2\frac{\partial^{\alpha}\mathcal{U}_1(s,t;C_1)}{\partial t^{\alpha}} \\ - C_2\frac{\partial^2 \mathcal{U}_1(s,t;C_1)}{\partial s^2} - \frac{C_2}{s}\frac{\partial \mathcal{U}_1(s,t;C_1)}{\partial s} \\ + C_3\frac{\partial^{\alpha}\mathcal{U}_0(s,t)}{\partial t^{\alpha}} - C_3\frac{\partial^2 \mathcal{U}_0(s,t)}{\partial s^2} \\ - \frac{C_3}{s}\frac{\partial \mathcal{U}_0(s,t)}{\partial s}, \quad \mathcal{U}_3(s,0) = 0. \tag{42}$$

Respective solutions of Equations (39)–(42) after apply fractional integral and initial condition are given:
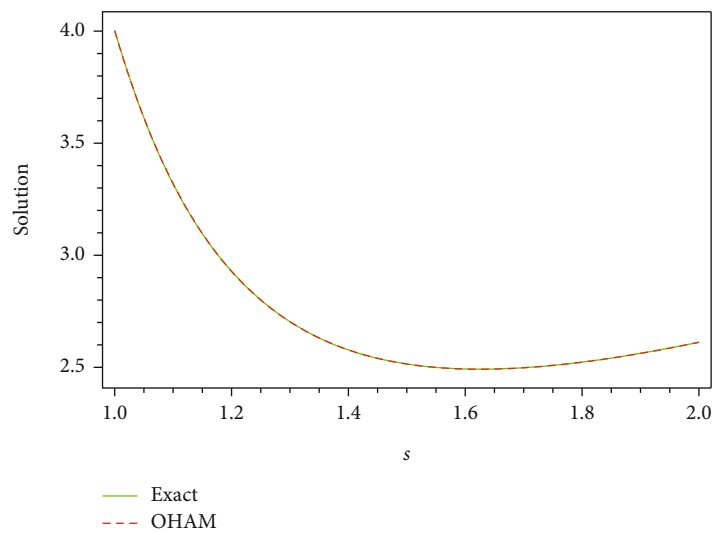
$$\mathcal{U}_0(s,t) = s,$$

$$\mathcal{U}_1(s,t;C_1) = -\frac{C_1 t^{\alpha}}{s\Gamma(1+\alpha)},$$

$$\mathcal{U}_2(s,t;C_1,C_2) = -\frac{(C_1 + C_1^2 + C_2)t^{\alpha}}{s\Gamma(1+\alpha)} + \frac{C_1^2 t^{2\alpha}}{s^3\Gamma(1+2\alpha)},$$
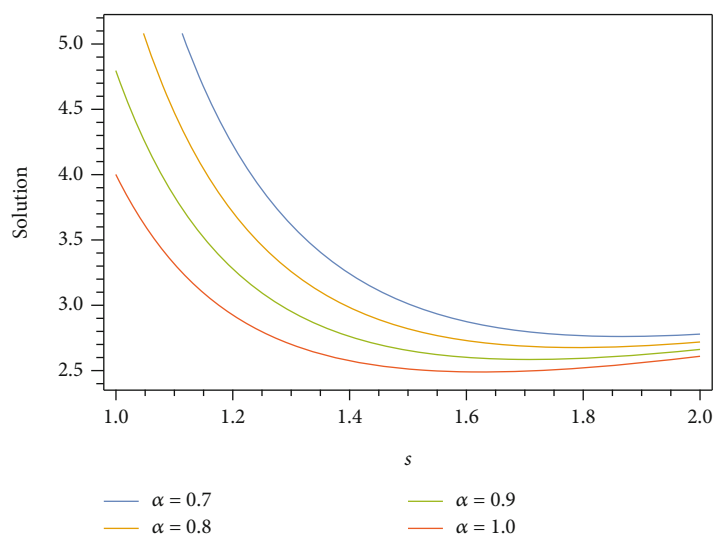
$$\mathcal{U}_3(s,t;C_1,C_2,C_3) = -\frac{(C_1 + 2C_1^2 + C_1^3 + 2C_1C_2 + C_2 + C_3)(t)^{\alpha}}{s\Gamma(1+\alpha)} \\ + \frac{2C_1(C_1 + C_1^2 + C_2)t^{2\alpha}}{s^3\Gamma(1+2\alpha)} - \frac{9C_1^3 t^{3\alpha}}{s^5\Gamma(1+3\alpha)}.$$

$$\vdots \tag{43}$$

One can calculate the next order problem solutions by above similar process. In the end, we can get the expression:
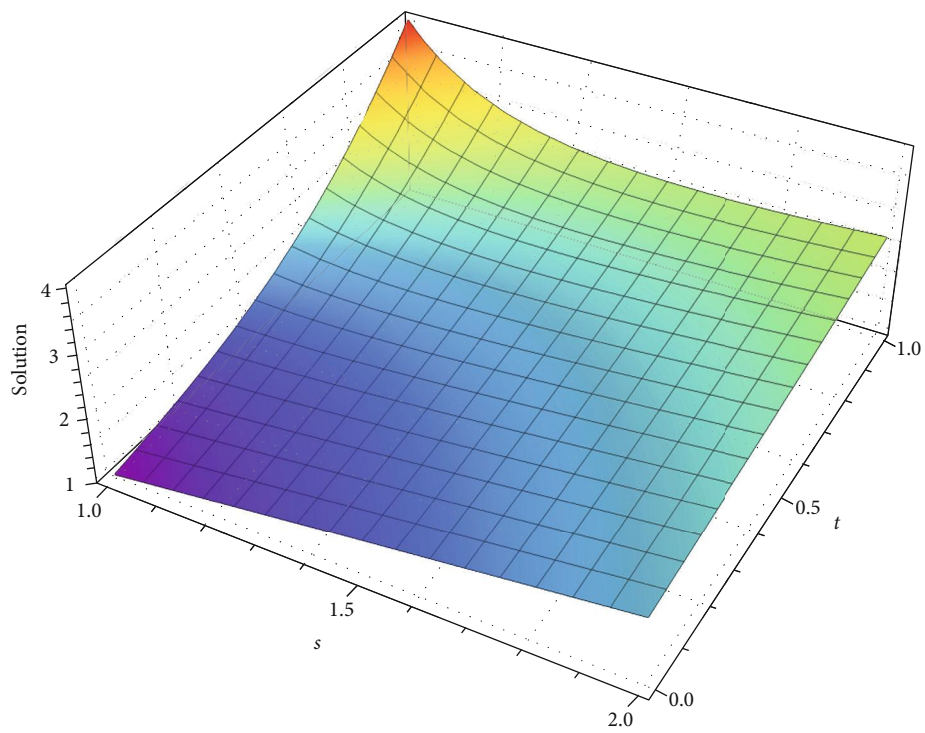
(a) Exact vs. OHAM solution

(b) Solution behavior for different values of α

Figure 2: Continued.
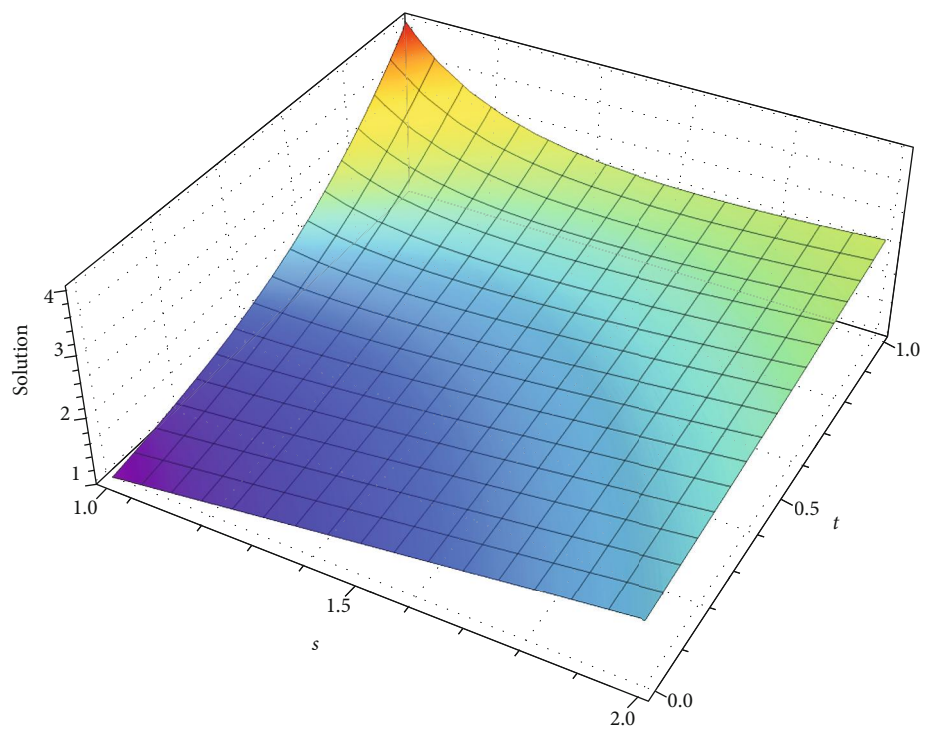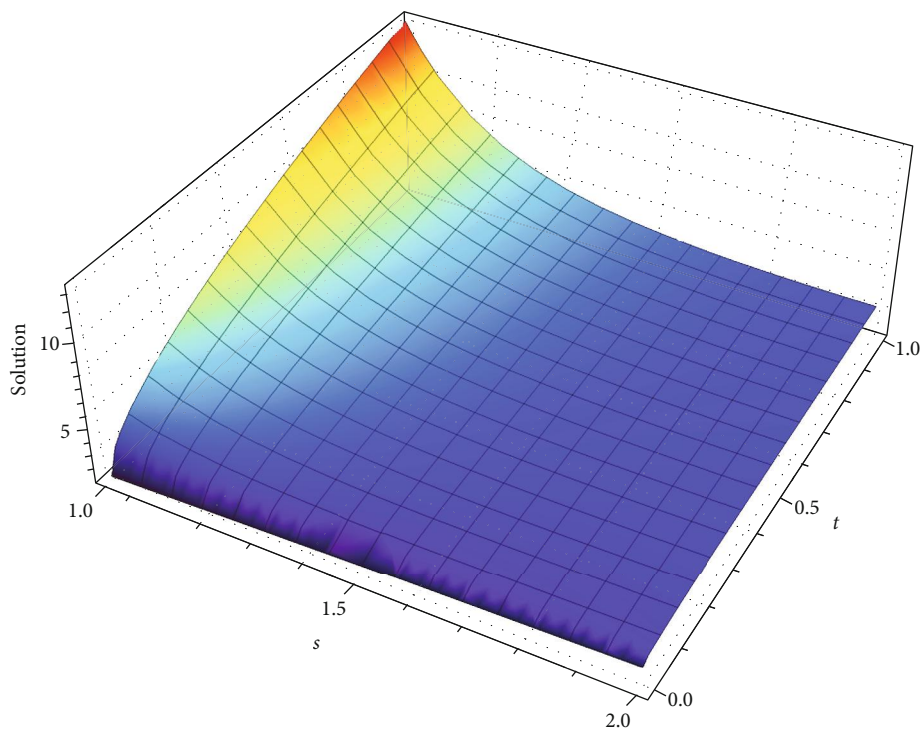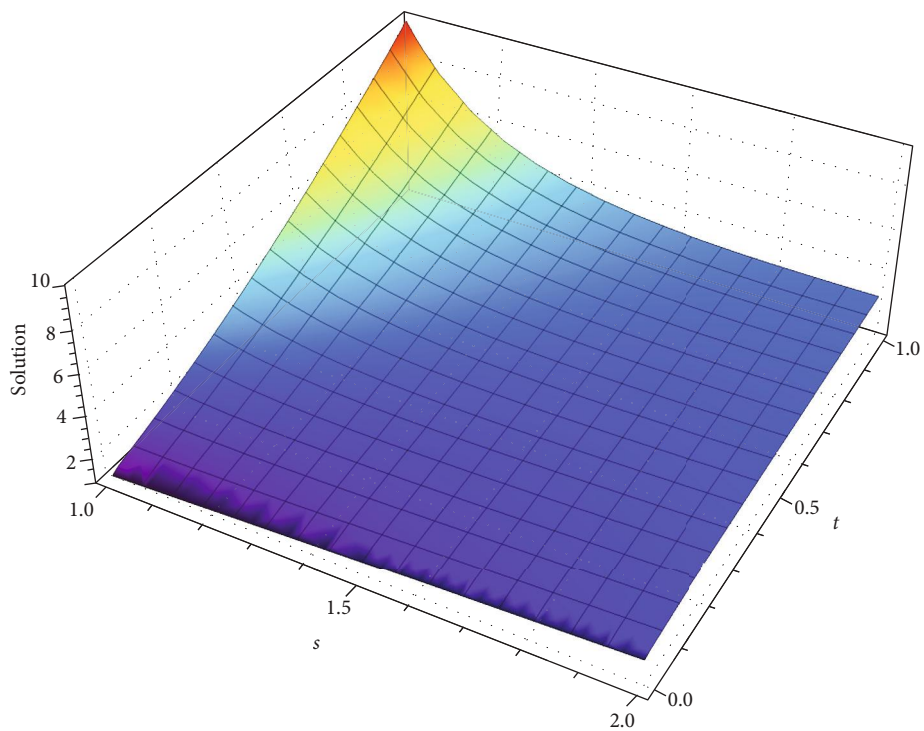
(c) Exact solution

(d) OHAM solution for $\alpha = 1$

Figure 2: Continued.

(e) Solution for $\alpha = 0.2$

(f) Solution for $\alpha = 0.5$

Figure 2: Exact solution and OHAM solution behavior of Example 2.

$$\tilde{\mathcal{U}}(s, t) = \mathcal{U}_0(s, t) + \mathcal{U}_1(s, t; C_1) + \mathcal{U}_2(s, t; C_1, C_2)$$
$$+ \mathcal{U}_3(s, t; C_1, C_2, C_3) + \cdots$$
$$= s - \frac{\left(3C_1 + 3C_1^2 + C_1^3 + 2C_2 + 2C_1 C_2 + C_3\right) t^\alpha}{s\Gamma(1 + \alpha)}$$
$$+ \frac{C_1\left(3C_1 + 2C_1^2 + 2C_2\right) t^{2\alpha}}{s^3 \Gamma(1 + 2\alpha)} - \frac{9C_1^3 t^{3\alpha}}{s^5 \Gamma(1 + 3\alpha)} \cdots.$$

$$(44)$$

We used the least square method after finding the residual and then got the auxiliary constant value for α (see Table 1). The absolute error of both solutions can be seen in Table 2.

Putting the values of C_1, C_2, and C_3 for $\alpha = 1$ in Equation (44), we get

$$\tilde{\mathcal{U}}(s, t) = s + \frac{t^\alpha}{s\Gamma(1 + \alpha)} + \frac{1.00008 t^{2\alpha}}{s^3 \Gamma(1 + 2\alpha)} + \frac{8.99873 t^{3\alpha}}{s^5 \Gamma(1 + 3\alpha)}. \quad (45)$$

Figure 2 shows the solution behavior of the time-fractional order Navier-Stokes equation by OHAM. In Figure 2(a), the solution curves are decreasing rapidly for higher fractional orders until we get the standard motion of fluid for $\alpha = 1$, and Figure 2(b) shows the proposed method's effectiveness and reliability. Figures 2(e) and 2(f) show the OHAM solution behavior for $\alpha = 0.2$ and $\alpha = 0.5$, respectively. The obtained results reveal that the method mentioned above is an efficient tool to study such types of fractional order fluid mechanics problems, which can be seen in Table 2.

5. Conclusion

In this study, fractional-order OHAM is successfully implemented to obtain the optimal solutions of time-fractional Navier-Stokes equation. From the acquired results, it can be seen that OHAM is an efficient and reliable semianalytical technique to approximate the solution of different fractional-order linear and nonlinear problems appearing in engineering and science. The above-mentioned technique provides a simple approach to control and adjust the convergence of the series solution utilizing the constants C_p' s which are determined optimally. Two examples have been studied to illustrate the efficiency and versatility of this approach. The OHAM solution of the first example is the same as the exact solution, and for the OHAM solution of the second example, the obtained numerical approximation of the solution has a strong agreement with the exact solution. Besides, when the order of approximation increases, the error accuracy of the numerical solution decreases and becomes closer to the exact solutions. The proposed technique's fast accuracy and convergence are valid reasons for the researcher to use it for various problems in science and technology. It has been noted that the semianalytical solutions by extended formulation are in remarkable agreement with the exact solutions.

Data Availability

No real data were used to support this study.

Conflicts of Interest

This work does not have any conflicts of interest.

Authors' Contributions

The authors have contributed equally to this manuscript.

Acknowledgments

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