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# Nonstandard Null Lagrangians and Gauge Functions for Newtonian Law of Inertia

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**Abstract:** New null Lagrangians and gauge functions are derived and they are called nonstandard because their forms are different than those previously found. The invariance of the action is used to make the Lagrangians and gauge functions exact. The first exact nonstandard null Lagrangian and its gauge function for the law of inertia are obtained, and their physical implications are discussed.

**Keywords:** Lagrangian formalism; null Lagrangians; gauge functions; law of inertia

## 1. Introduction

Most classical and quantum theories of modern physics are formulated using the standard Lagrangians and Lagrangian formalism [1]. There are also nonstandard Lagrangians, whose applications to physics are considered in [2], and null Lagrangians, whose relevance to ordinary differential equations (ODEs) has already been studied [3]. The main objective of this paper is to derive a new set of nonstandard null Lagrangians and their gauge functions, make them exact, and apply them to the law of inertia.

In the calculus of variations, the action  $\mathcal{A}[x(t)]$ , where  $x(t)$  is a dynamical variable (or classical object's trajectory) that depends on time  $t$ , is defined as an integral over a local real function called a Lagrangian [4], which, for a second-order ODE, is denoted as  $L(\dot{x}, x, t)$ , where the time derivative  $\dot{x}$  represents the particle's velocity in dynamics. The Hamilton principle [1,4] requires that  $\mathcal{A}[x]$  be stationary (to have either a minimum or maximum or saddle point), which is mathematically expressed as  $\delta\mathcal{A} = 0$ , where  $\delta$  is the functional (Fréchet) derivative of  $\mathcal{A}[x(t)]$  with respect to  $x$ . The necessary condition that  $L(\dot{x}, x, t)$  satisfies the Hamilton principle is  $\hat{E}L[L(\dot{x}, x, t)] = 0$ , where  $\hat{E}L$  is the Euler–Lagrange operator [4].

A general second-order ODE with constant coefficients is of the form  $\hat{D}x(t) = 0$ , where  $\hat{D} = d^2/dt^2 + bd/dt + c$  is a linear operator and  $b$  and  $c$  are constants. Let  $\hat{D}_0$  be the operator with  $b = c = 0$ , and  $\hat{D}_c$  be the operator with  $b = 0$ . Then, for  $\hat{D}_0x(t) = 0$ , its Lagrangian depends only on  $\dot{x}^2$ . However, for  $\hat{D}_cx(t) = 0$ , its Lagrangian depends on both  $\dot{x}^2$ , the kinetic energy-like term, and  $x^2$ , the potential energy-like term, as originally shown by Lagrange [5]. Moreover, if  $\hat{D}x(t) = 0$ , its Lagrangian becomes the Caldirola–Kanai Lagrangian [6,7]. Thus, Lagrangians that depend either on  $\dot{x}^2$  or on  $\dot{x}^2$  and  $x^2$  are called *standard* Lagrangians (SLs).

Since the standard Lagrangians are not unique, it is also possible to construct other Lagrangians that typically depend on  $\dot{x}$  and  $x$  but not on powers of these variables, and depend also on arbitrary functions of the independent variable  $t$ ; Arnold [8] refers to such Lagrangians as non-natural Lagrangians. Here, Lagrangians that depend on  $\dot{x}$  and  $x$ , and on functions of  $t$  are called *nonstandard* Lagrangians (NSLs).

The existence of the standard and nonstandard Lagrangians is guaranteed by the Helmholtz conditions [9]. The procedure of finding these Lagrangians for given ODEs is called the inverse (or Helmholtz) problem of the calculus of variations [10,11]. There are different methods to find the standard [12–16] and nonstandard [15–20] Lagrangians. Generalized nonstandard Lagrangians can also be obtained [21] and applied to the ODEs, whose solutions are special functions of mathematical physics [22]. Other generalizations

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of the nonstandard Lagrangians have been applied to the Riccati equation [23] and to a Liénard-type nonlinear oscillator [24].

In addition to the SLs and NSLs, there is also a family of null (or trivial) Lagrangians (NLs) for which the Euler-Lagrange (E-L) equation vanishes identically [3]. Another property of these NLs is that they can be expressed as the total derivative of a scalar function [3], which is called a gauge or gauge function [25,26]; in other words, all NLs have their corresponding gauge functions (GFs). The NLs were constructed and investigated in mathematics, specifically in Cartan and Lapage forms, symmetries of Lagrangians, in Carathéodory's theory of fields, and extremals and integral invariants [3,27–32]. The NLs were also applied to elasticity, where they represent the energy density function of materials [33,34].

The fact that the NLs and their GFs may also play an important role in physics was shown recently by using them to restore Galilean invariance of the standard Lagrangian for Newton's laws of dynamics [35,36] and to add forces to an undriven harmonic oscillator [37]. Since those previously constructed NLs resemble the standard Lagrangians, they are called here *standard* NLs, and their corresponding GFs become *standard* GFs. In this paper, the standard and nonstandard NLs and their GFs are constructed and applied to the law of inertia in Galilean space and time with the Galilean group of the metric [8,25].

The outline of the paper is as follows. In Section 2, previously obtained standard null Lagrangians and their gauge functions are described. In Section 3, the derived new nonstandard null Lagrangians are presented and compared to the standard null Lagrangians. Invariance of the action is used to define the exact nonstandard gauge functions in Section 4. Applications of the obtained results to the Newtonian law of inertia are presented and discussed in Section 5. Section 6 summarizes the results of this paper.

## 2. Standard Null Lagrangians

For the considered ODEs of the form  $\hat{D}x(t) = 0$ , its standard Lagrangian is given by

$$L_s(\dot{x}, x, t) = \frac{1}{2}(\dot{x}^2 - cx^2)e^{bt}. \quad (1)$$

This Lagrangian was first derived by Caldirola [6] and Kanai [7], and it reduces to that given originally by Lagrange [5], if  $b = 0$ .

One of the well-known null Lagrangians [4] is

$$L_{sn1}(\dot{x}, x) = c_1\dot{x}x, \quad (2)$$

where  $c_1$  is an arbitrary constant. In this Lagrangian, the power of the dependent variables is the same as in the standard Lagrangian given by Equation (1); however, the dependent variable and its derivative are mixed.

Recently [35],  $L_{sn1}(\dot{x}, x)$  was generalized to

$$L_{sn2}(\dot{x}, x, t) = c_1\dot{x}x + c_2(\dot{x}t + x) + c_3\dot{x} + c_4, \quad (3)$$

where  $c_2$ ,  $c_3$ , and  $c_4$  are arbitrary constants, and  $L_{sn2}$  becomes  $L_{sn1}$  if  $c_2 = c_3 = c_4 = 0$ . This Lagrangian was constructed based on the principle that the power of the terms with the dependent or independent variable, or their combination, does not exceed the power of the terms in the original standard Lagrangian given by Equation (1).

Since

$$L_{sn2}(\dot{x}, x, t) = \frac{d\Phi_{sn2}(x, t)}{dt}, \quad (4)$$

the gauge function  $\Phi_{sn2}(x, t)$  is given [35] by

$$\Phi_{sn2}(x, t) = \frac{1}{2}c_1x^2 + c_2xt + c_3x + c_4t. \quad (5)$$

Following [36], the derived  $\Phi_{sn2}(x, t)$  is generalized by replacing its constant coefficients by arbitrary functions that depend only on the independent variable. Then, the standard GF,  $\Phi_{sn3}(x, t)$ , can be written as:

$$\Phi_{sn3}(x, t) = \frac{1}{2}f_1(t)x^2 + f_2(t)xt + f_3(t)x + f_4(t)t. \tag{6}$$

Since  $\Phi_{sn3}(x, t)$  is a function of the variables  $x$  and  $t$ , and its total derivative results in the following general standard null Lagrangian

$$L_{sn3}(\dot{x}, x, t) = [f_1(t)\dot{x} + \frac{1}{2}\dot{f}_1(t)x]x + [f_2(t)\dot{x} + \dot{f}_2(t)x]t + f_2(t)x + [f_3(t)\dot{x} + \dot{f}_3(t)x] + [f_4(t) + \dot{f}_4(t)t], \tag{7}$$

where  $f_1(t)$ ,  $f_2(t)$ ,  $f_3(t)$ , and  $f_4(t)$  are arbitrary but at least twice differentiable functions of the independent variable [37]. Additional constraints on these functions are presented in Section 4, where invariance of the action is considered.

The generalization of the gauge function given by Equation (6), is one natural way to obtain a new NL, but there is also another way, namely, by replacing the constant coefficients in  $L_{sn2}(\dot{x}, x, t)$  (see Equation (3)) by the functions  $f_1(t)$ ,  $f_2(t)$ ,  $f_3(t)$ , and  $f_4(t)$ . The result is:

$$L_{sn4}(\dot{x}, x, t) = f_1(t)\dot{x}x + f_2(t)(\dot{x}t + x) + f_3(t)\dot{x} + f_4(t). \tag{8}$$

Applying  $\hat{E}L\{L_{s,test}(\dot{x}, x, t)\} = 0$ , it is found that  $L_{sn4}(\dot{x}, x, t)$  is a NL if, and only if, the following condition

$$\dot{f}_1(t)x + \dot{f}_2(t)t + \dot{f}_3(t) = 0. \tag{9}$$

on the functions  $f_1(t)$ ,  $f_2(t)$ , and  $f_3(t)$  is imposed.

There are several different solutions to Equation (9); the simplest one is  $f_1(t) = c_1$ ,  $f_2(t) = c_2$ , and  $f_3(t) = c_3$ , which reduces  $L_{sn4}(\dot{x}, x, t)$  to  $L_{sn2}(\dot{x}, x, t)$  without any generalization, but with an additional requirement that  $f_4(t) = c_4$ . More interesting cases are: (i)  $f_1(t) = c_1$ , which gives  $\dot{f}_2(t)t = -\dot{f}_3(t)$ ; (ii)  $f_2(t) = c_2$  and  $\dot{f}_1(t)x = -\dot{f}_3(t)$ ; and (iii)  $f_3(t) = c_3$  with  $\dot{f}_1(t)x = -\dot{f}_2(t)t$ . In all three cases, three new standard NLs are obtained.

Since the functions  $f_1(t)$ ,  $f_2(t)$ , and  $f_3(t)$  in  $L_{sn4}(\dot{x}, x, t)$  are limited by the auxiliary condition, given by Equation (9), and since  $L_{sn3}(\dot{x}, x, t)$  does not require any restrictions on these functions, the standard NL, given by Equation (7), is more general than  $L_{sn4}(\dot{x}, x, t)$ ; thus, the standard general NL becomes  $L_{sgn}(\dot{x}, x, t) = L_{sn3}(\dot{x}, x, t)$ .

The following Corollary summarizes (without a formal proof) the results obtained in this Section.

**Corollary 1.** *The Lagrangian  $L_{sgn}(\dot{x}, x, t)$  is the general null Lagrangian among all null Lagrangians that can be constructed based on the principle that the power of the dependent variable cannot exceed the power of this variable in the SL, given by Equation (1).*

### 3. Nonstandard Null Lagrangians

Any Lagrangian different from  $L_s(\dot{x}, x)$  is a nonstandard Lagrangian. Among different known nonstandard Lagrangians, the most commonly used [15–19] is:

$$L_{ns}(\dot{x}, x, t) = \frac{1}{g_1(t)\dot{x} + g_2(t)x + g_3(t)}, \tag{10}$$

where  $g_1(t)$ ,  $g_2(t)$ , and  $g_3(t)$  are arbitrary and differentiable functions to be determined.

Since there are no nonstandard NLs in the literature, the objective of this paper is to find them. The procedure is based on the two following conditions. First, for a null Lagrangian to be called *nonstandard*, it must be of different form than the standard NLs, given by Equations (3) and (7), and its form must be similar to that of Equation (10).

The latter means that it must contain  $\dot{x}$ ,  $x$ , and arbitrary functions of  $t$ , or constants. The second condition is similar to that used to construct  $L_{\text{sn}2}(\dot{x}, x, t)$ ,  $L_{\text{sn}3}[\dot{x}, x, t]$  and  $L_{\text{sn}4}(\dot{x}, x, t)$ , namely, the power of the dependent variable and its derivative must not exceed their order in the nonstandard Lagrangian given by Equation (10). The obtained nonstandard null Lagrangians are presented in Propositions 1 and 2, and in the Corollaries that follow them.

**Proposition 1.** Let  $a_1, a_2, a_3$ , and  $a_4$  be constants in the following nonstandard test-Lagrangian

$$L_{\text{ns,test}1}(\dot{x}, x, t) = \frac{a_1 \dot{x}}{a_2 x + a_3 t + a_4}. \tag{11}$$

Then,  $L_{\text{ns,test}1}(\dot{x}, x, t)$  is a null Lagrangian if, and only if,  $a_3 = 0$ .

**Proof.** Since this Lagrangian must satisfy the E-L equation,  $\hat{E}L\{L_{\text{ns,test}1}(\dot{x}, x, t)\} = 0$ , the required condition is  $a_1 a_3 = 0$ . With  $a_1 \neq 0$ , then  $a_3 = 0$ , and  $L_{\text{ns,test}1}(\dot{x}, x, t) = L_{\text{nsn}1}(\dot{x}, x, t)$ , where the latter is the nonstandard NL. This concludes the proof.  $\square$

**Corollary 2.** Let  $L_{\text{nsn}1}[\dot{x}, x]$  be the nonstandard null Lagrangian given by:

$$L_{\text{nsn}1}(\dot{x}, x, t) = \frac{a_1 \dot{x}}{a_2 x + a_4}, \tag{12}$$

then its gauge function  $\Phi_{\text{nsn}1}(x)$  is:

$$\Phi_{\text{nsn}1}(x) = \frac{a_1}{a_2} \ln |a_2 x + a_4|. \tag{13}$$

**Corollary 3.** Another nonstandard null Lagrangian that can be constructed is  $L_{\text{nsn}2}(t) = b_1 / (b_2 t + b_3)$  and the corresponding gauge function is  $\Phi_{\text{nsn}2}(t) = (b_1 / b_2) \ln |b_2 t + b_3|$ ; however, the Lagrangian and gauge function do not obey the first condition; thus, they will not be further considered.

Generalization of  $L_{\text{nsn}1}(\dot{x}, x)$  is now presented in Proposition 2.

**Proposition 2.** Let  $h_1(t), h_2(t)$ , and  $h_4(t)$  be at least twice differentiable functions and  $L_{\text{nsn}1}(\dot{x}, x)$  be the nonstandard null Lagrangian given by Equation (12), with the corresponding nonstandard gauge function given by Equation (13). A more general nonstandard null Lagrangian is obtained if, and only if, the constants in  $\Phi_{\text{nsn}1}(x)$  are replaced by the functions  $h_1(t), h_2(t)$ , and  $h_4(t)$ .

**Proof.** Replacing the constant coefficients  $a_1, a_2$ , and  $a_4$  in  $L_{\text{nsn}1}(\dot{x}, x)$  by the functions  $h_1(t), h_2(t)$ , and  $h_4(t)$ , respectively, the resulting Lagrangian is:

$$L_{\text{ns,test}2}(\dot{x}, x, t) = \frac{h_1(t) \dot{x}}{h_2(t)x + h_4(t)}. \tag{14}$$

Using  $\hat{E}L\{L_{\text{ns,test}2}(\dot{x}, x, t)\} = 0$ , it is found that  $L_{\text{ns,test}2}(\dot{x}, x, t)$  is the nonstandard NL only when  $h_1(t) = a_1, h_2(t) = a_2$  and  $h_4(t) = a_4$ , which reduces  $L_{\text{ns,test}2}(\dot{x}, x, t)$  to  $L_{\text{nsn}1}(\dot{x}, x, t)$  and shows that no generalization of  $L_{\text{nsn}1}(\dot{x}, x, t)$  can be accomplished this way.

Now, replacing the constant coefficients in  $\Phi_{\text{nsn}1}(x)$  by the functions  $h_1(t), h_2(t)$  and  $h_4(t)$  generalizes the gauge function to

$$\Phi_{\text{nsn}1}(x) = \frac{h_1(t)}{h_2(t)} \ln |h_2(t)x + h_4(t)|. \tag{15}$$

Since the total derivative of any differentiable scalar function that depends on  $x$  and  $t$  is a null Lagrangian, the following nonstandard NL is obtained:

$$L_{\text{nsgn}}(\dot{x}, x, t) = \frac{h_1(t)[h_2(t)\dot{x} + \dot{h}_2(t)x] + h_4(t)}{h_2(t)[h_2(t)x + h_4(t)]} + \left[ \frac{\dot{h}_1(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2^2(t)} \right] \ln |h_2(t)x + h_4(t)|. \quad (16)$$

As expected,  $\hat{E}L\{L_{\text{nsgn}}(\dot{x}, x, t)\} = 0$ ; thus,  $L_{\text{nsgn}}(\dot{x}, x, t)$  is the general nonstandard null Lagrangian when compared to Equation (12). This concludes the proof.  $\square$

The derived  $L_{\text{nsgn}}(\dot{x}, x, t)$  and  $\Phi_{\text{nsgn}}(\dot{x}, x, t)$  represent new families of nonstandard general NLs and their GFs. These NLs and GFs were derived based on the condition that the power of the dependent variable in the nonstandard general NLs is either the same as, or lower than, that displayed in the original NSL given by Equation (10).

#### 4. Action Invariance and Conditions for Exactness

Since the functions in the standard and nonstandard general null Lagrangians are arbitrary, they require either mathematical or physical constraints, or both. Among possible mathematical constraints is invariance of the action, which is used to introduce exact gauge functions [36], and symmetries of Lagrangians and the resulting dynamical equations [38–42]. Moreover, by using Galilean invariance [35], the additional physical constraints are imposed on the GFs [36]. Here, only the invariance of the action is applied to the obtained standard and nonstandard general NLs and GFs; symmetries of Lagrangians are also briefly discussed.

In the calculus of variations, the action is defined as:

$$\begin{aligned} A[x; t_e, t_o] &= \int_{t_o}^{t_e} (L + L_{\text{null}})dt = \int_{t_o}^{t_e} Ldt + \int_{t_o}^{t_e} \left[ \frac{d\Phi_{\text{null}}(t)}{dt} \right] dt \\ &= \int_{t_o}^{t_e} Ldt + [\Phi_{\text{null}}(t_e) - \Phi_{\text{null}}(t_o)], \end{aligned} \quad (17)$$

where  $t_o$  and  $t_e$  denote initial and final times,  $L$  is a Lagrangian that can be either any SL or any NSL,  $L_{\text{null}}$  is any null Lagrangian and  $\Phi_{\text{null}}$  is its gauge function. Since both  $\Phi_{\text{null}}(t_e)$  and  $\Phi_{\text{null}}(t_o)$  are constants, they do not affect the Hamilton principle that requires  $\delta A[x] = 0$ . However, the requirement that  $\Delta\Phi_{\text{null}} = \Phi_{\text{null}}(t_e) - \Phi_{\text{null}}(t_o) = \text{const}$  adds this constant to the value of the action. In other words, the value of the action is affected by the gauge function.

Using invariance of the action, the following definitions are introduced.

**Definition 1.** A null Lagrangian, whose  $\Delta\Phi_{\text{null}} = 0$ , is called the exact null Lagrangian (ENL).

**Definition 2.** A gauge function with  $\Delta\Phi_{\text{null}} = 0$  is called the exact gauge function (EGF).

The condition  $\Delta\Phi_{\text{null}} = 0$  is satisfied when either  $\Phi_{\text{null}}(t_e) - \Phi_{\text{null}}(t_o) = 0$ , or  $\Phi_{\text{null}}(t_e) = 0$  and  $\Phi_{\text{null}}(t_o) = 0$ ; let the latter be valid. Then, the exact null Lagrangians are those whose exact gauge functions make the action invariant.

Invariance of the action may now be used to establish constraints on the arbitrary functions in the standard NL,  $L_{\text{sgn}}(\dot{x}, x, t)$  (see Equation (7)), and its gauge function,  $\Phi_{\text{sgn}}(x, t)$  (see (6)), and make them exact. Taking  $\Phi_{\text{sgn}}(t_e) = 0$  and  $\Phi_{\text{sgn}}(t_o) = 0$ , the following conditions are obtained:

$$\frac{1}{2}f_1(t_e)x_e^2 + f_2(t_e)x_e t_e + f_3(t_e)x_e + f_4(t_e)t_e = 0, \quad (18)$$

and

$$\frac{1}{2}f_1(t_o)x_o^2 + f_2(t_o)x_o t_o + f_3(t_o)x_o + f_4(t_o)t_o = 0, \quad (19)$$

with  $x_e = x(t_e)$  and  $x_o = x(t_o)$  denoting the end points. If the arbitrary functions satisfy these conditions, then  $L_{\text{sgn}}(\dot{x}, x, t)$  and  $\Phi_{\text{sgn}}(x, t)$  are exact. The first condition may be solved by taking  $f_3(t_e) = -f_1(t_e)x_e/2$ , and  $f_4(t_e) = -f_2(t_e)x_e$ . Similar solutions are valid for  $t_o$  showing that the end values for the functions can be related to each other.

Applying the same procedure to  $L_{\text{nsgn}}(\dot{x}, x, t)$ , (see Equation (16)), and to the resulting general gauge function  $\Phi_{\text{nsgn}}(\dot{x}, x, t)$ , (see Equation (15)), the conditions on the arbitrary functions are:

$$\left[ \frac{h_1(t_e)}{h_2(t_e)} \right] \ln |h_2(t_e)x_e + h_4(t_e)| = 0, \tag{20}$$

and

$$\left[ \frac{h_1(t_o)}{h_2(t_o)} \right] \ln |h_2(t_o)x_o + h_4(t_o)| = 0. \tag{21}$$

Since  $\ln[h_2(t_e)x + h_4(t_e)] \neq 0$  and  $\ln[h_2(t_o)x + h_4(t_o)] \neq 0$ , both conditions set up stringent limits on the function  $h_1(t)$ , whose end values must be:  $h_1(t_e) = 0$  and  $h_1(t_o) = 0$ ; however, the procedure does not impose any constraint either on  $h_2(t)$  or on  $h_4(t)$ .

Further constraints on all arbitrary functions that appear in the standard and non-standard, general, exact null Lagrangians (ENLs) can be imposed by considering symmetries of these Lagrangians and the resulting dynamical equations. In general, Lagrangians possess less symmetry than the equations they generate [38]. Among different symmetries, Noether and non-Noether symmetries are identified [39–42]. The presence of NLs does not affect the Noether symmetries [38,41]; however, it may effect the non-Noether symmetries [42]. All these symmetries impose new constraints on the functions.

### 5. Applications to Newtonian Law of Inertia

Let  $(x, y, z)$  be a Cartesian coordinate system, and let  $t$  be time in all inertial frames; then the one-dimensional motion of a body in one inertial frame is given by  $\hat{D}_o x(t) = 0$ , which represents the law of inertia. Let  $t_o = 0$  and  $t_e = 1$  be the end conditions, and let  $x(0) = x_o = 1$ ,  $x(1) = x_e = 2$  and  $\dot{x}(0) = u_o$  be the initial conditions. Then, the solution to  $\hat{D}_o x(t) = 0$  is  $x = u_o t + 1$ .

The *standard* Lagrangian for this equation of motion is given by Equation (1), with the coefficients  $b = c = 0$ , and no arbitrary function to be determined. However, the standard general NL and the corresponding GF are given by Equations (7) and (6), respectively. To make the NL and GF exact, the following conditions (see Equations (18) and (19)) must be imposed on the arbitrary functions:

$$f_1(1) + f_2(1) + f_3(1) + f_4(1) = 0, \tag{22}$$

and

$$f_3(0) = -\frac{1}{2}f_1(0). \tag{23}$$

These conditions guarantee that  $L_{\text{sgn}}(\dot{x}, x, t)$  and  $\Phi_{\text{sgn}}(x, t)$  are the standard general ENL and the standard general EGF, respectively.

The *nonstandard* Lagrangian for the law of inertia is presented by the following Proposition.

**Proposition 3.** *Let  $g_1(t)$ ,  $g_2(t)$ , and  $g_3(t)$  be arbitrary but differentiable functions, and let  $\hat{D}_o x(t) = 0$  be the equation of motion for the law of inertia. Then, the nonstandard Lagrangian for this equation of motion is:*

$$L_{\text{ns}}(\dot{x}, x, t) = \frac{1}{C_1(a_o t + v_o)^2} \frac{1}{(a_o t + v_o)\dot{x} - a_o x + C_2}, \tag{24}$$

where  $a_o$ ,  $v_o$ ,  $C_1$ , and  $C_2$  are constants.



**Proof.** Following [18], the functions must satisfy

$$\frac{g_2(t)}{g_1(t)} + \frac{1}{3} \frac{\dot{g}_1(t)}{g_1(t)} = 0, \tag{25}$$

$$\frac{\dot{g}_2(t)}{g_1(t)} - \frac{1}{2} \frac{\dot{g}_1(t)}{g_1(t)} \frac{g_2(t)}{g_1(t)} + \frac{g_2^2(t)}{2g_1^2(t)} = 0, \tag{26}$$

and

$$\frac{\dot{g}_3(t)}{g_1(t)} - \frac{1}{2} \frac{\dot{g}_1(t)}{g_1(t)} \frac{g_3(t)}{g_1(t)} + \frac{g_3(t)}{g_1(t)} \frac{g_2(t)}{2g_1(t)} = 0. \tag{27}$$

Eliminating  $g_2(t)$  from Equations (25) and (26), and defining  $u(t) = \dot{g}_1(t)/g_1(t)$ , one obtains:

$$\dot{u}(t) + \frac{1}{3}u^2(t) = 0, \tag{28}$$

which is a special form of the Riccati equation. Following [22], the solution to Equation (28) is:

$$u(t) = 3 \frac{\dot{v}(t)}{v(t)}, \tag{29}$$

with  $v(t)$  representing a solution to  $\ddot{v}(t) = 0$ , which is the auxiliary condition [21,22].

The initial conditions  $v(t = 0) = v_0$  and  $\dot{v}(t = 0) = a_0$  are different from those used for  $\hat{D}_0 x(t) = 0$ . Then, the solution becomes  $v(t) = a_0 t + v_0$ , and the functions  $g_1(t)$ ,  $g_2(t)$  and  $g_3(t)$  become

$$g_1(t) = C_1(a_0 t + v_0)^3, \tag{30}$$

where  $C_1$  is an integration constant. Having obtained  $g_1(t)$ ,  $g_2(t)$  becomes

$$g_2(t) = -C_1 a_0 (a_0 t + v_0)^2. \tag{31}$$

Finally,  $g_3(t)$  can be found by eliminating  $g_1(t)$  and  $g_2(t)$  from Equation (26). The solution is

$$g_3(t) = C_1 C_2 (a_0 t + v_0)^2, \tag{32}$$

where  $C_2$  is an integration constant.

Substituting  $g_1(t)$ ,  $g_2(t)$ , and  $g_3(t)$  into Equation (10), for  $\hat{D}_0 x(t) = 0$ , the following final form of the NSL is obtained

$$L_{ns}(\dot{x}, x, t) = \frac{1}{C_1(a_0 t + v_0)^2} \frac{1}{(a_0 t + v_0)\dot{x} - a_0 x + C_2}, \tag{33}$$

which is the same as that given by Equation (24). This concludes the proof.  $\square$

The derived NSL depends on two constants,  $a_0$  and  $v_0$ , which are given by the initial conditions for  $\ddot{v}(t) = 0$ , and two arbitrary constants,  $C_1$  and  $C_2$ , which may be determined by the initial conditions for  $\hat{D}_0 x = 0$ . It is easy to verify that  $L_{ns}(\dot{x}, x, t)$  gives  $\hat{D}_0 x(t) = 0$  when substituted into the E-L equation. This is the first example of the nonstandard Lagrangian for the Newtonian law of inertia.

The nonstandard general null Lagrangian  $L_{nsgn}(\dot{x}, x, t)$ , and its gauge function,  $\Phi_{nsgn}(x, t)$ , are given by Equations (16) and (15), respectively. To make this NL and its GF exact, the following conditions must be obeyed (see Equations (20) and (21)):

$$\left[ \frac{h_1(1)}{h_2(1)} \right] \ln |2h_2(1) + h_4(1)| = 0, \tag{34}$$

and

$$\left[ \frac{h_1(0)}{h_2(0)} \right] \ln [h_2(0) + h_4(0)] = 0. \tag{35}$$

Since  $\ln[h_2(1) + h_4(1)] \neq 0$  and  $\ln[h_2(0) + h_4(0)] \neq 0$ , the end values of the function  $h_1(t)$  must be:  $h_1(1) = 0$  and  $h_1(0) = 0$ , but the end values of either  $h_2(t)$  or  $h_4(t)$  are not limited by the conditions for exactness. Further constraints on the functions  $h_1(t)$ ,  $h_2(t)$ , and  $h_4(t)$  may be imposed by considering symmetries and Lie groups [26] of the derived nonstandard general ENL (see Section 4). With these constraints, the first nonstandard EGF for the law of inertia is obtained.

The derived nonstandard null Lagrangians are of different forms when compared to the standard null Lagrangians obtained in Section 3; therefore, it is suggested that the NLs be divided into two separate sets. In previous work [35,36], it was shown that standard null Lagrangians and their gauge functions can be used to restore Galilean invariance of Lagrangians in classical mechanics, and to introduce classical forces. The main physical implication of the results obtained in this paper is that similar restoration of invariance of Lagrangians and definition of forces can also be carried out by using the derived general nonstandard NLs, and that the resulting forces will be of different forms from those previously determined [36]. The fact that not all general standard NLs contribute to the forces was shown by [37]; only NLs of special forms can be used to define forces [36,37]. In general, most NLs have no influence on these forces. It remains to be determined whether the derived general nonstandard null Lagrangian and its gauge function define forces, and whether they can be used to convert the first law of dynamics into the second law; however, such studies are beyond the scope of this paper.

The presented methods of finding general standard and nonstandard ENLs and their EGFs can be extended to all second-order ODEs of the form  $\hat{D}x(t) = 0$ , which includes the equations of motion of undamped and damped oscillators, and other dynamical systems. In previous work [43], the general standard ENLs and EGFs were derived for the Bateman oscillators; however, the nonstandard ENLs and EGFs are yet to be obtained. The presented methods may also be generalized to partial differential equations of quantum mechanics, such as the Schrödinger equation.

## 6. Conclusions

This paper presents methods to construct null Lagrangians. Using these methods, two different sets of null Lagrangians were obtained and classified as standard and nonstandard. The corresponding sets of gauge functions were also derived. The presented general standard null Lagrangians are known, but the general nonstandard null Lagrangians obtained in this paper are new. Since there are differences in the forms and properties of the two sets of null Lagrangians, it is suggested that these Lagrangians be divided into two classes that correspond to these sets.

The invariance of the action is used to introduce the exactness of both general standard and nonstandard gauge functions. Having obtained the exact gauge functions, they are used to derive the exact null Lagrangians. All null Lagrangians and gauge functions, derived in this paper are exact, which gives constraints on the end values of the arbitrary functions these Lagrangians and gauge functions depend on. Further constraints can be imposed by symmetries of the exact null Lagrangians and the corresponding exact gauge functions, as well as their underlying Lie groups [26].

The obtained results are applied to the ordinary differential equation (ODE) that represents the law of inertia for which the general exact nonstandard null Lagrangian and the corresponding general exact gauge function are derived. It is suggested that the derived Lagrangian and gauge function be used to restore Galilean invariance of the standard Lagrangian for this law, and to introduce forces as carried out in the previous work for the general standard null Lagrangians [35–37]. Since there are significant differences between the general standard and nonstandard null Lagrangians, the resulting forces must also be different, which may allow establishing a general procedure of defining forces in classical mechanics independently from Newton's law of dynamics.

Finally, it must be pointed out that the same method can be used to obtain the general exact nonstandard null Lagrangians and their gauge functions for any ODE given by



$\hat{D}x(t) = 0$ , and that it can be extended to homogeneous and inhomogeneous partial differential equations, and applied to physical problems described by these equations.

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