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Vector Potential, Magnetic Field, Mutual Inductance, Magnetic Force, Torque and Stiffness Calculation between Current-Carrying Arc Segments with Inclined Axes in Air

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Abstract: In this paper, the improved and the new analytical and semi-analytical expressions for calculating the magnetic vector potential, magnetic field, magnetic force, mutual inductance, torque, and stiffness between two inclined current-carrying arc segments in air are given. The expressions are obtained either in the analytical form over the incomplete elliptic integrals of the first and the second kind or by the single numerical integration of some elliptical integrals of the first and the second kind. The validity of the presented formulas is proved from the particular cases when the inclined circular loops are addressed. We mention that all formulas are obtained by the integral approach, except the stiffness, which is found by the derivative of the magnetic force. The novelty of this paper is the treatment of the inclined circular current-carrying arc segments for which the calculations of the previously mentioned electromagnetic quantities are given.

Keywords: vector potential; magnetic field; mutual inductance; magnetic force; torque; stiffness



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1. Introduction

In general, a field circular coil of arbitrary geometry may be made from finite circular current-carrying arc segments of the conductor for which the magnetic field can be calculated by the sum over the partial fields generated by each segment [1–16]. This assumption significantly simplifies studying the current-carrying coils of the complex shape. In this paper, we study the inclined circular current-carrying arc segments to calculate the magnetic force, the magnetic torque, the mutual inductance, and the stiffness between them. The goal of this paper is the calculation of these electromagnetic quantities using the magnetic field of the current-carrying arc segment. The magnetic vector potential and the magnetic field of the current-carrying segment can be considered as the auxiliary functions for the simplified calculation of the previously mentioned quantities. Even though in the literature one can find many methods for calculating the magnetic vector potential and magnetic field of different current coils, we give the simplified formulas for calculating them, which will be the crucial auxiliary functions because they are given in the analytical form over the incomplete elliptic integrals of the first and second kind.

Several analytical, semi-analytical and numerical methods in the calculation of parameters of electric circuits such as the self and the mutual inductance and force interaction between their elements play a significant role in power transfer, wireless communication, sensing and actuation, and are applied in different fields of science, including electrical and electronic engineering, medicine, physics, nuclear magnetic resonance, mechatronics, and robotics [17–38]. These calculations are obtained in the form of double integrals, complete or incomplete integrals of the first, second and third kind, Bessel and Struve functions, hypergeometric series, and other special functions, so that it is difficult for the potential user to obtain fast and precise calculation. Additionally, many of these methods use filament methods which study coaxial circular coils or circular coils with parallel axes. In this paper we give a quite simple method for calculating the magnetic force and the mutual inductance between two inclined circular current-carrying arc segments in air,

which can be used to calculate these parameters for inclined circular coils by using the filament method [20]. We give simple formulas for the magnetic force and the mutual inductance in the form of the single integral whose kernel functions are the incomplete integrals of the first and second kind as well as the elementary functions. Finally, we give new formulas for the calculation of the magnetic torque and the stiffness between two inclined circular current-carrying arc segments in air. They are obtained in the form of the single integral whose kernel functions are the incomplete integrals of the first and second kind. To our knowledge, these formulas appear for the first time in the literature. In all formulas, the angles of the current-carrying arcs are arbitrary. The validity of all formulas is verified with the corresponding calculations for the inclined circular loops. For the convenience of the reader, all the derived formulas were programmed using Mathematica. The Mathematica files with the implemented formulas are available from the author.

2. Basic Expressions

Let us take into consideration two current-carrying arc segments as shown in Figure 1, where the center of the larger segment (primary coil) of the radius R_P is placed at the plane XOY whose center is $O(0, 0, 0)$. The smaller circular segment (secondary coil) of the radius R_S is placed in an inclined plane whose general equation is:

$$\lambda \equiv ax + by + cz + d = 0, \tag{1}$$

where a, b and c are the components of the normal \vec{N} on the inclined plane in the center of the secondary circular segment $C(x_C, y_C, z_C)$. The current-carrying arc segments are with currents I_P, I_S . For circular segments (see Figure 1), we define [22–24]:

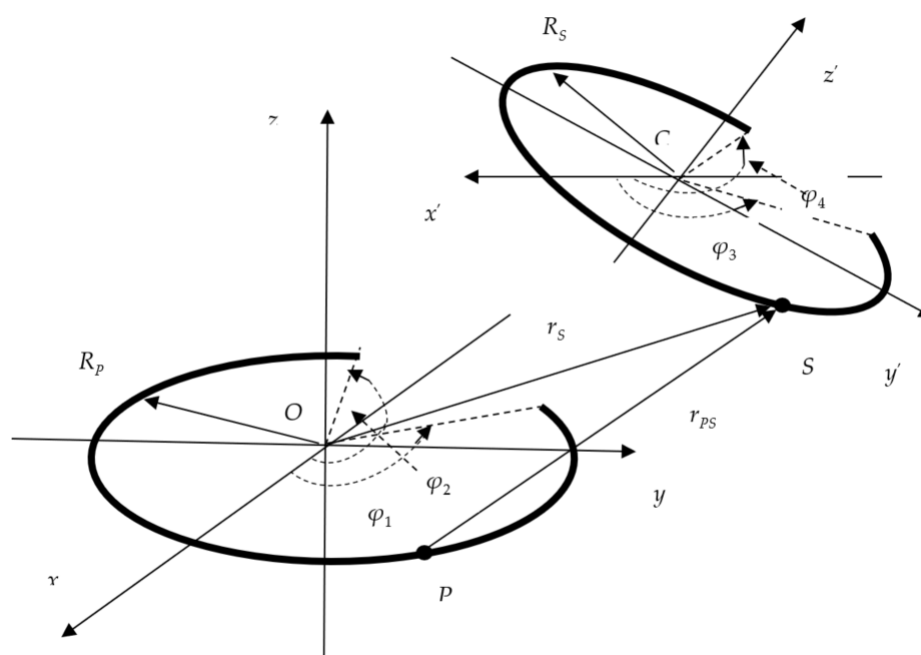


Figure 1. Two inclined current-carrying arc segments.

(1) The primary circular segment of radius R_P is placed in the plane XOY ($Z = 0$) with the center at $O(0, 0, 0)$. An arbitrary point $P(x_P, y_P, z_P)$ of this segment has parametric coordinates,

$$x_P = R_P \cos(t), \quad y_P = R_P \sin(t), \quad z_P = 0, \quad t \in (\varphi_1, \varphi_2). \tag{2}$$

(2) The differential of the primary circular segment is given by

$$d\vec{l}_P = R_P \{-\sin(t), \cos(t), 0\} dt, \quad t \in (\varphi_1, \varphi_2). \tag{3}$$

(3) The secondary circular segment of radius R_S is placed in the inclined plane (1) with the center at $C(x_c, y_c, z_c)$. The unit vector \vec{N} (the unit vector of the axis z') at the point C , which is the center of the secondary circular segment, lying in plane λ is defined by

$$\vec{N} = \left\{ \frac{a}{L}, \frac{b}{L}, \frac{c}{L} \right\}, \quad L = \sqrt{a^2 + b^2 + c^2}. \tag{4}$$

(4) The unit vector between two points C and S which are placed in plane (1) is:

$$\vec{u} = \{u_x, u_y, u_z\} = \left\{ -\frac{ab}{Ll}, \frac{l}{L}, -\frac{bc}{Ll} \right\}, \quad l = \sqrt{a^2 + c^2}. \tag{5}$$

(5) We define the unit vector \vec{v} as the cross product of the unit vectors \vec{N} and \vec{u} as follows:

$$\vec{v} = \vec{N} \times \vec{u} = \{v_x, v_y, v_z\} = \left\{ -\frac{c}{l}, 0, \frac{a}{l} \right\}. \tag{6}$$

(6) An arbitrary point $S(x_s, y_s, z_s)$ of the secondary circular segment has parametric coordinates,

$$\begin{aligned} x_s &= x_c + R_S u_x \cos(\theta) + R_S v_x \sin(\theta), \\ y_s &= y_c + R_S u_y \cos(\theta) + R_S v_y \sin(\theta), \\ z_s &= z_c + R_S u_z \cos(\theta) + R_S v_z \sin(\theta), \\ \theta &\in (\varphi_3, \varphi_4). \end{aligned} \tag{7}$$

This is the known parametric equation of a circle in 3D space. The filamentary circular segments are the part of this circle.

(7) The differential element of the secondary circular segment is given by

$$dl_S = R_S \{l_{xS}, l_{yS}, l_{zS}\} d\theta, \quad \theta \in (\varphi_3, \varphi_4), \tag{8}$$

where

$$\begin{aligned} l_{xS} &= -u_x \sin(\theta) + v_x \cos(\theta), \\ l_{yS} &= -u_y \sin(\theta) + v_y \cos(\theta), \\ l_{zS} &= -u_z \sin(\theta) + v_z \cos(\theta). \end{aligned}$$

3. Magnetic Vector Potential Calculation at Point S (x_S, y_S, z_S)

The magnetic vector potential $\vec{A}(S)$ produced by the primary circular arc segment of radius R_P carrying current I_P , can be calculated at the point $S(x_S, y_S, z_S)$ by [6]

$$\vec{A}(S) = \frac{\mu_0}{4\pi} \int \frac{I_P d\vec{l}_P}{r_{PS}}, \tag{9}$$

where

$$\begin{aligned} \vec{r}_{PS} &= (x_S - x_P)\vec{i} + (y_S - y_P)\vec{j} + (z_S - z_P)\vec{k}, \\ r_{PS}^2 &= (x_S - x_P)^2 + (y_S - y_P)^2 + (z_S - z_P)^2 = x_S^2 + y_S^2 + z_S^2 + R_P^2 - \\ &\quad - 2R_P \sqrt{x_S^2 + y_S^2} \cos(t - \gamma), \\ \cos(\gamma) &= \frac{x_S}{p}, \quad \sin(\gamma) = \frac{y_S}{p}, \quad \tan(\gamma) = \frac{y_S}{x_S}, \quad p = \sqrt{x_S^2 + y_S^2}. \end{aligned}$$

\vec{i} , \vec{j} and \vec{k} are the unit vectors of axes x , y , and z , respectively.

From Equations (3) and (9) one has:

$$A_x (S) = -\frac{\mu_0 I_P R_P}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\sin(t)}{r_{PS}} dt, \tag{10}$$

$$A_y (S) = \frac{\mu_0 I_P R_P}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\cos(t)}{r_{PS}} dt, \tag{11}$$

$$A_z (S) = 0. \tag{12}$$

Let us introduce the following substitution $t - \gamma = \pi - 2\beta$. Equations (10)–(12) become:

$$A_x (S) = -\frac{\mu_0 I_P \sqrt{R_P}}{4\pi p \sqrt{p}} k \int_{\beta_1}^{\beta_2} \frac{\sin(\gamma - 2\beta)}{\Delta} d\beta, \tag{13}$$

$$A_y (S) = \frac{\mu_0 I_P \sqrt{R_P}}{4\pi p \sqrt{p}} k \int_{\beta_1}^{\beta_2} \frac{\cos(\gamma - 2\beta)}{\Delta} d\beta, \tag{14}$$

$$A_z (S) = 0, \tag{15}$$

$$\Delta = \sqrt{1 - k^2 \sin^2(\beta)}, \quad k^2 = \frac{4R_P p}{[R_P + p]^2 + z_S^2}, \quad p = \sqrt{x_S^2 + y_S^2}.$$

The final solutions for Equations (13)–(15) can be obtained analytically in the form of the incomplete elliptic integrals of the first and the second kind and the simple elementary functions (Appendix A).

Finally,

$$A_x (S) = -\frac{\mu_0 I_P \sqrt{R_P}}{4\pi k p \sqrt{p}} I_x, \tag{16}$$

$$A_y (S) = \frac{\mu_0 I_P \sqrt{R_P}}{4\pi k p \sqrt{p}} I_y, \tag{17}$$

$$A_z (S) = 0, \tag{18}$$

where

$$I_x = \left\{ y_S \left[(k^2 - 2) F(\beta, k) + 2E(\beta, k) \right] + 2x_S \Delta \right\} \Big|_{\beta_1}^{\beta_2},$$

$$I_y = \left\{ x_S \left[(k^2 - 2) F(\beta, k) + 2E(\beta, k) \right] + 2y_S \Delta \right\} \Big|_{\beta_1}^{\beta_2},$$

$$\beta_1 = \frac{\pi}{2} + \frac{\gamma - \varphi_1}{2}, \quad \beta_2 = \frac{\pi}{2} + \frac{\gamma - \varphi_2}{2}.$$

$F(\beta, k)$ and $E(\beta, k)$ [39,40] are the incomplete elliptic integrals of the first and the second kind.

These expressions are valid for $z = 0$ and $x_S \neq R_P \cos(t)$, $y_S \neq R_P \sin(t)$.

3.1. Special Cases

3.1.1. $\varphi_1 = 0, \varphi_2 = 2\pi$

$$A_x (S) = -A_0 \sin(\gamma), \tag{19}$$

$$A_y(S) = A_0 \cos(\gamma), \tag{20}$$

$$A_z(S) = 0, \tag{21}$$

$$A_0 = \frac{\mu_0 I_P \sqrt{R_P}}{2\pi k \sqrt{p}} \left[(2 - k^2) K(k) - 2E(k) \right], \quad k^2 = \frac{4R_P p}{[R_P + p]^2 + z_S^2},$$

where $K(k)$ and $E(k)$, refs. [39,40] are the complete integrals of the first and the second kind. Expressions (19)–(21) are valid for $z_S = 0$.

3.1.2. Z-axis ($x_S = y_S = 0, z_S \neq 0$)

$$A_x(z_S) = \frac{\mu_0 I_P R_P}{4\pi \sqrt{z_S^2 + R_P^2}} [\cos(\varphi_2) - \cos(\varphi_1)], \tag{22}$$

$$A_y(z_S) = \frac{\mu_0 I_P R_P}{4\pi \sqrt{z_S^2 + R_P^2}} [\sin(\varphi_2) - \sin(\varphi_1)], \tag{23}$$

$$A_z(S) = 0. \tag{24}$$

3.1.3. $x_S = R_P \cos(t), y_S = R_P \sin(t), z_S = 0, \varphi \in (\varphi_1, \varphi_2)$

This is the singular case. The point S is between φ_1 and φ_2 on the circle what is shown in Figure 2.

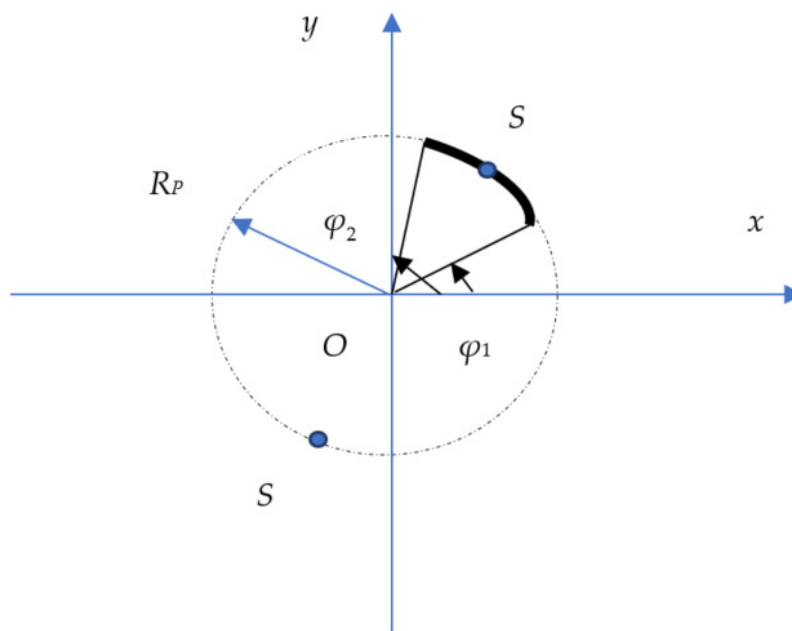


Figure 2. Point S lies between φ_1 and φ_2 where $x_S^2 + y_S^2 = R_P^2$.

3.1.4. $x_S = R_P \cos(t), y_S = R_P \sin(t), z_S = 0, \varphi \in (\varphi_2, \varphi_1 + 2\pi)$

$$A_x(S) = -\frac{\mu_0 I_P}{4\pi} \left\{ \sin(\gamma) \ln \left| \tan \frac{\varphi_1 - \gamma}{\varphi_2 - \gamma} \right| - 2 \sin \frac{\varphi_1 + \gamma}{2} + 2 \sin \frac{\varphi_2 + \gamma}{2} \right\}, \tag{25}$$

$$A_y(S) = \frac{\mu_0 I_P}{4\pi} \left\{ \sin(\gamma) \ln \left| \tan \frac{\varphi_1 - \gamma}{\varphi_2 - \gamma} \right| + 2 \cos \frac{\varphi_1 + \gamma}{2} - 2 \cos \frac{\varphi_2 + \gamma}{2} \right\}, \tag{26}$$

$$A_z(S) = 0. \tag{27}$$

Point S is between φ_2 and $\varphi_1 + 2\pi$ on the circle (See Figure 2).

3.1.5. For $x_S = 0$, Plane $x = 0$. One Needs to Put $\gamma = \pi/2$ and Use Equations (16)–(18)

Thus, all results are obtained in the closed form over the incomplete elliptic integrals of the first and the second kind as well as over some elementary functions.

4. Magnetic Field Calculation at the Point S (x_S, y_S, z_S)

The magnetic field $\vec{B}(S)$ produced by the primary circular segment of the radius R_P carrying the current I_P , can be calculated at an arbitrary point S (x_S, y_S, z_S) by [6],

$$\vec{B}(S) = \frac{\mu_0 I_P}{4\pi} \int \frac{d\vec{l}_P \times \vec{r}_{PS}}{r_{PS}^3} . \tag{28}$$

From Equations (2), (3) and (28) the components of the magnetic field are:

$$B_x(S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\cos(t)}{r_{PS}^3} dt, \tag{29}$$

$$B_y(S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\sin(t)}{r_{PS}^3} dt, \tag{30}$$

$$B_z(S) = \frac{\mu_0 I_P R_P}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{R_P - \sqrt{x_S^2 + y_S^2} \cos(t - \gamma)}{r_{PS}^3} dt, \tag{31}$$

where r_{PS} , γ and p are previously given.

Let us introduce the following substitution $t - \gamma = \pi - 2\beta$. Equations (29)–(31) become:

$$B_x(S) = \frac{\mu_0 I_P z_S k^3}{16\pi p \sqrt{p R_P}} \int_{\varphi_1}^{\varphi_2} \frac{\cos(\gamma - 2\beta)}{\Delta^3} d\beta, \tag{32}$$

$$B_y(S) = \frac{\mu_0 I_P z_S k^3}{16\pi p \sqrt{p R_P}} \int_{\varphi_1}^{\varphi_2} \frac{\sin(\gamma - 2\beta)}{\Delta^3} d\beta, \tag{33}$$

$$B_z(S) = -\frac{\mu_0 I_P k^3}{16\pi p \sqrt{p R_P}} \int_{\beta_1}^{\beta_2} \frac{R_P + \sqrt{x_S^2 + y_S^2} \cos(2\beta)}{\Delta^3} d\beta, \tag{34}$$

where β_1 and β_2 are previously given.

The final solutions for Equations (32)–(34) can be obtained analytically in the form of the incomplete elliptic integrals of the first and second kind and simple elementary functions (See Appendix B).

$$B_x(S) = \frac{\mu_0 I_P z_S k}{16\pi p^2 \sqrt{R_P p} (1 - k^2)} I_{xx}, \tag{35}$$

$$B_y(S) = \frac{\mu_0 I_P z_S k}{16\pi p^2 \sqrt{R_P p} (1 - k^2)} I_{yy}, \tag{36}$$

$$B_z(S) = -\frac{\mu_0 I_P k}{16\pi p \sqrt{R_P p} (1 - k^2)} I_{zz}, \tag{37}$$

where

$$I_{xx} = x_S \left\{ (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) + k^2(2 - k^2) \frac{\sin(\beta) \cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2} + \frac{2y_S}{\Delta} (1 - k^2) \Big|_{\beta_1}^{\beta_2},$$

$$I_{yy} = y_S \left\{ (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) + k^2(2 - k^2) \frac{\sin(\beta) \cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2} - \frac{2x_S}{\Delta} (1 - k^2) \Big|_{\beta_1}^{\beta_2},$$

$$I_{zz} = \left\{ [k^2(R_p + p) - 2p]E(\beta, k) + (2p - 2pk^2)F(\beta, k) + k^2(2p - (R_p + p)k^2) \frac{\sin(\beta) \cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2},$$

where Δ and k^2 are previously given.

Thus, for the given point $S(x_S, y_S, z_S)$ the magnetic field produced by the circular segment with the current I_p can be calculated analytically over the incomplete elliptic integrals of the first and the second kind Equations (35)–(37).

4.1. Special Cases

4.1.1. $z_S = 0$

$$B_x(S) = 0, \tag{38}$$

$$B_y(S) = 0, \tag{39}$$

$$B_z(S) = -\frac{\mu_0 I_p k_0}{16\pi p \sqrt{R_p p} (1 - k_0^2)} I_z(k_0), \tag{40}$$

$$k_0^2 = \frac{4R_p p}{[R_p + p]^2}, \quad x_S^2 + y_S^2 = R_p^2.$$

$I_z(k_0)$ is given by I_{zz} from (34) where $z_S = 0$.

4.1.2. $z_S = 0, x_S^2 + y_S^2 = R_p^2$ and $\varphi \in [\varphi_1, \varphi_2]$

This is the singular case, (see Figure 2) where point S is between φ_1 and φ_2 .

4.1.3. $z_S = 0, x_S^2 + y_S^2 = R_p^2$ and $\varphi \in (\varphi_2, \varphi_1 + 2\pi)$

$$B_x(S) = 0, \tag{41}$$

$$B_y(S) = 0, \tag{42}$$

$$B_z(S) = \frac{\mu_0 I_p}{4\pi R_p} \ln \left| \tan \frac{\varphi_1 - \gamma}{\varphi_2 - \gamma} \right|. \tag{43}$$

Point S is between φ_2 and $\varphi_1 + 2\pi$ on the circle (see Figure 2).

4.1.4. Z – axis $S(0, 0, z_S)$

$$B_x(S) = \frac{\mu_0 I_p R_p z_S}{4\pi \sqrt{(R_p^2 + z_S^2)^3}} [\sin(\beta_2) - \sin(\beta_1)], \tag{44}$$

$$B_y(S) = \frac{\mu_0 I_p R_p z_S}{4\pi \sqrt{(R_p^2 + z_S^2)^3}} [\cos(\beta_1) - \cos(\beta_2)], \tag{45}$$

$$B_z(S) = \frac{\mu_0 I_P R_P^2}{4\pi \sqrt{(R_P^2 + z_S^2)^3}} [\beta_2 - \beta_1]. \tag{46}$$

4.1.5. $\varphi_1 = 0$ and $\varphi_2 = 2\pi$

$$B_x(S) = B_0 \frac{z_S}{p} \left\{ \left[\frac{R_P^2 - p^2 - z_S^2}{(R_P - p)^2 + z_S^2} \right] E(k) - K(k) \right\} \cos(\gamma), \tag{47}$$

$$B_y(S) = B_0 \frac{z_S}{p} \left\{ \left[\frac{R_P^2 - p^2 - z_S^2}{(R_P - p)^2 + z_S^2} \right] E(k) - K(k) \right\} \sin(\gamma), \tag{48}$$

$$B_y(S) = B_0 \left\{ \left[\frac{R_P^2 + p^2 + z_S^2}{(R_P - p)^2 + z_S^2} \right] E(k) + K(k) \right\}, \tag{49}$$

$$B_0 = \frac{\mu_0 I_P k}{4\pi \sqrt{R_P p}}, \quad k^2 = \frac{4R_P p}{[R_P + p]^2 + z_S^2}, \quad p = \sqrt{x_S^2 + y_S^2}. \tag{50}$$

4.1.6. For $x_S = 0$, Plane $x = 0$. One Needs to Put $\gamma = \pi/2$ and Use Equations (35)–(37)

This is a known expression [11] obtained in the form of the complete elliptic integrals of the first and second kind $K(k)$ and $E(k)$ [39,40].

5. Magnetic Force Calculation between Two Inclined Current-Carrying Arc Segments

The magnetic force between two inclined arc segments with the radii R_P and R_S , and the corresponding currents I_P and I_S , can be calculated by [25,26]

$$\vec{F} = \frac{\mu_0 I_P I_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \int_{\varphi_3}^{\varphi_4} \frac{d\vec{l}_s \times \left(d\vec{l}_P \times \vec{r}_{PS} \right)}{r_{PS}^3}, \tag{51}$$

where \vec{r}_{PS} is the vector between point P of the primary arc segment and point S of the second arc segment (oriented to S) and $d\vec{l}_P$ and $d\vec{l}_s$ are the elementary current-carrying elements of the primary and the secondary arc segment given by Equations (3) and (7) (see Figure 1).

Equation (51) can be written as follows:

$$\vec{F} = I_S \int_{\varphi_3}^{\varphi_4} d\vec{l}_S \times \vec{B}(S), \tag{52}$$

where $\vec{B}(S)$ is the magnetic field produced by primary current I_P in the first arc segment, acting at point S of the second arc segment.

Previously, we calculated the magnetic field whose components are given by Equations (35)–(37). Using Equations (7), (35)–(37) and (52) the components of the magnetic forces are as follows:

$$F_x = I_S R_S \int_{\varphi_3}^{\varphi_4} [l_{yS} B_z(S) - l_{zS} B_y(S)] d\theta, \tag{53}$$

$$F_y = -I_S R_S \int_{\varphi_3}^{\varphi_4} [l_{xS} B_z(S) - l_{zS} B_x(S)] d\theta, \tag{54}$$

$$F_z = I_S R_S \int_{\varphi_3}^{\varphi_4} [l_{xS} B_y(S) - l_{yS} B_x(S)] d\theta. \tag{55}$$

Thus, the calculation of the magnetic force is obtained by the simple integration where the kernel functions are given in the analytical form over the incomplete elliptic integrals of the first and the second kind. These expressions are much easier than those in [25,26].

5.1. Special Cases

5.1.1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane $y = \text{constant}$. There are two possibilities for this case because of two symmetric points of the inclined segment regarding its center C .

5.1.2. $\vec{u} = \{-1, 0, 0\}, \vec{v} = \{0, 0, -1\}$

Unit vector for the singular case.

5.1.3. $\vec{u} = \{0, 0, -1\}, \vec{v} = \{-1, 0, 0\}$

Unit vector for the singular case.

These vectors must be used in Equations (53)–(55).

6. Magnetic Torque Calculation between Two Inclined Current-Carrying Arc Segments

Torque is defined as the cross product of a displacement and a force. The displacement is from the center for taking torque, which is arbitrarily defined, to point S of the application of the force to the body experiencing the torque [20],

$$d \vec{\tau} = \vec{r}_{CS} \times d \vec{F}(S). \tag{56}$$

In Equation (56) $\vec{r}_{CS} = (x_S - x_C)\vec{i} + (y_S - y_C)\vec{j} + (z_S - z_C)\vec{k}$ is the vector of displacement between the center C of the second arc segment and the point S of the application of this segment.

Previously, we calculated the magnetic force between two current-carrying arc segments.

Where the analytical expressions of the magnetic field at the at the point S of the second arc segment were used. The magnet field is produced by the current in the primary arc segment. We use the same reasoning for the torque and then from Equation (56):

$$d \vec{\tau} = I_S R_S \vec{r}_{CS} \times \left(d \vec{l}_S \times \vec{B}(S) \right), \tag{57}$$

or,

$$\vec{\tau} = I_S R_S \int_{\varphi_3}^{\varphi_4} \vec{r}_{CS} \times \left(d \vec{l}_S \times \vec{B}(S) \right). \tag{58}$$

Using Equations (7) and (35)–(37) and developing the double cross product in Equation (58), one obtains the final components of the torque between two inclined current segments with the radii R_p and R_S , and the corresponding currents I_p and I_S :

$$\tau_x = I_S R_S \int_{\varphi_3}^{\varphi_4} J_x d\theta, \tag{59}$$

$$\tau_y = I_S R_S \int_{\varphi_3}^{\varphi_4} J_y d\theta, \tag{60}$$

$$\tau_z = I_S R_S \int_{\varphi_3}^{\varphi_4} J_x d\theta, \tag{61}$$

where

$$\begin{aligned} J_x &= -[(y_S - y_C)l_{yS} + (z_S - z_C)l_{zS}]B_x(S) + (y_S - y_C)l_{xS}B_y(S) + (z_S - z_C)l_{xS}B_z(S), \\ J_y &= (x_S - x_C)l_{yS}B_x(S) - [(z_S - z_C)l_{zS} + (x_S - x_C)l_{xS}]B_y(S) + (z_S - z_C)l_{yS}B_z(S), \\ J_z &= (x_S - x_C)l_{zS}B_x(S) + (y_S - y_C)l_{zS}B_y(S) - [(x_S - x_C)l_{zS} + (y_S - y_C)l_{yS}]B_z(S). \end{aligned}$$

Thus, the calculation of the magnetic torque is obtained by the simple integration where the kernel functions are given in the analytical form over the incomplete elliptic integrals of the first and the second kind. As we know, these expressions appear for the first time in the literature.

6.1. Special Cases

6.1.1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane $y = \text{constant}$. There are two possibilities for this case.

6.1.2. $\vec{u} = \{-1, 0, 0\}, \vec{v} = \{0, 0, -1\}$

Unit vector for the singular case.

6.1.3. $\vec{u} = \{0, 0, -1\}, \vec{v} = \{-1, 0, 0\}$

Unit vector for the singular case.

These vectors must be used in Equations (59)–(61).

7. Mutual Inductance Calculation between Two Current-Carrying Arc Segments with Inclined Axes

The mutual inductance between two current-carrying arc segments with inclined axes with the radii R_P and R_S , and the corresponding currents I_P and I_S , in air can be calculated by [1]

$$M = \frac{\mu_0}{4\pi} \int_{\varphi_1}^{\varphi_2} \int_{\varphi_3}^{\varphi_4} \frac{d\vec{l}_P \cdot d\vec{l}_S}{r_{PS}}, \tag{62}$$

where $d\vec{l}_P$, $d\vec{l}_S$ and r_{PS} are previously given.

From, Equations (3), (7) and (62) the mutual inductance can be calculated by

$$M = \frac{\mu_0 R_P R_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \int_{\varphi_3}^{\varphi_4} \frac{-l_{xS} \sin(t) + l_{yS} \cos(t)}{\sqrt{x_S^2 + y_S^2 + z_S^2 + R_P^2 - 2R_P \sqrt{x_S^2 + y_S^2} \cos(t - \gamma)}} dt d\theta. \tag{63}$$

We take the substitution $t - \gamma = \pi - 2\beta$ that leads to final solution for the mutual inductance (see Appendix C):

$$M = \frac{\mu_0 R_S \sqrt{R_P}}{4\pi} \int_{\varphi_3}^{\varphi_4} \frac{V}{k p \sqrt{p}} d\theta, \tag{64}$$

where

$$V = [l_{ys}x_S - l_{xs}y_S] \left\{ \left[(k^2 - 2)F(\beta, k) + 2E(\beta, k) \right] \frac{\beta_2}{\beta_1} - 2\Delta[l_{ys}y_S + l_{xs}x_S] \frac{\beta_2}{\beta_1} \right\}.$$

Thus, the calculation of the mutual inductance is obtained by the simple integration where the kernel functions are given in the analytical form over the incomplete elliptic integrals of the first and the second kind.

7.1. Special Cases

7.1.1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane $y = \text{constant}$. There are two possibilities for this case.

7.1.2. $\vec{u} = \{-1, 0, 0\}, \vec{v} = \{0, 0, -1\}$

Unit vector for the singular case.

7.1.3. $\vec{u} = \{0, 0, -1\}, \vec{v} = \{-1, 0, 0\}$

Unit vector for the singular case.

These vectors must be used in Equation (64).

All previous electromagnetic quantities are obtained by using the integral approach.

8. Stiffness Calculation between Two Inclined Current-Carrying Arc Segments

The stiffness is the extent to which an object resists deformation in response to an applied force. Knowing the magnetic force between two inclined current-carrying arc segments with the radii R_P and R_S , and the corresponding currents I_P and I_S , the corresponding stiffness between them can be calculated by the derivate of the corresponding components as follows [27]:

$$k_{xx} = -\frac{\partial F_x}{\partial x}, \quad k_{xy} = -\frac{\partial F_x}{\partial y}, \quad k_{xz} = -\frac{\partial F_x}{\partial z}, \tag{65}$$

$$k_{yy} = -\frac{\partial F_y}{\partial y}, \quad k_{yx} = -\frac{\partial F_y}{\partial x}, \quad k_{yz} = -\frac{\partial F_y}{\partial z}, \tag{66}$$

$$k_{zz} = -\frac{\partial F_z}{\partial z}, \quad k_{zx} = -\frac{\partial F_z}{\partial x}, \quad k_{zy} = -\frac{\partial F_z}{\partial y}. \tag{67}$$

Thus, the first derivative of the corresponding force components over the corresponding variable leads to the corresponding stiffness. Obviously, it is not easy work because of the complicate kernel functions which are the analytical functions given in the form of incomplete elliptic integrals of the first and the second kind and some elementary functions. Even though this is tedious work, we give only the stiffness k_{zz} from Equation (66) which is the axial stiffness. This developed formula can serve potential readers in calculating other stiffness, by Mathematica or MATLAB programming. The calculation of other stiffness will be the subject of our future work. In this paper we give the benchmark example for calculating the axial stiffness between two coaxial current circular loops.

The magnetic force between two coaxial circular loops is [26]:

$$F_x = 0, \tag{68}$$

$$F_y = 0, \tag{69}$$

$$F_z = \frac{\mu_0 I_P I_S}{4\sqrt{R_P R_S}} \frac{zk}{1 - k^2} \Phi(k), \tag{70}$$

where

$$k^2 = \frac{4R_P R_S}{[R_P + R_S]^2 + z^2},$$

$$\Phi(k) = 2(1 - k^2)K(k) - (2 - k^2)E(k).$$

I_P and I_S are the currents in the primary and secondary loop.

R_P and R_S are the corresponding radii of loops.

Obviously, we can find analytically only the stiffness k_{zz} because others are zero.

This stiffness k_{zz} for the coaxial loops is given by,

$$k_{zz} = -\frac{\partial F_z}{\partial z} \tag{71}$$

or

$$k_{zz} = -\frac{\mu_0 I_P I_S}{4\sqrt{R_P R_S}} T_0 \tag{72}$$

where

$$\frac{dk}{dz} = \frac{zk^3}{4R_P R_S} \frac{1 + k^2}{(1 - k^2)^2}, \tag{73}$$

$$T_0 = \frac{d}{dz} \left[\frac{zk}{1 - k^2} \Phi(k) \right] = \frac{k}{1 - k^2} \Phi(k) + z \frac{d}{dk} \left[\frac{k}{1 - k^2} \right] \frac{dk}{dz} \Phi(k) + \frac{zk}{1 - k^2} \frac{d\Phi(k)}{dk} \frac{dk}{dz} =$$

$$= \frac{k}{1 - k^2} \Phi(k) - \frac{z^2 k^3}{4R_P R_S} \frac{1 + k^2}{(1 - k^2)^2} \Phi(k) - \frac{z^2 k^4}{4R_P R_S} \frac{1}{1 - k^2} \frac{d\Phi(k)}{dk}. \tag{74}$$

From Equations (71)–(74) the axial stiffness k_{zz} is:

$$k_{zz} \text{ (coaxial loops)} = -\frac{\mu_0 I_P I_S}{4\sqrt{R_P R_S}} \frac{k}{1 - k^2} \left\{ \left[1 - \frac{z^2 k^2}{4R_P R_S} \frac{1 + k^2}{(1 - k^2)} \right] \Phi(k) - \frac{3z^2 k^4}{4R_P R_S} \Psi(k) \right\}, \tag{75}$$

where

$$\Psi(k) = E(k) - K(k).$$

As mentioned before, this Formula (74) will serve as the benchmark example to verify the validity of the general expression for the stiffness k_{zz} . In Appendix D, the complete expressions of this axial stiffness are given.

Here, we give only final expressions of k_{zz} :

$$k_{zz} = -\frac{\partial F_z}{\partial z_s} = -\frac{\mu_0 I_P I_S R_S}{16\pi\sqrt{R_P}} \int_{\varphi_3}^{\varphi_4} \frac{k}{(1 - k^2)^2 \sqrt{p^5}} [I_{xS} T_{zz1} - I_{yS} T_{zz2}] d\theta, \tag{76}$$

where

$$T_{zz1} = I_{yy} - \frac{z_S^2 k^3}{pR_P} \frac{1 + k^2}{1 - k^2} I_{yy} - \frac{z_S^2 k^4}{pR_P} V_y \tag{77}$$

$$T_{zz2} = I_{xx} - \frac{z_S^2 k^2}{pR_P} \frac{1 + k^2}{1 - k^2} I_{xx} - \frac{z_S^2 k^4}{pR_P} V_x,$$

$$I_{xx} = x_S S + 2y_S (1 - k^2) \left[\frac{1}{\Delta} \right], \tag{78}$$

$$I_{yy} = y_S S - 2x_S (1 - k^2) \left[\frac{1}{\Delta} \right], \tag{79}$$

$$\begin{aligned}
 S &= (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) + k^2(2 - k^2)\frac{\sin(2\beta)}{2\Delta}, \\
 V_y &= y_S b_1 + 2x_S b_2, \\
 V_x &= x_S b_1 - 2y_S b_2, \\
 b_1 &= 3[E(\beta, k) - F(\beta, k)] \frac{\beta_2}{\beta_1} + \left\{ \left[\frac{\sin(2\beta)}{\Delta} \right] (1 - 2k^2) + \left[\frac{\sin(2\beta) \sin^2(\beta)}{\Delta^3} \right] \frac{k^2(2 - k^2)}{2} \right\} \frac{\beta_2}{\beta_1}, \\
 b_2 &= \left[\frac{2}{\Delta} \right] \frac{\beta_2}{\beta_1} - (1 - k^2) \left[\frac{\sin^2(\beta)}{\Delta^3} \right] \frac{\beta_2}{\beta_1}.
 \end{aligned}$$

8.1. Special Cases

8.1.1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane $y = \text{constant}$. There are two possibilities for this case.

8.1.2. $\vec{u} = \{-1, 0, 0\}, \vec{v} = \{0, 0, -1\}$

Unit vector for the singular case.

8.1.3. $\vec{u} = \{0, 0, -1\}, \vec{v} = \{-1, 0, 0\}$

Unit vector for the singular case.

These vectors must be used in Equation (76).

9. Numerical Validation

To verify the validity of the new formulas, the following set of examples are considered. The particular cases are discussed. The results obtained using the presented formulas are compared with those known in the literature.

Example 1. Calculate the magnetic vector potential produced by the current-carrying arc segment of the radius $R_P = 3 \text{ m}$ at the point $S(x_S, y_S, z_S) = S(3 \text{ m}, 4 \text{ m}, 5 \text{ m})$. The current $I_P = 1 \text{ A}$.

Let us begin with the circular loop for which is $\varphi_1 = 0$ and $\varphi_2 = 2\pi$.

From Equations (16)–(18), one has the components, and the total magnetic vector potential as follows:

$$A_x(S) = -28.61844373019504 \text{ nT}\cdot\text{m},$$

$$A_y(S) = 21.46383279764628 \text{ nT}\cdot\text{m},$$

$$A_z(S) = 0 \text{ T}\cdot\text{m},$$

$$A(S) = 35.7730546627438 \text{ nT}\cdot\text{m}.$$

From [11], one obtains the same results for the total magnetic vector potential.

Obviously, these are the known formulas for the current loop.

Let us take $\varphi_1 = \pi/3$ and $\varphi_2 = 5\pi/4$. From Equations (16)–(18), one finds:

$$A_x(S) = -60.73902566793771 \text{ nT}\cdot\text{m},$$

$$A_y(S) = -54.76725580732807 \text{ nT}\cdot\text{m},$$

$$A_z(S) = 0 \text{ T}\cdot\text{m},$$

$$A(S) = 81.78436004368871 \text{ nT}\cdot\text{m}.$$

Thus, the analytic form of the magnetic vector potential is found for the different angle positions.

Example 2. Calculate the magnetic field produced by the current-carrying arc segment of the radius $R_P = 3 \text{ m}$ at the point $S(x_S, y_S, z_S) = S(3 \text{ m}, 4 \text{ m}, 5 \text{ m})$. The current $I_P = 1 \text{ A}$.

Let us start with the circular loop for which is $\varphi_1 = 0$ and $\varphi_2 = 2\pi$.

From Equations (35)–(37), one finds the components, and the total magnetic field as follows:

$$B_x(S) = 6.590422756026894 \text{ nT},$$

$$B_y(S) = 8.787230341369193 \text{ nT},$$

$$B_z(S) = 5.554432293082448 \text{ nT},$$

$$B(S) = 12.30856641830695 \text{ nT}.$$

The magnetic field produced by the circular loop can be considered as axisymmetric so that we need to calculate only the radial and azimuthal component. Applying equations from [11], these components as well as the total magnetic field are as follows:

$$B_r(S) = 10.98403792671149 \text{ nT},$$

$$B_z(S) = 5.554432293082448 \text{ nT},$$

$$B(S) = 12.30856641830695 \text{ nT}.$$

From the previous calculations, the radial component of the magnetic field is:

$$B_r(S) = \sqrt{B_x^2(S) + B_y^2(S)} = 10.98403792671149 \text{ nT}.$$

Thus, we show the validity of Equations (35)–(37).

From [11], we obtained the same results for the magnetic field. Obviously, these are the known formula for the current loop.

Now, let us apply these equations for the same problem but with the various positions of angles, for example, $\varphi_1 = \pi/6$ and $\varphi_2 = 3\pi/4$. We obtain:

$$B_x(S) = 3.204077158320579 \text{ nT},$$

$$B_y(S) = 11.48651408884254 \text{ nT},$$

$$B_z(S) = -3.013457271456703 \text{ nT},$$

$$B(S) = 12.29987971797063 \text{ nT}.$$

Thus, these examples for the arbitrary angles may serve as the benchmark example. As one can see, the calculations of the magnetic vector potential and the magnetic field of the current-carrying segments with arbitrary angles are obtained in the closed form and expressed by the incomplete elliptic integrals of the first and the second kind. In the case of the circular loops, these calculations are the known and obtained over the complete elliptical integral of the first and the second kind. The analytical formula for the magnetic field is crucial for calculating other electromagnetic quantities such as the magnetic force, the magnetic torque, the mutual inductance, and the stiffness between inclined circular current-carrying arc segments.

Example 3. Calculate the magnetic force between two arc carrying-current segments whose radii are $R_P = 0.2 \text{ m}$ and $R_S = 0.1 \text{ m}$, respectively. The first arc segment is placed in the plane XOY with the center at the origin and the second in the plane $x + y + z = 0.3$ with the center $C(0.1 \text{ m}; 0.1 \text{ m}; 0.1 \text{ m})$. The currents are units.

We begin with two inclined circular loops; see Figure 3.

By using Ren’s method, [20], the components of the magnetic force are:

$$\begin{aligned} F_x &= -0.10807277 \mu\text{N}, \\ F_y &= -0.10807276 \mu\text{N} \\ F_z &= -1.4073547 \mu\text{N}. \end{aligned}$$

By using Poletkin’s method [31], the components of the magnetic force are as follows:

$$\begin{aligned} F_x &= -0.108072965612845 \mu\text{N}, \\ F_y &= -0.108072965612845 \mu\text{N}, \\ F_z &= -1.40737206031365 \mu\text{N}. \end{aligned}$$

From [25,26], the components of the magnetic force are:

$$\begin{aligned} F_x &= -0.1080729656128444 \mu\text{N}, \\ F_y &= -0.1080729656128444 \mu\text{N}, \\ F_z &= -1.407372060313649 \mu\text{N}. \end{aligned}$$

From the calculations, presented in this paper, using Equations (53)–(55), one has:

$$\begin{aligned} F_x &= -0.1080729656128444 \mu\text{N}, \\ F_y &= -0.1080729656128444 \mu\text{N}, \\ F_z &= -1.407372060313649 \mu\text{N}. \end{aligned}$$

Thus, the validity of the approach presented here is confirmed.

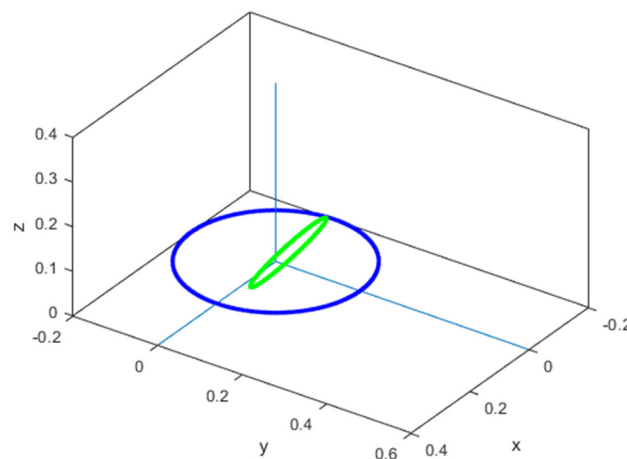


Figure 3. Two inclined circular loops. General case.

Now, let us apply these equations for the same problem but with the various positions of the segment angles, for example, $\varphi_1 = \varphi_3 = \pi/6$ and $\varphi_2 = \varphi_4 = 3\pi/4$. We obtain:

$$\begin{aligned} F_x &= -137.7416772905457 \mu\text{N}, \\ F_y &= -6.783844980209707 \mu\text{N}, \\ F_z &= 32.30984917651751 \mu\text{N}. \end{aligned}$$

Example 4. The center of the primary coil of the radius $R_P = 0.4 \text{ m}$ is $O(0; 0; 0)$ and the center of the secondary coil of the radius $R_S = 0.05 \text{ m}$ is $C(0.1 \text{ m}; 0.15 \text{ m}; 0.0 \text{ m})$. The secondary coil is in the plane $3x + 2y + z = 0.6$. Calculate the magnetic force between coils. All currents are units. The angles of segments are, respectively, $\varphi_1 = 0, \varphi_2 = 2\pi$ and $\varphi_3 = 0, \varphi_4 = 19\pi/10, 195\pi/100, 19,999 \pi/10,000, 2\pi$. Investigate four cases for angle φ_4 .

The first coil is the current loop. Using the method presented here, one has:

For $\varphi_4 = 19\pi/10$,

$$F_x = -1.030225970922242 \text{ nN},$$

$$F_y = -5.151227163000918 \text{ nN},$$

$$F_z = 27.14297688555945 \text{ nN}.$$

For $\varphi_4 = 195\pi/100$,

$$F_x = 2.692181753461003 \text{ nN},$$

$$F_y = 1.173665675174731 \text{ nN},$$

$$F_z = 27.52894004960609 \text{ nN}.$$

For $\varphi_4 = 19,999\pi/10,000$,

$$F_x = 4.171134702846683 \text{ nN},$$

$$F_y = 6.514234771668451 \text{ nN},$$

$$F_z = 27.71528704863114 \text{ nN}.$$

For $\varphi_4 = 2\pi$,

$$F_x = 4.171776672650815 \text{ nN},$$

$$F_y = 6.523855691357912 \text{ nN},$$

$$F_z = 27.7154997521196 \text{ nN}.$$

The last results for $\varphi_4 = 2\pi$, are obtained in [25,26].

Thus, we show that the presented formulas for the magnetic force between two inclined current-carrying segments with arbitrary angles are correct which is proved by the limit case for the two inclined circular loops.

Example 5. The center of the primary coil of the radius $R_P = 0.3 \text{ m}$ is $O(0; 0; 0)$ and the center of the secondary coil of the radius $R_S = 0.3 \text{ m}$ is $C(0.1 \text{ m}; -0.3 \text{ m}; 0.2 \text{ m})$. The secondary coil is in the plane $x - 2y + z = 0.9$. All currents are units but of the opposite sign. The angles of segments are, respectively, $\varphi_1 = 0$, $\varphi_2 = \pi$, $3\pi/2$, $7\pi/24$, $90\pi/46$, $1999\pi/1000$, 2π and $\varphi_3 = 0$, $\varphi_4 = \pi$, $3\pi/2$, $7\pi/24$, $90\pi/46$, $1999\pi/1000$, 2π . Calculate the magnetic force between these current segments.

Using the presented method here, one finds:

$$\varphi_1 = 0, \varphi_2 = \pi, \varphi_3 = 0, \varphi_4 = \pi,$$

$$F_x = 0.1434856008022091 \text{ }\mu\text{N},$$

$$F_y = -0.1326852649109414 \text{ }\mu\text{N},$$

$$F_z = 0.02679590119992052 \text{ }\mu\text{N}.$$

$$\varphi_1 = 0, \varphi_2 = 3\pi/2, \varphi_3 = 0, \varphi_4 = 3\pi/2,$$

$$F_x = 0.125846805955475 \text{ }\mu\text{N},$$

$$F_y = -0.1412059414594633 \text{ }\mu\text{N},$$

$$F_z = -0.008568308516912724 \text{ }\mu\text{N}.$$

$$\varphi_1 = 0, \varphi_2 = 7\pi/4, \varphi_3 = 0, \varphi_4 = 7\pi/4,$$

$$F_x = 0.1971372403346838 \text{ }\mu\text{N},$$

$$F_y = -0.3359342255993592 \text{ }\mu\text{N},$$

$$F_z = -0.1029523519216523 \text{ }\mu\text{N}.$$

$$\varphi_1 = 0, \varphi_2 = 90\pi/46, \varphi_3 = 0, \varphi_4 = 90\pi/46,$$

$$\begin{aligned} F_x &= 0.2298232863299166 \mu\text{N}, \\ F_y &= -0.5316472767567059 \mu\text{N}, \\ F_z &= -0.094553128442032 \mu\text{N}. \end{aligned}$$

$$\varphi_1 = 0, \varphi_2 = 1999\pi/1000, \varphi_3 = 0, \varphi_4 = 1999\pi/1000,$$

$$\begin{aligned} F_x &= 0.2292493650352244 \mu\text{N}, \\ F_y &= -0.5614719226361647 \mu\text{N}, \\ F_z &= -0.09253078729453428 \mu\text{N}. \end{aligned}$$

Let us take the limit case of two inclined current loops. This approach gives:

$$\begin{aligned} F_x &= 0.2292455704933025 \mu\text{N}, \\ F_y &= -0.5621415690326643 \mu\text{N}, \\ F_z &= -0.09249247340323912 \mu\text{N}. \end{aligned}$$

The last results are obtained in [25,26].

Thus, when the segments lead to the circular loops, we can see the results that converge to those of the circular loops.

Example 6. The center of the primary coil of the radius $R_P = 1 \text{ m}$ is $O(0; 0; 0)$ and the center of the secondary coil of the radius $R_S = 0.5 \text{ m}$ is $C(2 \text{ m}; 2 \text{ m}; 2 \text{ m})$. Coils have perpendicular axes (see Figure 4). The secondary coil is in the plane $y = 2 \text{ m}$. Calculate the magnetic force between the coils. All currents are units.

This case is the singular case because $a = c = 0$. Let us begin with two perpendicular current loops [25,26], for which we found:

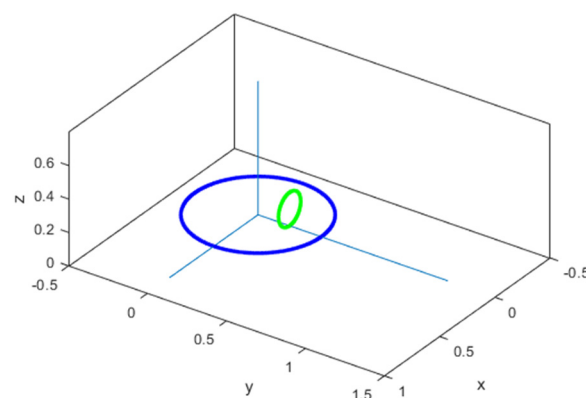


Figure 4. Two perpendicular circular loops. Singular case, $a = c = 0, l = 0$.

$$\begin{aligned} F_x &= -4.901398177052345 \text{ nN}, \\ F_y &= -1.984872313200137 \text{ nN}, \\ F_z &= -2.582265710169336 \text{ nN}. \end{aligned}$$

According to [20],

$$\begin{aligned} F_x &= -4.9013835 \text{ nN}, \\ F_y &= -1.9848816 \text{ nN}, \\ F_z &= -2.5821969 \text{ nN}. \end{aligned}$$

Using this paper, case 5.1.2 [$\vec{u} = \{-1, 0, 0\}$, $\vec{v} = \{0, 0, -1\}$] and Equations (53)–(55), one has:

$$\begin{aligned}F_x &= 4.901398177052345 \text{ nN}, \\F_y &= 1.984872313200137 \text{ nN}, \\F_z &= 2.582265710169336 \text{ nN}.\end{aligned}$$

Using this paper, case 5.1.3 [$\vec{u} = \{0, 0, -1\}$, $\vec{v} = \{-1, 0, 0\}$] and Equations (53)–(55) one has:

$$\begin{aligned}F_x &= -4.901398177052345 \text{ nN}, \\F_y &= -1.984872313200137 \text{ nN}, \\F_z &= -2.582265710169336 \text{ nN}.\end{aligned}$$

Thus, we obtained for case 5.1.3 the same results as in [25,26,31]. For case 5.1.2 we obtain the same results as in [25,26] but with opposite signs for each component, because in this case we did not take into consideration other unit vectors.

Let us take case 5.1.3. and $\varphi_1 = \pi/6$, $\varphi_2 = 5\pi/6$, $\varphi_3 = \pi/4$, $\varphi_4 = 5\pi/4$. The approach presented here gives:

$$\begin{aligned}F_x &= -12.06294047887778 \text{ nN}, \\F_y &= 5.242872781049669 \text{ nN}, \\F_z &= 7.708406091689127 \text{ nN}.\end{aligned}$$

Let us take case 5.1.3. and $\varphi_1 = \pi/1000$, $\varphi_2 = 1999\pi/1000$, $\varphi_3 = \pi/1000$, $\varphi_4 = 1999\pi/1000$. The approach presented here gives:

$$\begin{aligned}F_x &= -4.901398087973561 \text{ nN}, \\F_y &= -1.977166719062928 \text{ nN}, \\F_z &= -2.553525470247053 \text{ nN}.\end{aligned}$$

For $\varphi_1 = \varphi_3 = 0$ and $\varphi_2 = \varphi_4 = 2\pi$ we approach the limit case (see the first calculation in this example).

Thus, this singular case, where the angles are arbitrary, can be used as the benchmark example which in the limit leads to the case of the perpendicular circular loops.

Example 7. Calculate the torque between two inclined current-carrying arc segments for which $R_P = 0.2 \text{ m}$ and $R_S = 0.1 \text{ m}$. The first arc segment is placed in the plane XOY and the second in the plane $x + y + z = 0.3$ with center C (0.1 m; 0.1 m; 0.1 m). The currents are units.

Let us begin with two inclined circular loops for which $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$ (see Figure 3).

By using Ren's method [20], the components of the magnetic torque are as follows:

$$\begin{aligned}\tau_x &= -27.861249 \text{ nN}\cdot\text{m}, \\ \tau_y &= 27.861249 \text{ nN}\cdot\text{m}, \\ \tau_z &= 0 \text{ N}\cdot\text{m}.\end{aligned}$$

By using Poletkin's method [31], the components of the magnetic force are as follows:

$$\begin{aligned}\tau_x &= -27.8620699713 \text{ nN}\cdot\text{m}, \\ \tau_y &= 27.8620699713 \text{ nN}\cdot\text{m}, \\ \tau_z &= -5.65233285126159 \times 10^{-14} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

Using the approach presented in this paper, Equations (59)–(61), one finds:

$$\begin{aligned} \tau_x &= -27.86206997129496 \text{ nN}\cdot\text{m}, \\ \tau_y &= 27.86206997129496 \text{ nN}\cdot\text{m}, \\ \tau_z &= 5.007385868157401 \times 10^{-64} \approx 0 \text{ N}\cdot\text{m}. \end{aligned}$$

All the results are in an excellent agreement. Thus, the validity of the approach presented here is confirmed.

Let us take $\varphi_1 = \pi/12$, $\varphi_2 = \pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.

The approach presented here gives:

$$\begin{aligned} \tau_x &= -0.4295228631728361 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0.3155545746006545 \text{ nN}\cdot\text{m}, \\ \tau_z &= 0.1139682885721816 \text{ nN}\cdot\text{m}. \end{aligned}$$

As it was mentioned above, the magnetic torque calculation represents novelty in the literature.

Example 8. Let us consider two arc segments of the radii $R_P = 1 \text{ m}$ and $R_S = 0.5 \text{ m}$. The primary loop lies in the plane $z = 0 \text{ m}$, and it is centered at $O (0 \text{ m}; 0 \text{ m}; 0 \text{ m})$. The secondary loop lies in the plane $x = 1 \text{ m}$, with its center located at $C (1 \text{ m}; 2 \text{ m}; 3 \text{ m})$. Calculate the torque between these inclined coils. All currents are units. Investigate the point (a) $C (1 \text{ m}; 2 \text{ m}; 3 \text{ m})$, (b) $C (1 \text{ m}; 2 \text{ m}; 0 \text{ m})$, (c) $C (1 \text{ m}; 0 \text{ m}; 0 \text{ m})$, (d) $C (0 \text{ m}, 0 \text{ m}, 0 \text{ m})$.

Obviously, these coils are perpendicular (see Figures 5–8) but by the presented method these cases are the not singular case because $a = 1, b = c = 0 (L = l = 1)$.

Let us take into configuration two perpendicular loops. The approach presented here gives:

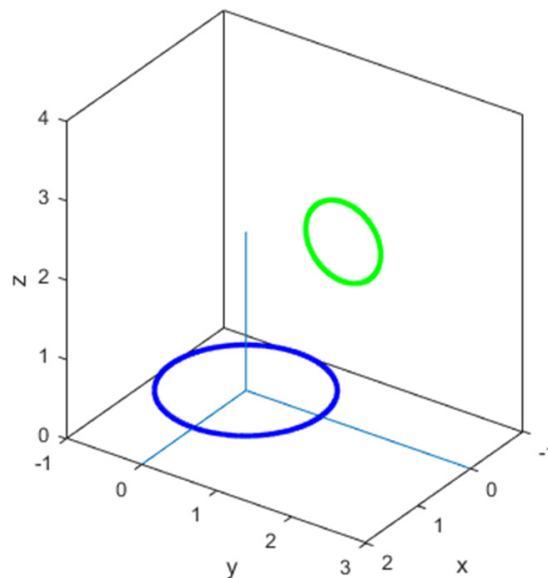


Figure 5. Case (a): two perpendicular circular loops. Not a singular case, $(a = 1, b = c = 0, L = l = 1)$.

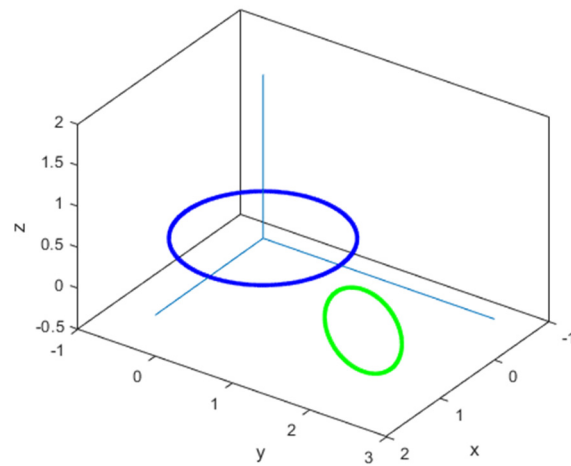


Figure 6. Case (b): two perpendicular circular loops. Not a singular case, $(a = 1, b = c = 0, L = l = 1)$.

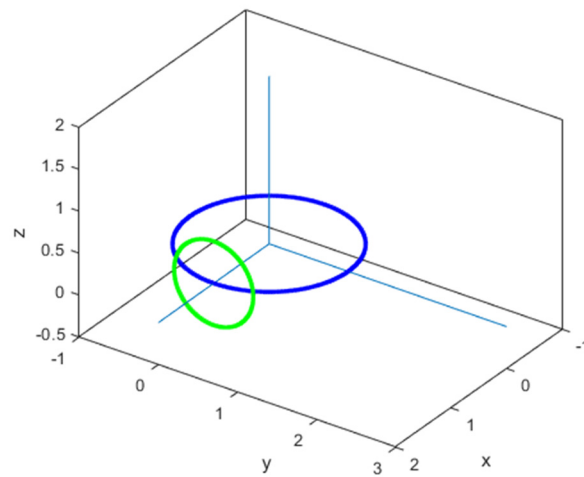


Figure 7. Case (c): two perpendicular circular loops. Not a singular case, $(a = 1, b = c = 0, L = l = 1)$.

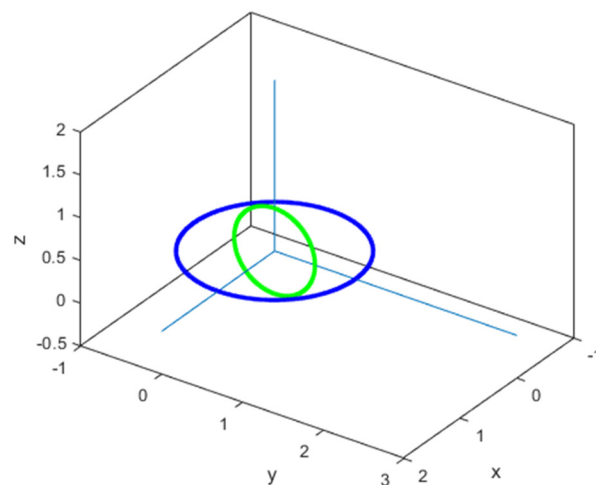


Figure 8. Case (d): two perpendicular circular loops. Not a singular case, $(a = 1, b = c = 0, L = l = 1)$.

(a) C (1 m; 2 m; 3 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -4.668729435430873 \text{ nN}\cdot\text{m}, \\ \tau_z &= 5.739664477343296 \text{ nN}\cdot\text{m}.\end{aligned}$$

According to [20],

$$\begin{aligned}\tau_x &= 4.1994108 \times 10^{-14} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -4.6686544 \text{ nN}\cdot\text{m}, \\ \tau_z &= 5.7396408 \text{ nN}\cdot\text{m}.\end{aligned}$$

In [31], this case is the singular case for which we have:

$$\begin{aligned}\tau_x &= -2.858735873626482 \times 10^{-25} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -4.668669979976015 \text{ nN}\cdot\text{m}, \\ \tau_z &= 5.739664477343291 \text{ nN}\cdot\text{m}.\end{aligned}$$

(b) C (1 m; 2 m; 0 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= 27.83604705327234 \text{ nN}\cdot\text{m}, \\ \tau_z &= 1.527368681710413 \times 10^{-146} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

By [20],

$$\begin{aligned}\tau_x &= 1.5403435 \times 10^{-16} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= 27.835798 \text{ nN}\cdot\text{m}, \\ \tau_z &= -1.3063032 \times 10^{-11} \text{ N}\cdot\text{m}.\end{aligned}$$

In [31], this case is the singular for which we have:

$$\begin{aligned}\tau_x &= 1.704466532265166 \times 10^{-24} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= 27.83605090793333 \text{ nN}\cdot\text{m}, \\ \tau_z &= 0 \text{ N}\cdot\text{m}.\end{aligned}$$

(c) C (0 m; 0 m; 0 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ nN}, \tau_y = -185.0045402475441 \text{ nN}\cdot\text{m}, \\ \tau_z &= 7.657062549408825 \times 10^{-138} \approx 0 \text{ nN}\cdot\text{m}.\end{aligned}$$

According to [20],

$$\begin{aligned}\tau_x &= -1.6956645 \times 10^{-14} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -184.99891 \text{ nN}\cdot\text{m}, \\ \tau_z &= 2.2736774 \times 10^{-12} \text{ N}\cdot\text{m}.\end{aligned}$$

In [31], this case is the singular for which we have:

$$\begin{aligned}\tau_x &= -1.132826091097256 \times 10^{-24} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -185.0045403925399 \text{ nN}\cdot\text{m}, \\ \tau_z &= 0 \text{ N}\cdot\text{m}.\end{aligned}$$

(d) C (1 m; 0 m; m)

By the presented method,

$$\begin{aligned} \tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -435.2765381474917 \text{ nN}\cdot\text{m}, \\ \tau_z &= -3.403343626298156 \times 10^{-128} \approx 0 \text{ N}\cdot\text{m}. \end{aligned}$$

According to [20],

$$\begin{aligned} \tau_x &= 4.3207387 \times 10^{-20} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -435.27404 \text{ nN}\cdot\text{m}, \\ \tau_z &= 5.5087283 \times 10^{-13} \text{ N}\cdot\text{m}. \end{aligned}$$

In [31], this case is the singular for which we have:

$$\begin{aligned} \tau_x &= -2.665551203021357e \times 10^{-23} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -435.3175470473964 \text{ nN}\cdot\text{m}, \\ \tau_z &= 0 \text{ N}\cdot\text{m}. \end{aligned}$$

All the results are in exceptionally good agreement. We state that the presented method is exact. In [2], the authors use the small segments to approximate the circular loops. In [31], this case is singular as previously mentioned.

Example 9. Let us consider two arc segments of the radii $R_P = 1 \text{ m}$ and $R_S = 0.5 \text{ m}$. The primary loop lies in the plane $z = 0 \text{ m}$, and it is centered at $O (0 \text{ m}; 0 \text{ m}; 0 \text{ m})$. The secondary loop lies in the plane $x = 0 \text{ m}$, with its center located at $C (0 \text{ m}; 2 \text{ m}; 3 \text{ m})$. Calculate the torque between these inclined coils. All currents are units. Investigate the point (a) $C (0 \text{ m}; 2 \text{ m}; 3 \text{ m})$, (b) $C (0 \text{ m}; 2 \text{ m}; 0 \text{ m})$, (c) $C (0 \text{ m}; 0 \text{ m}; 3 \text{ m})$, (d) $C (0 \text{ m}; 0 \text{ m}; 0 \text{ m})$.

Obviously, these coils are perpendicular (see Figures 9–12) but according to the presented method these cases are not singular cases because $a = 1, b = c = 0, (L = l = 1)$. In [31] this case is the singular case, and it was studied with special attention. Grover’s formula A.8 given in [2] was corrected to obtain the correct results for this singular case.

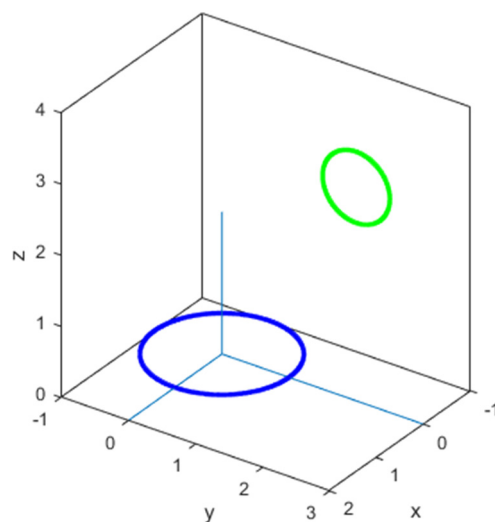


Figure 9. Case (a): two perpendicular circular loops. Not a singular case, $(a = 1, b = c = 0, L = l = 1)$.

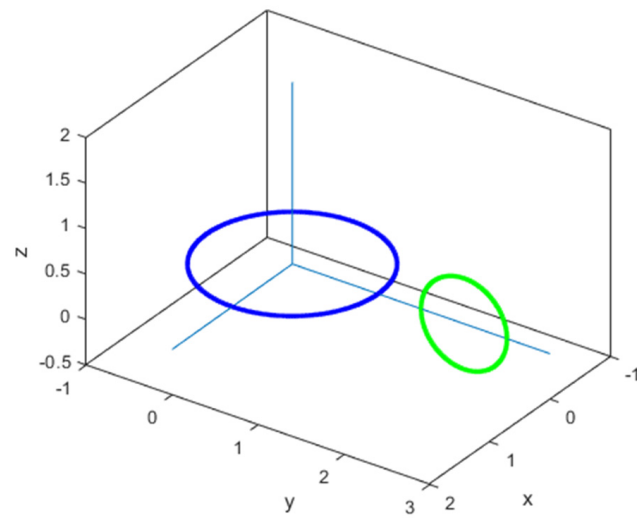


Figure 10. Case (b): two perpendicular circular loops. Not a singular case, ($a = 1, b = c = 0, L = l = 1$).

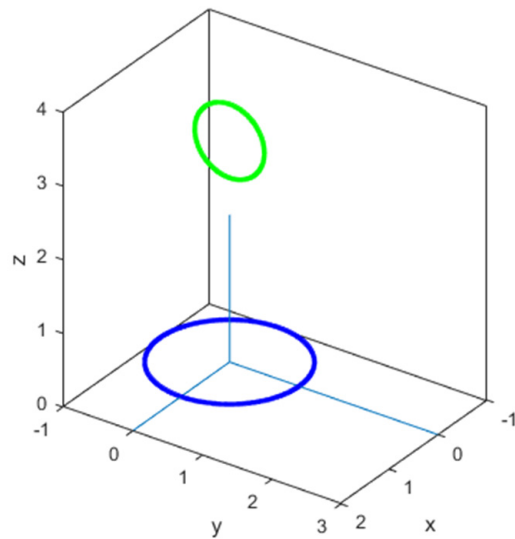


Figure 11. Case (c): two perpendicular circular loops. Not a singular case, ($a = 1, b = c = 0, L = l = 1$).

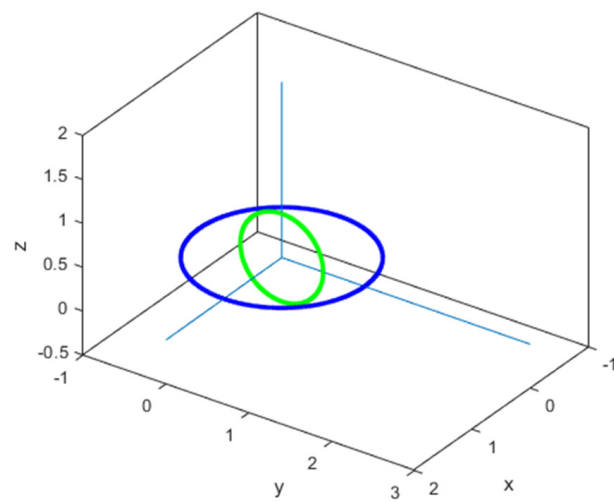


Figure 12. Case (b): two perpendicular circular loops. Not a singular case, ($a = 1, b = c = 0, L = l = 1$).

(a) C (0 m; 2 m; 3 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -6.03647173178846 \text{ nN}\cdot\text{m}, \\ \tau_z &= 6.860953527497661 \text{ nN}\cdot\text{m}.\end{aligned}$$

According to [31] we obtain.

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -6.036471731788459 \text{ nN}\cdot\text{m}, \\ \tau_z &= 6.860953527497644 \text{ nN}\cdot\text{m}.\end{aligned}$$

(b) C (0 m; 2 m; 0 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= 46.60910437567855 \text{ nN}\cdot\text{m}, \\ \tau_z &= 1.027871789475573 \times 10^{-136} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

According to [31], we obtain:

$$\begin{aligned}\tau_x &= 2.853984524239991 \times 10^{-24} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= 46.6091043756787 \text{ nN}\cdot\text{m}, \\ \tau_z &= 1.282404413152518 \times 10^{-23} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

(c) C (0 m; 0 m; 3 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -16.3969954478874 \text{ nN}\cdot\text{m}, \\ \tau_z &= 1.682963244063953 \times 10^{-128} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

According to [31], we obtain:

$$\begin{aligned}\tau_x &= 7.300027041557918 \times 10^{-18} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -16.39567517228915 \text{ nN}\cdot\text{m}, \\ \tau_z &= 1.682963244063953 \times 10^{-9} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

(d) C (0 m; 0 m; 0 m)

By the presented method,

$$\begin{aligned}\tau_x &= 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -435.2765381474917 \text{ nN}\cdot\text{m}, \\ \tau_z &= 1.731874227122655 \times 10^{-128} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

According to [31], we have:

$$\begin{aligned}\tau_x &= -2.66513932049866 \times 10^{-23} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_y &= -435.2502815267608 \text{ nN}\cdot\text{m}, \\ \tau_z &= 5.416141293918174 \times 10^{-16} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

Thus, we investigated all possible cases in this example where the coils are the perpendicular, but the general formula for the torque treats this case ($b = c = 0$) as the regular case but in [31] it is the singular case. All the results are in excellent agreement.

Example 10. Let us consider two arc segments of the radii $R_P = 40$ cm and $R_S = 10$ cm. The primary arc segment lies in the plane $z = 0$ cm, and it is centered at O (0 cm; 0 cm; 0 cm). The secondary arc segment lies in the plane $y = 20$ cm, with its center located at C (0 cm; 20 cm; 10 cm). Calculate the torque between two arc segments with $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = \pi$.

This case is the singular case because $a = c = 0$. Let us begin with two inclined circular loops (see Figure 4).

Using case 6.1.2 [$\vec{u} = \{-1, 0, 0\}$, $\vec{v} = \{0, 0, -1\}$] and Equations (59)–(61), one has:

$$\begin{aligned}\tau_x &= -0.498395165432447 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= 3.696785155039511 \times 10^{-137} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

Using case 6.1.3 [$\vec{u} = \{0, 0, -1\}$, $\vec{v} = \{-1, 0, 0\}$] and Equations (59)–(61), one has:

$$\begin{aligned}\tau_x &= 0.498395165432447 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= -3.696785155039511 \times 10^{-137} \approx 0 \text{ N}\cdot\text{m}.\end{aligned}$$

Thus, we obtained the same results with case 6.1.3 and case 6.1.2 but with opposite signs for each component. This was explained in the previous examples, where the singularities appear.

Let us take case 6.1.2. and $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = 2\pi$. The approached here gives:

$$\begin{aligned}\tau_x &= -24.91975827162235 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= -0.9803004730404883 \text{ nN}\cdot\text{m}.\end{aligned}$$

Let take us case 6.1.3. and $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = 2\pi$. The approach here gives:

$$\begin{aligned}\tau_x &= 24.91975827162235 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= 0.9803004730404883 \text{ nN}\cdot\text{m}.\end{aligned}$$

These results were expected.

Example 11. The center of the primary coil of the radius $R_P = 1$ m is O (0 m; 0 m; 0 m) and the center of the secondary coil of the radius $R_S = 0.5$ m is C (2 m; 2 m; 2 m). The secondary coil is in the plane $y = 2$ m which means that the coils are with perpendicular axes. Calculate the magnetic torque between coils for which is $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = \pi$ and $\varphi_4 = 2\pi$. All currents are units.

This case is the singular case because $a = c = 0$. Let us start with two perpendicular current loops, (see Figure 4).

Using case 6.1.2 [$\vec{u} = \{-1, 0, 0\}$, $\vec{v} = \{0, 0, -1\}$] and Equations (59)–(61), one finds:

$$\begin{aligned}\tau_x &= -0.3526562725465321 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= 5.833051727704416 \text{ nN}\cdot\text{m}.\end{aligned}$$

Using case 6.1.3 [$\vec{u} = \{0, 0, -1\}$, $\vec{v} = \{-1, 0, 0\}$] and Equations (59)–(61), one finds:

$$\begin{aligned}\tau_x &= 0.3526562725465321 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= -5.833051727704416 \text{ nN}\cdot\text{m}.\end{aligned}$$

Thus, we obtained the same results with case 6.1.3 and case 6.1.2 but with opposite signs for each component.

By [20],

$$\begin{aligned}\tau_x &= 0.35628169 \text{ nN}\cdot\text{m}, \\ \tau_y &= -0.40169339 \times 10^{-15} \approx 0 \text{ N}\cdot\text{m}, \\ \tau_z &= -5.8330053 \text{ nN}\cdot\text{m}.\end{aligned}$$

All the results are in excellent agreement.

For $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = \pi$ and $\varphi_4 = 2\pi$:

Using case 6.1.2 [$\vec{u} = \{-1, 0, 0\}$, $\vec{v} = \{0, 0, -1\}$] and (59)–(61), one has:

$$\begin{aligned}\tau_x &= 7.24520528470814 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= 3.767244177134524 \text{ nN}\cdot\text{m}.\end{aligned}$$

Using case 6.1.3 [$\vec{u} = \{0, 0, -1\}$, $\vec{v} = \{-1, 0, 0\}$] and Equations (59)–(61), one has:

$$\begin{aligned}\tau_x &= -7.245205284708146 \text{ nN}\cdot\text{m}, \\ \tau_y &= 0 \text{ N}\cdot\text{m}, \\ \tau_z &= -3.767244177134524 \text{ nN}\cdot\text{m}.\end{aligned}$$

Thus, we obtained the same results with case 6.1.3 and case 6.1.2 but with opposite signs for each component that was proved in previous singular cases.

Example 12. Calculate the mutual inductance between two inclined current-carrying arc segments for which is $R_P = 0.2 \text{ m}$ and $R_S = 0.1 \text{ m}$. The first arc segment is placed in the plane XOY and the second in the plane $x + y + z = 0.3$ with the center C (0.1 m; 0.1 m; 0.1 m) which lies within.

Let us begin with $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$ (see Figure 3).

Applying Equation (64), the mutual inductance for inclined circular loops is:

$$M = 81.31862021231823 \text{ nH}.$$

We find the same result in [24].

Now, let us change the positions of the arc segments, for example, $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\varphi_3 = \pi$ and $\varphi_4 = 3\pi/2$. Applying Equation (63), the mutual inductance is:

$$M = 17.38258810896817 \text{ nH}.$$

Example 13. Let us consider two arc segments of the radii $R_P = 40 \text{ cm}$ and $R_S = 10 \text{ cm}$. The primary arc segment lies in the plane $z = 0 \text{ cm}$, and it is centered at O (0 cm; 0 cm; 0 cm). The secondary arc segment lies in plane $y = 20 \text{ cm}$, with its center is located at C (0 cm; 20 cm; 10 cm). Calculate the mutual inductance between two arc segments.

This is the singular case, $a = c = 0$. Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$, (see Figure 4).

$$M = -10.72715167866112 \text{ nH}.$$

We find the same result in [24].

For $y = -20$ cm the mutual inductance is:

$$M = 10.72715167866112 \text{ nH.}$$

For $y = 0$ cm the mutual inductance is:

$$M = 0 \text{ H.}$$

This result is found in [30].

Example 14. Let us consider two arc segments of the radii $R_P = 40$ cm and $R_S = 10$ cm, which are mutually perpendicular to each other. The primary arc segment lies in the plane $z = 0$ m, and it is centered at $O(0 \text{ m}; 0 \text{ m}; 0 \text{ m})$, and the center of the secondary coil is located at origin, thus $C = O(0;0;0)$. Calculate the mutual inductance between two arc segments, [30].

Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$. Here, we taste three cases (1) $a = 1$, $b = c = 1$; (2) $a = 0$, $b = 1$, $c = 0$; (3) $a = b = 0$, $c = 1$. For all cases, the mutual inductance [24] gives:

$$M = 0 \text{ H.}$$

From this paper calculations, we obtained the same value. This means that for any position of the secondary loop the mutual inductance is zero when the center of the second loop is positioned in the origin O . The same results are obtained in [30].

Example 15. Let us consider the previous example, but the center of the secondary coil is at the plane XOY with the following coordinates $x_C = y_C = 10$ cm and $z_C = 0$ cm, [30]. Calculate the mutual inductance between these coils.

From this approach the mutual inductance gives:

$$M = 1.78729016039874 \times 10^{-143} \approx 0 \text{ H.}$$

In [30], the mutual is found to be zero.

Example 16. Calculate the stiffness between two coaxial circular loops for which is $R_P = 2$ m, and $R_S = 1$ m. The axial distance between loops is 1 m.

Let us begin with two parallel loops (see Figure 13) for which $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.

Obviously, there is only the stiffness k_{zz} and other stiffnesses are zero because of the coaxial loops.

From Equation (75), the stiffness is:

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -0.2064021172440473 \times 10^{-6} \text{ N/m.}$$

From developed general Formula (76),

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -0.2064021172440473 \times 10^{-6} \text{ N/m.}$$

Thus, with the benchmark formula we confirmed the validity of the general formula for k_{zz} .

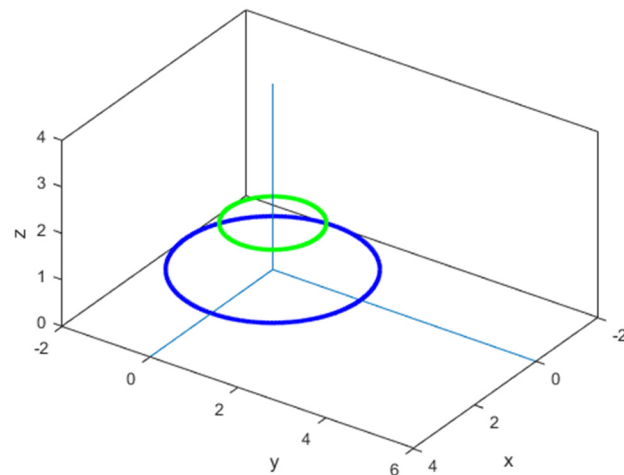


Figure 13. Two parallel circular loops, ($a = 1, b = c = 0, L = l = 1$).

Example 17. Calculate the stiffness between two inclined current-carrying arc segments (see Figure 3) for which $R_P = 0.2\text{ m}$ and $R_S = 0.1\text{ m}$. The first arc segment is placed in the plane XOY and the second in the plane $x + y + z = 0.3$ with the center $C (0.1\text{ m}; 0.1\text{ m}; 0.1\text{ m})$ which lies in this plane.

Let us begin with two circular loops for which $\varphi_1 = 0, \varphi_2 = 2\pi, \varphi_3 = 0$ and $\varphi_4 = 2\pi$. From this approach (76), one has:

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -57.36862305837861 \times 10^{-6}\text{ N/m}.$$

This example could be used as the benchmark example to test other methods in which the axial stiffness is calculated. Now, let us take $\varphi_1 = \pi/4, \varphi_2 = \pi/2, \varphi_3 = 3\pi/4$ and $\varphi_4 = 3\pi/2$.

From Equation (76), one has:

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -28.70523534358855 \times 10^{-6}\text{ N/m}.$$

Example 18. Let us consider two arc segments of the radii $R_P = 40\text{ cm}$ and $R_S = 10\text{ cm}$. The primary arc segment lies in the plane $z = 0\text{ cm}$, and it is centered at $O (0\text{ cm}; 0\text{ cm}; 0\text{ cm})$. The secondary arc segment lies in the plane $y = 20\text{ cm}$, with its center is located at $C (10\text{ cm}; 20\text{ cm}; 10\text{ cm})$. Calculate the stiffness between two arc segments.

This case is the singular case because $a = c = 0$. Let us start with two perpendicular current circular loops (see Figure 4).

Using case 8.1.2 [$\vec{u} = \{-1, 0, 0\}, \vec{v} = \{0, 0, -1\}$] and Equation (76), one finds:

$$k_{zz} = -1.322488731905245\ \mu\text{N/m}.$$

Using case 8.1.3 [$\vec{u} = \{0, 0, -1\}, \vec{v} = \{-1, 0, 0\}$] and Equation (76) one finds:

$$k_{zz} = 1.322488731905245\ \mu\text{N/m}.$$

Thus, we obtained with case 8.1.2 and case 8.1.3 the same results but with opposite signs for each component.

Now, let us take $\varphi_1 = \pi, \varphi_2 = 2\pi, \varphi_3 = \pi$ and $\varphi_4 = 2\pi$.

Using case 8.1.2 [$\vec{u} = \{-1, 0, 0\}$, $\vec{v} = \{0, 0, -1\}$] and Equation (76), one has:

$$k_{zz} = 34.21629087092779 \mu\text{N/m}.$$

Using case 8.1.3 [$\vec{u} = \{0, 0, -1\}$, $\vec{v} = \{-1, 0, 0\}$] and Equation (76), one has:

$$k_{zz} = -34.21629087092779 \mu\text{N/m}.$$

These results were expected.

10. Conclusions

In this paper, we give some ameliorated and new formulas for calculating important electromagnetic quantities such as the magnetic vector potential, the magnetic field, the magnetic force, the mutual inductance, and the stiffness between two inclined current-carrying arc segments in air. The angles of arc segments are arbitrary. All formulas are developed in the close form over the incomplete elliptic integrals of the first and the second kind (the magnetic vector potential and the magnetic field) and in the simple integral form whose kernel functions are also given in the close form over the incomplete elliptic integrals of the first and the second kind (the magnetic force, the magnetic torque, the mutual inductance, and the stiffness). The magnetic vector potential and the magnetic field calculations are given in the analytical form expressed by the incomplete elliptic integrals of the first and second kind. All particular cases are included. Even though these calculations exist in the literature, those presented here, where the angles of coils are arbitrary, present the ameliorated and simplified formulas which are easy to use. The formulas for calculation of the magnetic force and the mutual inductance are significantly simplified regarding those known in the literature. Singular cases are included. Finally, the formulas for calculating the magnetic torque and the stiffness between inclined circular loops or segments appear for the first time in this form in the literature. All electromagnetic quantities are given in quite simple form so that potential readers can easily program them in MATLAB or Mathematica. Many examples confirmed the validity of the presented formulas for inclined circular current-carrying arc segments, and they can be used as the benchmark examples for testing other methods concerning this subject.

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Appendix A

$$\begin{aligned} I_1 &= \int_{\beta_1}^{\beta_2} \frac{\sin(\gamma-2\beta)}{\Delta} d\beta = \sin(\gamma) \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{\Delta} d\beta - \cos(\gamma) \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{\Delta} d\beta \\ &= \frac{y_S}{p} \int_{\beta_1}^{\beta_2} \frac{1-2\sin^2(\beta)}{\Delta} d\beta - 2 \frac{x_S}{p} \int_{\beta_1}^{\beta_2} \frac{\sin(\beta)\cos(\beta)}{\Delta} d\beta. \end{aligned}$$

From [39,40] we obtain the final expression in the analytical form,

$$\begin{aligned}
 I_1 &= \frac{1}{pk^2} \left\{ y_s \left[(k^2 - 2)F(\beta, k) + 2E(\beta, k) \right] + 2x_s \Delta \right\} \Big|_{\beta_1}^{\beta_2}, \\
 I_2 &= \int_{\beta_1}^{\beta_2} \frac{\cos(\gamma - 2\beta)}{\Delta} d\beta = \cos(\gamma) \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{\Delta} d\beta + \sin(\gamma) \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{\Delta} d\beta \\
 &= \frac{x_s}{p} \int_{\beta_1}^{\beta_2} \frac{1 - 2\sin^2(\beta)}{\Delta} d\beta + 2\frac{y_s}{p} \int_{\beta_1}^{\beta_2} \frac{\sin(\beta)\cos(\beta)}{\Delta} d\beta, \\
 I_2 &= \frac{1}{pk^2} \left\{ x_s \left[(k^2 - 2)F(\beta, k) + 2E(\beta, k) \right] - 2y_s \Delta \right\} \Big|_{\beta_1}^{\beta_2}.
 \end{aligned}$$

Appendix B

$$\begin{aligned}
 I_3 &= \int_{\beta_1}^{\beta_2} \frac{\cos(\gamma - 2\beta)}{r_{PS}^3} d\beta = \left\{ \cos(\gamma) \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{r_{TP}^3} d\beta + \sin(\gamma) \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{r_{TP}^3} d\beta \right\} \\
 &= \frac{x_s}{p} \int_{\beta_1}^{\beta_2} \frac{1 - 2\sin^2(\beta)}{\Delta^3} d\beta + \frac{y_s}{p} \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{\Delta^3} d\beta = \frac{x_s}{pk^2(1-k^2)} \left\{ (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) \right. \\
 &\quad \left. + k^2(2 - k^2)k^2(2 - k^2) \frac{\sin(\beta)\cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2}, \\
 I_3 &= \frac{x_s}{pk^2(1-k^2)} \left\{ (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) \right. \\
 &\quad \left. + k^2(2 - k^2) \frac{\sin(\beta)\cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2} + \frac{2y_s}{pk^2\Delta} \Big|_{\beta_1}^{\beta_2}. \\
 I_4 &= \int_{\beta_1}^{\beta_2} \frac{\sin(\gamma - 2\beta)}{r_{PS}^3} d\beta \left\{ \sin(\gamma) \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{r_{TP}^3} d\beta - \cos(\gamma) \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{r_{TP}^3} d\beta \right\} \\
 &= \frac{y_s}{pk^2(1-k^2)} \left\{ (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) + k^2(2 - k^2) \frac{\sin(\beta)\cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2} - \frac{2x_s}{pk^2\Delta} \Big|_{\beta_1}^{\beta_2}. \\
 I_5 &= \int_{\beta_1}^{\beta_2} \frac{R_p + \sqrt{x_s^2 + y_s^2} \cos(2\beta)}{\Delta^3} d\beta = \int_{\beta_1}^{\beta_2} \frac{R_p + p}{\Delta^3} d\beta - 2p \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{\Delta^3} d\beta \\
 &= \frac{1}{k^2(1-k^2)} \left\{ [k^2(R_p + p) - 2p]E(\beta, k) + (2p - 2pk^2)F(\beta, k) \right. \\
 &\quad \left. + k^2(2p - (R_p + p)k^2) \frac{\sin(\beta)\cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2}.
 \end{aligned}$$

Appendix C

$$I_6 = \int_{\varphi_1}^{\varphi_2} \frac{-I_{Sx} \sin(t) + I_{Sy} \cos(t)}{\sqrt{x_S^2 + y_S^2 + z_S^2 + R_p^2 - 2R_p \sqrt{x_S^2 + y_S^2} \cos(t - \gamma)}} dt.$$

The substitution $t - \gamma = \pi - 2\beta$ gives,

$$\begin{aligned}
 I_6 &= -\frac{2l_{xS}}{\sqrt{(R_p + \sqrt{x_s^2 + y_s^2})^2 + z_s^2}} \left\{ \sin(\gamma) \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{\Delta} d\beta - \cos(\gamma) \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{\Delta} d\beta \right\} \\
 &+ \frac{2l_{yS}}{\sqrt{(R_p + \sqrt{x_s^2 + y_s^2})^2 + z_s^2}} \left\{ \cos(\gamma) \int_{\beta_1}^{\beta_2} \frac{\cos(2\beta)}{\Delta} d\beta + \sin(\gamma) \int_{\beta_1}^{\beta_2} \frac{\sin(2\beta)}{\Delta} d\beta \right\} \\
 &= -\frac{l_{xS}k}{p\sqrt{R_p p}} \left\{ y_S \int_{\beta_1}^{\beta_2} \frac{1-2\sin^2(\beta)}{\Delta} d\beta - 2x_S \int_{\beta_1}^{\beta_2} \frac{\sin(\beta)\cos(\beta)}{\Delta} d\beta \right\} \\
 &+ \frac{l_{yS}k}{p\sqrt{R_p p}} \left\{ x_S \int_{\beta_1}^{\beta_2} \frac{1-2\sin^2(\beta)}{\Delta} d\beta + 2y_S \int_{\beta_1}^{\beta_2} \frac{\sin(\beta)\cos(\beta)}{\Delta} d\beta \right\} \\
 &= \frac{l_{yS}}{kp\sqrt{R_p p}} [l_{yS}x_S - l_{xS}y_S] \left\{ [(k^2 - 2)F(\beta, k) + 2E(\beta, k)] \right\} \Big|_{\beta_1}^{\beta_2} - 2\Delta [l_{yS}y_S + l_{xS}x_S] \Big|_{\beta_1}^{\beta_2}.
 \end{aligned}$$

Appendix D

$$\begin{aligned}
 k_{zz} &= -\frac{\partial F_z}{\partial z_s} = -I_S R_S \frac{\partial}{\partial z_s} \int_{\varphi_3}^{\varphi_4} [l_{xS} B_y(S) - l_{yS} B_x(S)] d\theta \\
 &= -I_S R_S \int_{\varphi_3}^{\varphi_4} \left[\frac{\partial l_{xS}}{\partial z_s} B_y(S) + l_{xS} \frac{\partial B_y(S)}{\partial z_s} - \frac{\partial l_{yS}}{\partial z_s} B_x(S) - l_{yS} \frac{\partial B_x(S)}{\partial z_s} \right] d\theta \\
 &= -I_S R_S \int_{\varphi_3}^{\varphi_4} \left[l_{xS} \frac{\partial B_y(S)}{\partial z_s} - l_{yS} \frac{\partial B_x(S)}{\partial z_s} \right] d\theta = \left[\text{because } \frac{\partial l_{xS}}{\partial z_s} = \frac{\partial l_{yS}}{\partial z_s} = 0 \right] \\
 &= -\frac{\mu_0 I_p I_S R_S}{16\pi\sqrt{R_p}} \int_{\varphi_3}^{\varphi_4} \left[l_{xS} \frac{\partial \left(\frac{z_s k}{(1-k^2)\sqrt[2]{p^5}} I_{yy} \right)}{\partial z_s} - l_{yS} \frac{\partial \left(\frac{z_s k}{(1-k^2)\sqrt[2]{p^5}} I_{xx} \right)}{\partial z_s} \right] d\theta \\
 &= -\frac{\mu_0 I_p I_S R_S}{16\pi\sqrt{R_p}} \int_{\varphi_3}^{\varphi_4} \frac{k}{(1-k^2)\sqrt[2]{p^5}} [l_{xS} T_{zz1} - l_{yS} T_{zz2}] d\theta.
 \end{aligned}$$

$$\begin{aligned}
 T_{z1} &= \frac{\partial}{\partial z_s} \left[\frac{z_s k}{(1-k^2)} I_{yy} \right] = \frac{k}{(1-k^2)} I_{yy} + \frac{\partial}{\partial k} \left[\frac{k}{(1-k^2)} \right] \frac{\partial k}{\partial z_s} I_{yy} + \frac{z_s k}{(1-k^2)} \frac{\partial I_{yy}}{\partial z_s} \\
 &\qquad \frac{\partial k}{\partial z_s} = -\frac{z_s k^3}{4pR_p}, \\
 T_{z1} &= \frac{k}{(1-k^2)} T_{zz1}, \\
 T_{zz1} &= c_1 I_{yy} + z_S \frac{\partial (I_{yy})}{\partial z_s}, \\
 T_{zz2} &= c_1 I_{xx} + z_S \frac{\partial (I_{xx})}{\partial z_s}.
 \end{aligned}$$

I_{xx} and I_{yy} are given in (76) and (77).

$$c_1 = 1 - \frac{z_S^2 k^2}{4pR_P} \frac{1 + k^2}{1 - k^2},$$

$$\begin{aligned} \frac{\partial (I_{yy})}{\partial z_S} &= \frac{\partial}{\partial z_S} \left\{ y_S (A_2 - A_1) - 2x_S (1 - k^2) (\Delta_2^{-1} - \Delta_1^{-1}) \right\} = y_S \frac{\partial}{\partial z_S} (A_2 - A_1) \\ &\quad - \frac{x_S z_S}{pR_P} k^4 (\Delta_2^{-1} - \Delta_1^{-1}) + \frac{x_S z_S}{2pR_P} k^4 (1 - k^2) \left[\sin^2(\beta_2) \Delta_2^{-3} - \sin^2(\beta_1) \Delta_1^{-3} \right]. \\ \frac{\partial A_2}{\partial z_S} &= 2kE(\beta_2, k) \frac{\partial k}{\partial z_S} + (k^2 - 2) \frac{dE(\beta_2, k)}{dk} \frac{\partial k}{\partial z_S} + (2 - 2k^2) F(\beta_2, k) \frac{\partial k}{\partial z_S} \\ &\quad - 4k \frac{dF(\beta_2, k)}{dk} \frac{\partial k}{\partial z_S} + \frac{\sin(2\beta_2)}{2} \frac{\partial k}{\partial z_S} \left[4k(1 - k^2) \frac{1}{\Delta_2} + k^3(2 - k^2) \frac{\sin^2(\beta_2)}{\Delta_2^3} \right] \\ &= \frac{\partial k}{\partial z_S} \left\{ 2k[E(\beta_2, k) - 2F(\beta_2, k)] + (k^2 - 2) \frac{dE(\beta_2, k)}{dk} + (2 - 2k^2) \frac{dF(\beta_2, k)}{dk} \right. \\ &\quad \left. + \frac{k \sin(2\beta_2)}{2\Delta_2} \left[4(1 - k^2) + k^2(2 - k^2) \frac{\sin^2(\beta_2)}{\Delta_2^2} \right] \right\}. \\ \frac{\partial A_1}{\partial z_S} &= \frac{\partial A_2(\beta_2 \rightarrow \beta_1, k)}{\partial z_S}. \end{aligned}$$

$$\frac{\partial (I_{yy})}{\partial z_S} = y_S b_1 + 2x_S b_2.$$

$$b_1 = 3[E(\beta, k) - F(\beta, k)] \frac{\beta_2}{\beta_1} + \left\{ \left[\frac{\sin(2\beta)}{\Delta} \right] (1 - 2k^2) + \left[\frac{\sin(2\beta) \sin^2(\beta)}{\Delta^3} \right] \frac{k^2(2 - k^2)}{2} \right\} \frac{\beta_2}{\beta_1}.$$

$$b_2 = \left[\frac{2}{\Delta} \right] \frac{\beta_2}{\beta_1} - (1 - k^2) \left[\frac{\sin^2(\beta)}{\Delta^3} \right] \frac{\beta_2}{\beta_1}.$$

$$T_{zz1} = I_{yy} + z_S \frac{\partial (I_{xx})}{\partial z_S} \quad T_{zz1} = I_{yy} - \frac{z_S^2 k^3}{pR_P} \frac{1 + k^2}{1 - k^2} I_{yy} - \frac{z_S^2 k^4}{pR_P} \frac{\partial (I_{yy})}{\partial z_S}.$$

Similarly,

$$T_{zz2} = I_{xx} - \frac{z_S^2 k^3}{pR_P} \frac{1 + k^2}{1 - k^2} I_{xx} - \frac{z_S^2 k^4}{pR_P} \frac{\partial (I_{xx})}{\partial z_S},$$

$$\frac{\partial (I_{xx})}{\partial z_S} = x_S b_1 - 2y_S b_2.$$

References

1. Maxwell, J.C. *A Treatise on Electricity and Magnetism*, 3rd ed.; Dover Publications Inc.: New York, NY, USA, 1954; Volume 2.
2. Grover, F.W. *Inductance Calculations*; Dover: New York, NY, USA, 1964; Chapters 2 and 13.
3. Snow, C. *Formulas for Computing Capacitance, and Inductance*; National Bureau of Standards Circular 544: Washington, DC, USA, 1954.
4. Kalantarov, P.L. *Inductance Calculations*; National Power Press: Moscow, USSR, 1955. (In Russian)
5. Kalantarov, P.L.; Zeitlin, L.A. *Raschet Induktivnostey [Calculation of Inductances]*; Energoatomizdat: Leningrad, USSR, 1986. (In Russian)
6. Urankar, L. Vector potential and magnetic field of current-carrying finite arc segment in analytical form, part I: Filament approximation. *IEEE Trans. Magn.* **1980**, *16*, 1283–1288. [CrossRef]
7. Urankar, L. Vector potential and magnetic field of current-carrying finite arc segment in analytical form, part II: Thin sheet approximation. *IEEE Trans. Magn.* **1982**, *18*, 911–917. [CrossRef]
8. Urankar, L. Vector potential and magnetic field of current-carrying finite arc segment in analytical form, part III: Exact computation for rectangular cross section. *IEEE Trans. Magn.* **1982**, *18*, 1860–1867. [CrossRef]
9. Urankar, L. Vector potential and magnetic field of current-carrying finite arc segment in analytical form, part IV: General three-dimensional current density. *IEEE Trans. Magn.* **1984**, *20*, 2145–2150. [CrossRef]

10. Urankar, L. Vector potential and magnetic field of current-carrying circular finite arc segment in analytical form. V. Polygon cross section. *IEEE Trans. Magn.* **1990**, *26*, 1171–1180. [[CrossRef](#)]
11. Walstrom, P.L. *Algorithms for Computing the Magnetic Field, Vector Potential, and Field Derivatives for Circular Current Loops in Cylindrical Coordinates*; National Laboratory: Los Alamos, NM, USA, 2017. [[CrossRef](#)]
12. Christodoulides, C. Comparison of the Ampere and Biot-Savart magnetostatic force laws in their line-current-element forms. *Am. J. Phys.* **1988**, *56*, 357–362. [[CrossRef](#)]
13. Babic, S.; Krstajic, B.; Milojkovic, S.; Andjelic, Z. Magnetostatic Field of Thin Current-Carrying Arc Filament. In Proceedings of the Fifth International Symposium on High Voltage Engineering, Braunschweig, Germany, 24–28 August 1987.
14. Smith, M.; Fokas, N.; Hart, K.; Babic, S.I.; Selvaggi, J.P. The magnetic field produced from a conical current sheet and from a thin and tightly wound conical coil. *Prog. Electromagn. Res. B* **2021**, *90*, 1–20. [[CrossRef](#)]
15. Conway, J. Exact solutions for the magnetic fields of axisymmetric solenoids and current distributions. *IEEE Trans. Magn.* **2001**, *37*, 2977–2988. [[CrossRef](#)]
16. Conway, J.T. Trigonometric Integrals for the magnetic field of the coil of rectangular cross section. *IEEE Trans. Magn.* **2006**, *42*, 1538–1548. [[CrossRef](#)]
17. Conway, J.T. Inductance calculations for circular coils of rectangular cross section and parallel axes using Bessel and Struve functions. *IEEE Trans. Magn.* **2009**, *46*, 75–81. [[CrossRef](#)]
18. Ren, Y.; Wang, F.; Kuang, G.; Chen, W.; Tan, Y.; Zhu, J.; He, P. Mutual inductance and force calculations between coaxial bitter coils and superconducting coils with rectangular cross section. *J. Supercond. Nov. Magn.* **2010**, *24*, 1687–1691. [[CrossRef](#)]
19. Ren, Y.; Zhu, J.; Gao, X.; Shen, F.; Chen, S. Electromagnetic, mechanical, and thermal performance analysis of the CFETR magnet system. *Nucl. Fusion* **2015**, *55*, 093002. [[CrossRef](#)]
20. Wang, Z.J.; Ren, Y. Magnetic force and torque calculation between circular coils with nonparallel axes. *IEEE Trans. Appl. Supercond.* **2014**, *24*, 4901505. [[CrossRef](#)]
21. Babic, S.I.; Akyel, C. Torque calculation between circular coils with inclined axes in air. *Int. J. Numer. Model. Electron. Netw. Devices Fields* **2011**, *24*, 230–243. [[CrossRef](#)]
22. Babic, S.; Akyel, C. New formulas for calculating torque between filamentary circular coil and thin wall solenoid with inclined axes whose axes are at the same plane. *Prog. Electromagn. Res. M* **2018**, *73*, 141–151. [[CrossRef](#)]
23. Babic, S.; Akyel, C. Magnetic force between inclined circular filaments placed in any desired position. *IEEE Trans. Magn.* **2011**, *48*, 69–80. [[CrossRef](#)]
24. Babic, S.; Sirois, F.; Akyel, C.; Girardi, C. Mutual inductance calculation between circular filaments arbitrarily positioned in space: alternative to Grover's formula. *IEEE Trans. Magn.* **2010**, *46*, 3591–3600. [[CrossRef](#)]
25. Babic, S.I.; Akyel, C. Magnetic force calculation between thin coaxial circular coils in air. *IEEE Trans. Magn.* **2008**, *44*, 445–452. [[CrossRef](#)]
26. Babic, S.I.; Akyel, C. Magnetic force between inclined circular loops (Lorentz approach). *Prog. Electromagn. Res. B* **2012**, *38*, 333–349. [[CrossRef](#)]
27. Ravaut, R.; Lemarquand, G.; Lemarquand, V. Force and stiffness of passive magnetic bearings using permanent magnets. Part 1: Axial magnetization. *IEEE Trans. Magn.* **2009**, *45*, 2996–3302. [[CrossRef](#)]
28. Ravaut, R.; Lemarquand, G. Force and stiffness of passive magnetic bearings using permanent magnets. Part 2: Radial magnetization. *IEEE Trans. Magn.* **2009**, *45*, 3334–3342. [[CrossRef](#)]
29. Kim, K.-B.; Levi, E.; Zabar, Z.; Birenbaum, L. Restoring force between two noncoaxial circular coils. *IEEE Trans. Magn.* **1996**, *32*, 478–484. [[CrossRef](#)]
30. Poletkin, K.V.; Korvink, J. Efficient calculation of the mutual inductance of arbitrarily oriented circular filaments via a generalisation of the Kalantarov-Zeitlin method. *J. Magn. Magn. Mater.* **2019**, *483*, 10–20. [[CrossRef](#)]
31. Poletkin, K.V. Calculation of force and torque between two arbitrarily oriented circular filaments using Kalantarov-Zeitlin's method. *arXiv* **2021**, arXiv:2106.09496.
32. Poletkin, K.V.; Chernomorsky, A.I.; Shearwood, C.; Wallrabe, U. An Analytical Model of Micromachined Electromagnetic Inductive Contactless Suspension. In Proceedings of the ASME 2013 International Mechanical Engineering Congress and Exposition, San Diego, CA, USA, 15–21 November 2013. [[CrossRef](#)]
33. Poletkin, K.; Chernomorsky, A.I.; Shearwood, C.; Wallrabe, U. A qualitative analysis of designs of micromachined electromagnetic inductive contactless suspension. *Int. J. Mech. Sci.* **2014**, *82*, 110–121. [[CrossRef](#)]
34. Lu, Z.; Poletkin, K.; Hartogh, B.D.; Wallrabe, U.; Badilita, V. 3D micro-machined inductive contactless suspension: Testing and modeling. *Sens. Actuators A Phys.* **2014**, *220*, 134–143. [[CrossRef](#)]
35. Ferreira da Rocha Gama, M.B. Modelling and Simulation of Inductive Levitation Micro-Actuators. Mestrado Integrado em Engenharia Mecânica (Faculdade De Engenharia, Universidade Do Porto, Porto, Portugal, 2021). Available online: <https://repositorio-aberto.up.pt/bitstream/10216/133306/2/453360.pdf> (accessed on 20 September 2021).
36. Poletkin, K.; Chernomorsky, A.; Shearwood, C. Proposal for Micromachined Accelerometer, Based on a Contactless Suspension with Zero Spring Constant. *IEEE Sens. J.* **2012**, *12*, 2407–2413. [[CrossRef](#)]
37. Lubin, T.; Berger, K.; Rezzoug, A. Inductance and force calculation for axisymmetric coil systems including an iron core of finite length. *Prog. Electromagn. Res. B* **2012**, *41*, 377–396. [[CrossRef](#)]

-
38. Theodoulidis, T.; Ditchburn, R.J. Mutual Impedance of Cylindrical Coils at an Arbitrary Position and Orientation above a Planar Conductor. *IEEE Trans. Magn.* **2007**, *43*, 3368–3370. [[CrossRef](#)]
 39. Abramowitz, M.; Stegun, I.A.; Miller, D.A.B. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables; (National Bureau of Standards Applied Mathematics Series No. 55). *J. Appl. Mech.* **1965**, *32*, 239. [[CrossRef](#)]
 40. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*; Dover: New York, NY, USA, 1972.