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Viewpoint

Energy Mechanisms of Free Vibrations and Resonance in Elastic Bodies

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Abstract: The scientific novelty of this work is determined by the rationale for the participation in transformations, along with the kinetic energy of particles, of four types of elastic energy, identified by the peculiarities of their phase changes in the oscillation process. Two types are converted into kinetic energy, while the other two types change the deformed state of particles in accordance with the equations of motion due to internal sources. The result is obtained based on the use of the superposition principle in the space of Lagrange variables with the imposition of forced and free oscillations, as well as a new model of mechanics based on the concepts of space, time, and energy with a new scale of average stresses that takes into account the energy of particles in the initial state. In such a model of mechanics, a generalized measure of the elastic energy of particles is a quadratic invariant of asymmetric tensor whose components are partial derivatives of Euler variables with respect to Lagrange variables. The concept of kinematic energy parameters is introduced, which differ from the corresponding volumetric energy densities by a multiplier equal to the modulus of elasticity, which is directly proportional to the density and heat capacity of the material, and inversely proportional to the volumetric compression coefficient. Comparison of the values of kinematic parameters shows that most of the energy required for oscillations is associated with the deformation of particles and comes from internal sources. The mechanisms of transformation of forced vibrations into their own for transverse, torsional, and longitudinal vibrations are considered, as well as the occurrence of resonance when free and forced vibrations are superimposed with the same or a similar frequency. The formation of a new free wave after each cycle of external influences with an increase in amplitude, which occurs mainly due to internal, and not external, energy sources is justified.

Keywords: energy model of mechanics; equations of motion; Lagrange variables; superposition principle; four types of elastic energy; kinematic parameters of elastic energy

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1. Introduction

Vibrations are among the most common processes, and no phenomenon in nature, none of the created mechanisms, can do without them. They should be considered when calculating, manufacturing, and operating building structures, transport systems, and in technological processes in mechanical engineering [1–7].

One of the founders of the theory of oscillations is J. Rayleigh [1]. His fundamental theorems, including those based on the comparison of the kinetic and potential energy of an oscillating system, have been successfully used to determine the natural frequencies in elastic bodies, optical phenomena, and acoustics. The examples of solving technical problems with a detailed analysis of various types of vibrations are given in [2]. A separate chapter is devoted to the longitudinal, transverse, and torsional vibrations considered in this work. The original concepts, paradoxes, and the most common errors in the analysis of oscillations are considered in [3]. In [4,5], the importance of the general methods of the theory of vibrations for dynamic calculations of engineering structures is noted.

The nature of free vibrations and resonance is still not fully understood. There are reasons to believe that the energy basis of free vibrations and resonance, along with the energy of external forces, are internal energy sources [8].

In most publications, the analysis is limited to the form, frequency, and period of oscillations. As a rule, the vibrations of material points are considered. For elastic bodies, and the relations of the theory of elasticity are used, but without analyzing the energy state of particles in the volume of the body [2,5]. While recognizing the essential role of internal energy, the mechanism of its participation in the occurrence and development of free vibrations and resonance is usually not considered.

The purpose of this work is: using a new concept in mechanics based on the concepts of space, time and energy [9,10], to analyze the change in the deformed and energetic state of particles; comparing their phase changes with changes in kinetic energy and energy of external forces, to substantiate the essential role of additional types of internal energy, providing a change only in the deformed state of particles determined by their equations of motion; check the fulfillment of the law of conservation of energy for local volumes and for the body as a whole.

2. Fundamentals of the Energy Model of Mechanics

Attempts to build an energy model of mechanics were made at the end of the 19th century. In particular, in the work [11], it is noted that the replacement of the abstract concepts of "force" and "mass" with energy allows us to clarify and obtain more information about motion than follows from the basic principles of classical mechanics. The work [12] was characterized by contemporaries as the kinetic basis of mechanics. The orientation of scientists of that time to absolutely solid bodies did not allow for using the general concept of energy (the generalized scalar measure of various types of motion), so their works did not receive support.

The question, "Why do all sections of physics begin with energy and only the first and basic 'mechanics' with an indefinite concept of force?" in the work [13] can be considered as the recommendation of physicists to change the initial prerequisites of mechanics.

The energy model of mechanics should use a description of motion of material particles in the form of Lagrange, since only Lagrange variables allow us to consider the change in the energy state of particles at any time interval, and also to consider the transformation of some types of energy into others, including due to deformation and temperature. Let us use the notations [8–10]

$$x_i = x_i(\alpha_p,\ t), \tag{1}$$

where t is the time, $x_i \in (x, y, z)$ are the current coordinates (Euler), and $\alpha_p \in (\alpha, \beta, \gamma)$ are the Lagrange variables, uniquely associated with the initial coordinates of the particles. They are the arguments of all the functions used in the future.

The different nature of the arguments in Equation (1) allows us to use two independent infinitesimal operators: the d operator for infinitesimal increments in time, for example, $dx_i/dt \equiv x_{i,t}$ and $d^2x_i/dt^2 \equiv x_{i,tt}$, which are components of velocity and acceleration in the directions of the x_i axes, and the δ operator for increments in space, for example, the volume of a particle $\delta V_0 = \delta\alpha\delta\beta\delta\gamma$.

Energy as a generalized scalar characteristic of any type of motion must consider all independent invariants of the system (1), including the invariants of the strain tensors:

$$x_{i,\alpha_p} \equiv \frac{\partial x_i}{\partial \alpha_p} = \begin{pmatrix} x_\alpha & x_\beta & x_\gamma \\ y_\alpha & y_\beta & y_\gamma \\ z_\alpha & z_\beta & z_\gamma \end{pmatrix}, \tag{2}$$

and strain rates $x_{i,t\alpha_p} = x_{i,\alpha_p t} \equiv \partial(dx_i/dt)/\partial\alpha_p$. Linear I_1, quadratic I_2, and cubic I_3 invariants for tensor (2) are defined via the equations:

$$I_1 = x_\alpha + y_\beta + z_\gamma,\ I_2 = x_\alpha^2 + y_\alpha^2 + z_\alpha^2 + x_\beta^2 + y_\beta^2 + z_\beta^2 + x_\gamma^2 + y_\gamma^2 + z_\gamma^2 = \Gamma_e^2,\ I_3 = |x_{i,\alpha_P}| = \delta V/\delta V_0 = R, \quad (3)$$

here and in what follows, when denoting functions $f(\alpha, \beta, \gamma, t)$, the subscripts correspond to the partial derivatives, $f_\alpha \equiv \partial f/\partial\alpha$, with respect to the Lagrange variables and with respect to time t, for example, $df_\alpha/dt \equiv f_{\alpha t} = f_{t\alpha}$.

The energy of the particle, $\delta E_m(\xi_i)$, associated with the motion, is represented as a sum:

$$\delta E_m = \sum \delta E_i(\xi_i) = \sum k_i \xi_i \delta V_0,$$

where the scalar factors k_i ensure equality of the dimensions of the terms.

The law of conservation of energy is written in the form of a balance of increments of the energy of motion, $d\delta E_m(\xi_i)$, and external forces, $d\delta E_\mathrm{e}$:

$$d\delta E = \sum_{i=1}^{13} k_i \xi_{i,t} \delta V_0 d\,t - d\delta E_\mathrm{e} = 0. \quad (4)$$

To determine $d\delta E_\mathrm{e}$ we use, by analogy with classical mechanics, the scalar product of the forces $\delta\mathbf{P}$ by the outer surfaces of the particle and their displacements $d\mathbf{r}$. Taking into account the rule of summation by a repeating index, using the concept of the surface force density τ_{pi}, one obtains [8]:

$$d\,\delta E_e = \sum (\delta\mathbf{P} \cdot d\,\mathbf{r}) = \sum (\delta\mathbf{P} \cdot \mathbf{v})\,d\,t = (\tau_{pi} x_{i,t\alpha_p} + x_{i,t} \partial\tau_{pi}/\partial\alpha_p) \delta V_0 d\,t.$$

The stresses τ_{pi} are similar to the Kirchhoff stresses [10] but differ in the range of the arguments (Lagrange variables) and the possibility of arbitrary selection of the reference point of the average stress scale.

The invariants obtained by time integration take into account the change in the energy state of the particles due to dissipative processes. In the field of reversible deformations, the invariants should not be considered, and the law of energy conservation (4) takes the form [10,14]:

$$k_{I_1} I_{1,t} + k_{I_2} I_{2,t} + k_{I_3} I_{3,t} = \tau_{pi} x_{i,tp} + x_{i,t}\left(\partial\tau_{pi}/\partial\alpha_p - \rho_0 x_{i,tt}\right), \quad (5)$$

where $I_{1,t}$, $I_{2,t}$, and $I_{3,t}$ are the time derivatives for the linear, quadratic, and cubic invariants (3) of the tensor (2), respectively.

From the condition that the energy is invariant with respect to the choice of the velocity reference system, the sum of the last three terms in Equation (5) must turn to 0, and the system (1) must satisfy the differential equation [14,15]:

$$x_t\left(\frac{\partial\tau_{px}}{\partial\alpha_p} - \rho_0 x_{tt}\right) + y_t\left(\frac{\partial\tau_{py}}{\partial\alpha_p} - \rho_0 y_{tt}\right) + z_t\left(\frac{\partial\tau_{pz}}{\partial\alpha_p} - \rho_0 z_{tt}\right) = 0, \quad (6)$$

where ρ_0 is the density of the material in the initial state, τ_{pi} is the surface density of forces on the faces of the infinitesimal parallelepiped, to which the normal is in the initial state that specifies the first subscript $p \in (\alpha, \beta, \gamma)$, and the direction of the voltage is—second $i \in (x, y, z)$. If each bracket is equated to zero, one gets analogs of the differential equations of motion of the classical mechanics of a deformable solid [2,3], but in this case the loss of some possible solutions is not excluded.

Equating the coefficients with the same multipliers $x_{i,t\alpha_p}$ in the remainder of Equation (5):

$$k_{I_1} I_{1,t} + k_{I_2} I_{2,t} + k_{I_3} I_{3,t} = \tau_{pi} x_{i,tp},$$

one obtains the dependences between the stress components τ_{pi}, derivatives $\partial x_i / \partial \alpha_p \equiv x_{i,\alpha_p}$, and coefficients k_{I_i} with a dimension of MPa, called the elastic modulus:

$$\tau_{pi} = k_{I_1} \delta_{pi} + 2k_{I_2} x_{i,p} + k_{I_3} \widetilde{x}_{i,p}. \tag{7}$$

In relations (7) and further on, $\widetilde{x}_{i,p}$—the algebraic complements elements $x_{i,p}$ of the matrix (2), the unit tensor δ_{pi} takes values $\delta_{pi} = 1$ for $\tau_{\alpha x}$, $\tau_{\beta y}$, $\tau_{\gamma z}$ and $\delta_{pi} = 0$ for all other stresses not located on the main diagonal of the tensor τ_{pi}. Equation (7) is derived from the law of energy conservation (5) and is analogous of Hooke's law for the elastic deformation of materials in the space of Lagrange variables.

The linear invariant I_1 cannot determine the energy, since it depends not only on the deformation of the particle, but also on its rotation as a rigid whole [14]. In this regard, the condition $k_{I_1} = 0$ should be accepted. The cubic invariant I_3 determines the ratio of the particle volumes in the current and initial states: $I_3 = |x_{i,p}| = \delta V / \delta V_0$. The coefficient corresponding to this invariant, k_{I_3}, is an additive component of the normal Cauchy stresses and determines the choice of the reference point of the average stress scale [10,14]. At $k_{I_3} = 0$, the average voltages in the initial state take values $\sigma_0 = 2k_{I_2}$.

When moving to a single modulus of elasticity and a new scale of average stresses, considered as the volume energy density of particles [10,14], the concept of stresses τ_{pi} becomes redundant, since they differ from the components of the tensor (2) only by a constant factor equal to twice the modulus of elasticity $k_{I_2} = \kappa$:

$$\tau_{pi} = 2\kappa x_{i,p}. \tag{8}$$

The remaining only one modulus of elasticity κ is directly proportional to the density ρ_0, the heat capacity of the material, c, and is inversely proportional to the volume compressibility, $\beta = 3\alpha$, where α is the coefficient of linear expansion [14]:

$$\kappa = \frac{\rho_0 c}{\beta} = \frac{\rho_0 c}{3\alpha}.$$

Instead of Equation (6), one gets:

$$x_t \left(\frac{\partial x_p}{\partial \alpha_p} - \mu^2 x_{tt} \right) + y_t \left(\frac{\partial y_p}{\partial \alpha_p} - \mu^2 y_{tt} \right) + z_t \left(\frac{\partial z_p}{\partial \alpha_p} - \mu^2 z_{tt} \right) = 0, \tag{9}$$

where $\mu^2 = \frac{\rho_0}{2\kappa}$. The differential equations of motion are transformed into Poisson equations for each of the functions (1):

$$\partial^2 x_i / \partial \alpha_p^2 = \mu^2 (\partial^2 x_i / \partial t^2). \tag{10}$$

The energy model of mechanics with a single modulus of elasticity leads to a significant reduction in mathematical difficulties, and the solutions do not contradict classical mechanics for any types of problems [14].

Under these assumptions, the quadratic invariant (3) of the tensor (2) is a kinematic parameter of the volume density of the elastic energy of particles, considering their initial state, $\Gamma_e^2 = \delta E_e / (\kappa \delta V_0)$ [10]. In the following, the designations $e_i = \delta E_i / (\kappa \delta V_0)$ correspond to the dimensionless kinematic parameters of the volume density of the corresponding modifications of the elastic energy δE_i, of a particle with a volume δV_0. The energy of the particle acquired ($e_{\text{def}} > 0$) or lost ($e_{\text{def}} < 0$) due to elastic deformation, $\delta E_{\text{def}} = \kappa e_{\text{def}} \delta V_0$, in comparison with the initial state, defines a local kinematic parameter $e_{\text{def}} = \Gamma_e^2 - 3$, the right part of which can be written in terms of the squares of the ratios of the lengths of the edges before δl_0 and after δl of the deformation, initially oriented in the direction of the corresponding axes,

$$l_p^2 = (\delta l/\delta l_0)_p^2 = x_p^2 + y_p^2 + z_p^2,\ p \in (\alpha, \beta, \gamma).$$

Then, the parameter e_{def} can be represented via the other dimensionless scalars e_e and e_s [8,15]:

$$\begin{gathered} e_{\mathrm{def}} = \Gamma_e^2 - 3 = 3(e^2 - 1) + e_s = e_e + e_s, \\ e = (l_\alpha + l_\beta + l_\gamma)/3,\ e_e = 3(e^2 - 1),\ e_s = (l_\alpha - e)^2 + (l_\beta - e)^2 + (l_\gamma - e)^2, \end{gathered} \tag{11}$$

where e is the average value of the relative lengths of the l_p edges of an infinitesimal parallelepiped before and after deformation. Parameter e_e depends on the average length of e and can be either positive or negative. The value e_s is always positive and coincides with the standard deviation of the lengths of the edges of the parallelepiped l_p from their average value e. Equation (11) allows a change in the deformed state of particles due to internal energy sources if $e_e + e_s = \mathrm{const}$.

Considering the features of phase changes, for the processes, discussed below, e_e and e_s can be represented in terms of additional kinematic parameters such as $e_e = e_{e1} + e_{e2}$ and $e_s = e_{s1} + e_{s2}$, which play a different role in fulfilling the law of conservation of energy for particles and the body as a whole [8,15]. In particular, the parameters e_{e1} and e_{s1} will include the energy shares that participate in the transformation of the kinetic energy of particles into elastic or vice versa, as well as in the implementation of the law of conservation of energy for an elastic body as a whole when considering external forces. The parameters e_{e2} and e_{s2} consider transitions only within the elastic deformation of particles, for example, the transition of the energy of volume change to the energy of shape change or vice versa. They do not require an influx of energy through its boundaries—as, by analogy, with the implementation of the law of conservation of energy for free vibrations in an elastic body—and do not affect the change in the energy of elastic deformation of the body.

Taking into account the current and the initial ($\Gamma_e^2 = 3$) states, the energy δE_{def} of a particle with a volume δV_0 due to deformation can be represented as:

$$\delta E_{\mathrm{def}} = \kappa\delta V_0(\Gamma_e^2 - 3) = \kappa\delta V_0[3(e^2 - 1) + e_s] = \kappa\delta V_0(e_e + e_s) = \kappa\delta V_0(e_{e1} + e_{e2} + e_{s1} + e_{s2}). \tag{12}$$

In all the cases discussed below, an elastic rod of length L with a cross section S_0 is considered as a physical body, the ends of which are fixed in fixed arrays that do not exchange energy with the oscillating system.

For the energy justification of the resonance, the transformation of forced vibrations into free ones is considered after the termination of external forces, as well as the subsequent occurrence of resonance, when periodic external forces with a frequency of natural vibrations or close to it appear.

3. Transverse Vibrations

Consider the energy features of transverse vibrations in accordance with the equations:

$$x(\alpha_p,\ t) = \alpha,\ y(\alpha_p,\ t) = \beta + v(\alpha,\ t),\ z(\alpha_p,\ t) = \gamma, \tag{13}$$

where $v(\alpha,\ t)$—moves in the direction of the y-axis. We start counting the time when there is no deformation $v(\alpha, \mathrm{t}) = 0$ and the Lagrangian coordinates coincide with the initial ones. The vibrations, allowed by the law of energy conservation, must correspond to Equation (9), which is converted to the form

$$y_{tt}(\alpha,\ t) = (2\kappa/\rho_0)y_{\alpha\alpha}. \quad (14)$$

This equation under the initial condition, $\alpha_i = x_i(\alpha_p, t = 0)$, and boundary conditions for displacements, $v(\alpha = 0, \mathrm{t}) = 0$, $v(\alpha = L, \mathrm{t}) = 0$, is fulfilled by the function

$$y(\alpha, t) = \beta + q\sin(\pi\alpha/L)\sin(\omega_0 t) \quad (15)$$

with derivatives

$$y_{\alpha\alpha}(\alpha, t) = -q\left(\frac{\pi}{L}\right)^2 \sin\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t),\ y_{tt}(\alpha, t) = -q\omega^2 \sin\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t), \quad (16)$$

where q is the maximum displacement of particles along the y axis in the cross section $\alpha = L/2$. Natural frequency,

$$\omega_0 = (\pi/L)\sqrt{2\kappa/\rho_0}, \quad (17)$$

considers the properties of the material and the size of the sample. Velocities of particles

$$y_t(\alpha, t) = v_t(\alpha, t) = q\omega_0 \sin(\pi\alpha/L)\cos(\omega_0 t) \quad (18)$$

at the ends of the rod are equal to 0, at $t = \frac{\pi n}{\omega_0}$ they are maximal in each of the sections along its entire length. The tensor corresponds to the system (13) considering Equation (15):

$$x_{i,p} = \begin{pmatrix} 1 & 0 & 0 \\ q\pi/L\cos(\pi\alpha/L)\sin(\omega_0 t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The deformation is carried out due to shifts, $\partial y/\partial\alpha \equiv y_\alpha$, and the quadratic invariant of the tensor determines the increment of the local energy of elastic deformation of the particles (11),

$$\delta E_{\mathrm{def}} = \kappa(\Gamma_e^2 - 3)\delta V_0 = \kappa\delta V_0(\pi q/L)^2\cos^2(\pi\alpha/L)\sin^2(\omega_0 t). \quad (19)$$

The energy for deformation in the volume of the rod is

$$E_{\mathrm{def}} = \frac{1}{2}\kappa V_0(\pi q/L)^2\sin^2(\omega_0 t). \quad (20)$$

For kinetic energy with velocity (18) one finds

$$\delta E_{\mathrm{kin}} = \frac{1}{2}\rho_0 v_t^2 \delta V_0 = \frac{1}{2}\delta V_0\rho_0[q\omega_0\sin(\pi\alpha/L)\cos(\omega_0 t)]^2. \quad (21)$$

After integration by volume, one gets

$$E_{\mathrm{kin}} = \frac{L}{4}S_0\rho_0 q^2\omega_0^2\cos^2(\omega_0 t) = 0.5V_0\kappa(\pi q/L)^2\cos^2(\omega_0 t). \quad (22)$$

Total kinetic energy and strain energy in the volume of the oscillating rod,

$$E_{\mathrm{sum}} = E_{def} + E_{\mathrm{kin}} = S_0\kappa\frac{\pi^2 q^2}{2L}\left[\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\right] = \frac{1}{2}V_0\kappa\left(\frac{\pi q}{L}\right)^2 = \mathrm{const}, \quad (23)$$

coincides with the kinetic energy (22) available in the system at the moment $t = 0$ and does not change in time, which indicates compliance with the law of conservation of energy, if there is no energy transfer to the fixed walls at the contacts with the ends of the sample or to the environment from the outer surface of the rod.

Condition (23) corresponds to proper transverse vibrations in an elastic rod without energy consumption from external forces and without changing the volume of particles:

$$R = \delta V/\delta V_0 = \begin{vmatrix} 1 & 0 & 0 \\ y_\alpha & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

At the end of the cycle ($t = T = \frac{2\pi}{\omega_0}$), the system returns to its initial state, the elastic energy is absent, and the kinetic energy (22) takes the maximum possible value.

To find out the energy features of the resonance, we consider vibrations with a driving force [1–3]:

$$F = F_0 \sin(\omega t), \tag{24}$$

acting in the central section along the length $\alpha = L/2$ with an amplitude F_0 and a circular frequency ω, which does not necessarily coincide with ω_0. The force F_0 can be determined from the integral law of conservation of energy in the form of equality of the power of the external force and the rate of change of kinetic and elastic energy in the volume of the rod.

Considering the velocity y_t in the cross section $\alpha = L/2$, where the force (24) is applied, one obtains the power

$$W_{\text{ext}} = F y_t|_{\alpha=L/2} = F_0 q \omega \sin(\omega t)\cos(\omega t). \tag{25}$$

The energy transferred to the system is converted into elastic (19) and kinetic (21) energies of particles, which characterize the specific powers δW_{kin} and δW_{def}:

$$\delta W_{\text{kin}} = d\delta E_{\text{kin}}/dt = 0.5\rho_0 \delta V_0 dy_t^2/dt = 2\kappa\delta V_0 y_t y_{tt}/\lambda_0^2, \delta W_{\text{def}} = d\delta E_{\text{def}}/dt = \kappa\delta V_0 dy_\alpha^2/dt = 2\kappa\delta V_0 y_\alpha y_{t\alpha}. \tag{26}$$

In the expression for the rate of change of the kinetic energy of the particle, δW_{kin}, the material constant is used $\lambda_0^2 = 2\kappa/\rho_0$, which determines the circular frequency (17). The ratio $\eta = \omega/\omega_0$ characterizes both the kinematic features of vibrations (ω) and the physical properties of the material (ω_0). Considering the derivatives (16) and (18) of the function (15) in time and Lagrange variables at the force frequency ω, the local powers of the kinetic, δW_{kin}, and elastic, δW_{def}, particle energy are determined by the following equations,

$$\frac{\delta W_{\text{kin}}}{2\kappa\delta V_0} = -\omega(q\pi\eta/L)^2 \sin^2(\pi\alpha/L)\sin(\omega t)\cos(\omega t),$$

$$\frac{\delta W_{\text{def}}}{2\kappa\delta V_0} = \omega(q\pi/L)^2 \cos^2(\pi\alpha/L)\sin(\omega t)\cos(\omega t).$$

The total power integral in the rod volume at any given time then is

$$\frac{W_s}{2\kappa V_0} = \frac{W_{\text{def}}}{2\kappa V_0} + \frac{W_{\text{kin}}}{2\kappa V_0} = 0.5\omega(q\pi/L)^2(1-\eta^2)\sin(\omega t)\cos(\omega t). \tag{27}$$

Note that when describing the motion in the Lagrange form, it is not necessary to monitor the change in the contour of an elastic body during the oscillation, since for a Lagrangian coordinate system, it coincides with the original configuration and does not change in time.

Equating the powers of external (25) and internal (27) forces, one finds the force F_0 corresponding to the vibrations with the frequency and amplitude considered:

$$F_0 = \kappa V_0 q(1-\eta^2)(\pi/L)^2. \tag{28}$$

Depending on the frequency ratio, the force F_0 can be positive ($\omega < \omega_0$), negative ($\omega > \omega_0$), and zero ($\omega = \omega_0$). The positive force, as follows from Equation (25), supplies energy to the rod in the first and third quarters of the cycle when the kinetic energy is converted into the elastic deformation of the particles. In the other two quarters, the power will be negative, and the energy of the elastic deformation of the particles is converted into kinetic energy and transmitted to an external source. A negative force F_0 occurs when the circular frequency of the periodic external force exceeds the natural frequency of the system. Then, in the first and third quarters of the cycle, the power is negative, and the kinetic energy is spent on the deformation of the particles and transferred to an external source.

Equations (24) and (25)–(27) correspond to forced vibrations in an elastic rod with a period $T = 2\pi/\omega$ under the action of an exciting force (28). After the cycle is completed, the system returns to its original state and, if the force (24) continues to operate, the cycle repeats.

If the external force ceases after the next cycle, regardless of the frequency ratio, in the system remains the kinetic energy of the particles,

$$\delta E_{\text{kin}}\Big|_{t=T} = 0.5\rho_0 v_t^2 \delta V_0 = 0.5\delta V_0 \rho_0 [q\omega \sin(\pi\alpha/L)]^2 = \kappa \delta V_0 (\pi\eta q/L)^2 \sin^2(\pi\alpha/L),$$

which will lead to continued fluctuations. The frequency and amplitude may vary, but Equations (26) and (27) remain valid.

It follows from Equation (27) that if the frequency of the external force is less than $\omega < \omega_0$, then the positive power $W_s > 0$ in the first quarter of the cycle will increase the actual frequency ω. Similarly, when $\omega > \omega_0$, the negative power of $W_s < 0$ will reduce the actual frequency. Only in the case that $\omega = \omega_0$, the volume integral power is equal to 0 over the entire cycle the sum of kinetic and elastic energy in the system remains unchanged, which corresponds to the definition of free vibrations that can continue without energy input from external sources [1–5].

From the point of view of resonance, the case is interesting when a cyclic force (24) acts creating a forced oscillation with a frequency of natural vibrations $\omega = \omega_0$ or close to it. Then the two waves will interact, forming a new wave.

In accordance with the superposition principle [9], to obtain the equations of joint motion it is sufficient to replace the Lagrange variables of external (superimposed) motion with expressions for the corresponding Euler variables of internal (nested) motion. In our case, the equations for natural and forced oscillations may differ in the circular frequency ω and the amplitude q, but they are equivalent in their effect on the resulting oscillation. Any of them can be considered external or internal.

We choose the amplitude of natural oscillations by the lower index q_0:

$$x(\alpha_p,\ t) = \alpha,\ y(\alpha, t) = \beta + q_0 \sin(\pi\alpha/L)\sin(\omega_0 t),\ z(\alpha_p,\ t) = \gamma. \tag{29}$$

For a forced oscillation considering the resonance, we use the equation

$$x(\alpha_p,\ t) = \alpha,\ y(\alpha, t) = \beta + q_1 \sin(\pi\alpha/L)\sin(\omega_0 t),\ z(\alpha_p,\ t) = \gamma. \tag{30}$$

Replacing the variable β in Equation (29) with the right-hand side of Equation (30), for joint motion one obtains:

$$x(\alpha_p,\ t) = \alpha,\ y(\alpha, t) = \beta + (q_0 + q_1)\sin(\pi\alpha/L)\sin(\omega_0 t),\ z(\alpha_p,\ t) = \gamma. \tag{31}$$

If the frequencies of external and natural oscillations on the new cycle are equal, the amplitude will be equal to the sum of the amplitudes of forced (on the current cycle) and natural (on the previous cycle) oscillations of the system. To clarify the question of the energy possibility of such oscillations, it is necessary to additionally determine the kinetic and elastic energies using derivatives of Equation (31),

$$y_t = (q_0 + q_1)\omega_0 \sin(\pi\alpha/L)\cos(\omega_0 t),\ y_\alpha = (q_0 + q_1)(\pi/L)\cos(\pi\alpha/L)\sin(\omega_0 t).$$

The integral values are equal to

$$E_{\mathrm{def}} = 0.5\kappa V_0(q_0 + q_1)^2(\pi/L)^2\sin^2(\omega_0 t),\ E_{\mathrm{kin}} = 0.5\kappa V_0(q_0 + q_1)^2(\pi\eta/L)^2\cos^2(\omega_0 t).$$

Only these equations, by analogy with Equations (20) and (22), can ensure the continuation of oscillations with the fulfillment of the law of conservation of energy for the system as a whole:

$$E_{\mathrm{def}} + E_{\mathrm{kin}} = 0.5\kappa V_0(q_0 + q_1)^2(\pi/L)^2 = \mathrm{const}. \tag{32}$$

As a result, one gets a new free oscillation with an increased amplitude which can interact with a new cycle of forced oscillation. An increase in the amplitude of free vibrations due to interaction with forced vibrations with the frequency of natural vibrations, or close to it, is the basis of resonance [15]. Equation (32) can be considered the energy justification of the resonance. The kinetic and elastic energies in the body volume increase in proportion to the square of the new amplitude due to internal sources determined by the elastic modulus of the material.

The superposition principle, which is successfully used in various problems for absolutely solid and deformable bodies [9,14], and the new model of mechanics with a single elastic modulus (8), confirm the kinematic and energy possibility of joint motion (31) in compliance with the law of conservation of energy. At equal amplitudes $q_0 = q_1$, all energy characteristics of the combined oscillation increase by four times in relation to the initial free oscillation.

To identify the role of internal energy sources, we pay attention to the kinematic parameters of the energy invariants (11), which depend on the relative average length e and standard deviations e_s. For each particle, one can point out four specific fractions of the dimensionless bulk energy density:

$$\begin{gathered}
e_e = e_{e1} + e_{e2} = \tfrac{1}{3}\left(\tfrac{\pi q}{L}\right)^2\cos^2\left(\tfrac{\pi\alpha}{L}\right)\sin^2(\omega_0 t) + \tfrac{4}{3}\left\{\left[1 + \left(\tfrac{\pi q}{L}\right)^2\cos^2\left(\tfrac{\pi\alpha}{L}\right)\sin^2(\omega_0 t)\right]^{1/2} - 1\right\},\\
e_s = e_{s1} + e_{s2} = \tfrac{2}{3}\left(\tfrac{\pi q}{L}\right)^2\cos^2\left(\tfrac{\pi\alpha}{L}\right)\sin^2(\omega_0 t) - \tfrac{4}{3}\left\{\left[1 + \left(\tfrac{\pi q}{L}\right)^2\cos^2\left(\tfrac{\pi\alpha}{L}\right)\sin^2(\omega_0 t)\right]^{1/2} - 1\right\},\\
e_{\mathrm{def}} = e_e + e_s = \left(\tfrac{\pi q}{L}\right)^2\cos^2\left(\tfrac{\pi\alpha}{L}\right)\sin^2(\omega_0 t).
\end{gathered} \tag{33}$$

These equations carry objective information, including that about the energy sources that are not related to external forces and are not converted into kinetic energy of particles. The most informative are the relative measures with respect to the kinematic parameter of the total elastic energy (19) of a particle. Considering the first two terms of the function expansion in a series $\sqrt{1+x} = 1 + 1/2x - 1/8x^2 + \ldots$, these relations remain the same for all particles at any given time:

$$\frac{e_{e1}}{e_{\mathrm{def}}}+\frac{e_{e2}}{e_{\mathrm{def}}}=\frac{1}{3}+\frac{2}{3},\ \frac{e_{s1}}{e_{\mathrm{def}}}+\frac{e_{s2}}{e_{\mathrm{def}}}=\frac{2}{3}-\frac{2}{3},$$

where the energy fractions $e_{e1}=1/3e_{\mathrm{def}}$ and $e_{s1}=2/3e_{\mathrm{def}}$ change synchronously and participate in the transformation of elastic energy into kinetic energy. The amount,

$$e_{\mathrm{def}}=e_e+e_s=e_{e1}+e_{s1}=\left(\frac{\pi q}{L}\right)^2\cos^2\left(\frac{\pi\alpha}{L}\right)\sin^2(\omega_0 t),$$

ensures the implementation of the law of energy conservation in the volume of the rod with an integral significance (20) and a phase shift of $\pi/2$ with respect to the change in the kinetic energy (22) of particles.

The energy shares of e_{e2} and e_{s2} vary in opposite phases, and the sum of these shares is always 0, although each of them is comparable to the total energy of e_{def}. In other words, the energy e_{e2}, determined by the change in the average length of the edges of the particle in the form of an infinitesimal parallelepiped, passes into the energy e_{s2}, which is associated with the standard deviation of the relative lengths of the edges of the particle from their average value, and vice versa. Such deformations do not change the energy state of the particle and the body as a whole. The addition of velocities in accordance with the principle of superposition of motions (31) provides the necessary kinetic energy for the deformation of particles during the cycle and the fulfillment of the law of conservation of energy for the system as a whole when taking into account external forces.

Further development of oscillation can occur in one of the following ways:

(1) the appearance of a new oscillation with an amplitude depending on the frequency ratio,

$$y(\alpha,t)=\beta+[q_0\sin(\omega_0 t)+qn\sin(\omega t)]\sin(\pi\alpha/L),$$

if, after the formation of free vibrations, the external force (24) begins to act again with a frequency ω significantly different from ω_0. The influence of the free wave will decrease and forced oscillations (30) with the frequency of force (24) will continue, which require power (25);

(2) continuation of free oscillations with an increase in the amplitude $q_0+q_1+\ldots+q_i$ after the next superposition if the frequency of forced oscillations is close to its own and there is no energy exchange with the external environment;

(3) decrease in the amplitude if the resonant system is used as an accumulator; energy stored in the system energy will go into the environment, including if due to displacement were considered the stationary supporting walls of a perfectly rigid body, in which is fixed the elastic rod;

(4) the most dangerous is the continuation of vibrations in the conditions of resonance with the achievement and subsequent exceeding of the limit values of the stresses acting in the system, and the occurrence of irreversible deformations or destruction of the system.

4. Torsional Vibrations

During torsional vibrations, circumferential and radial movements of particles can occur. In this regard, under the condition of plane deformation in the Cartesian coordinate system, two of the three equation in the system (10) must be considered:

$$y_{\alpha\alpha}+y_{\beta\beta}+y_{\gamma\gamma}=\mu^2 y_{tt},\ z_{\alpha\alpha}+z_{\beta\beta}+z_{\gamma\gamma}=\mu^2 z_{tt}. \tag{34}$$

As soon as one ignores the change of the radial coordinate and accepts the ratios

$$x=\alpha,\ y=\beta\cos\Delta\psi-\gamma\sin\Delta\psi,\ z=\beta\sin\Delta\psi+\gamma\cos\Delta\psi, \tag{35}$$

where ψ is the angle of rotation of the section relative to the x axis, system (34) takes the form,

$$-y\psi_\alpha^2 - z\psi_{\alpha\alpha} = -\mu^2 y\psi_t^2 - \mu^2 z\psi_{tt},$$

$$-z\psi_\alpha^2 + y\psi_{\alpha\alpha} = -\mu^2 z\psi_t^2 + \mu^2 y\psi_{tt}.$$

Squaring and summing the left and right parts, one obtains equation for the function $\psi(\alpha, t)$,

$$\psi_\alpha^4 + \psi_{\alpha\alpha}^2 = \mu^4\psi_t^4 + \mu^4\psi_{tt}^2.$$

The solution $\psi(\alpha, t) = C\sin(\alpha \pm qt)$ turns this equation into an identity, but it does not agree with the initial and boundary conditions

$$\begin{gathered} \psi(\alpha, \mathrm{t}=0) = \theta\sin(\pi\alpha/L),\ \psi(\alpha = L/2, \mathrm{t}=0) = \theta,\ \psi_t(\alpha, \mathrm{t}=0) = 0, \\ \psi(\alpha = 0, \mathrm{t}) = 0,\ \psi(\alpha = L, \mathrm{t}) = 0. \end{gathered} \tag{36}$$

If $\psi_\alpha^4 = \mu^4\psi_t^4$ is accepted, then for the function $\psi = \psi(\alpha, t)$ one obtains an equation similar to Equation (14) for transverse vibrations:

$$\psi_{\alpha\alpha}^2 - \mu^4\psi_{tt}^2 = 0 \tag{37}$$

with a solution

$$\Delta\psi(\alpha, t) = \theta\sin(\pi\alpha/L)\sin(\omega_0 t), \tag{38}$$

$$\omega_0 = (\pi/L)\sqrt{2\kappa/\rho_0}, \tag{39}$$

where ω_0 is the frequency of natural vibrations and θ is the angle of rotation of the rod in the cross section with the coordinate $\alpha = L/2$ at $t = \pi/(2\omega_0)$. In this case, system (35) as well as the boundary and initial conditions (36) are fulfilled.

Time and direction derivatives,

$$\psi_t(\alpha, t) = \theta\omega_0\sin(\pi\alpha/L)\cos(\omega_0 t),\ \psi_{tt}(\alpha, t) = -\theta\omega_0^2\sin(\pi\alpha/L)\sin(\omega_0 t),$$

$$\psi_\alpha(\alpha, t) = \theta\pi/L\cos(\pi\alpha/L)\sin(\omega_0 t),\ \psi_{\alpha\alpha}(\alpha, t) = -\theta(\pi/L)^2\sin(\pi\alpha/L)\sin(\omega_0 t),$$

$$\psi_{\alpha t}(\alpha, t) = \theta\omega_0\pi/L\cos(\pi\alpha/L)\cos(\omega_0 t),$$

determine the kinematic, deformation, and energy characteristics of the particles and the body as a whole. Note that the obtained solution satisfies not only the system (10), but also the more general Equation (9):

$$y_t(y_{\alpha\alpha} + y_{\beta\beta} + y_{\gamma\gamma} - \mu^2 y_{tt}) + z_t(z_{\alpha\alpha} + z_{\beta\beta} + z_{\gamma\gamma} - \mu^2 z_{tt}) = 0.$$

In the latter case, only the simplified condition (37) must be satisfied.

This can be considered as an additional argument about the acceptability of the obtained solution for analyzing the energy features of free torsional vibrations.

Considering the components of the tensor (2),

$$x_{i,p} = \begin{pmatrix} 1 & 0 & 0 \\ -\psi_\alpha z & \cos\Delta\psi & -\sin\Delta\psi \\ \psi_\alpha y & \sin\Delta\psi & \cos\Delta\psi \end{pmatrix},$$

one finds the value of the quadratic invariant, the specific energy of elastic deformation, and the kinetic energy of particles:

$$\Gamma_e^2 = l_\alpha^2 + l_\beta^2 + l_\gamma^2 = 3 + y_\alpha^2 + z_\alpha^2 = 3 + \psi_\alpha^2 r^2,$$
$$\delta E_{\text{def}} = \kappa\delta V_0 \psi_\alpha^2 r^2 = \kappa\delta V_0 \pi^2 \theta^2 (r/L)^2 \cos^2(\pi\alpha/L)\sin^2(\omega_0 t),$$
$$\delta E_{\text{kin}} = \kappa\mu^2 \delta V_0 \psi_t^2 r^2 = \kappa\delta V_0 \theta^2 \pi^2 (r/L)^2 \sin^2(\pi\alpha/L)\cos^2(\omega_0 t),$$

which depend on the particle radius, r. Volume integral energy values,

$$E_{\text{def}} = \kappa V_0 \pi^2 \theta^2 \frac{R^2}{4L^2}\sin^2(\omega_0 t),\ E_{\text{kin}} = \kappa V_0 \theta^2 \pi^2 \frac{R^2}{4L^2}\cos^2(\omega_0 t),$$

in sum, correspond to the law of conservation of energy in the volume of the oscillating rod and coincide with the work of external forces transmitted to the body at the time of the beginning of vibrations, which corresponds to the concept of natural vibrations

$$E_{\text{sum}} = E_{\text{def}} + E_{\text{kin}} = \kappa V_0 \theta^2 \pi^2 \frac{R^2}{4L^2} = \text{const}. \tag{40}$$

In accordance with Equations (35) and (38), similar to the case of transverse vibrations, the elastic deformation is carried out due to shifts, while the volume of material particles and the density of the material remain unchanged, regardless of the magnitude of the rotation angle ψ:

$$R = \delta V/\delta V_0 = \begin{vmatrix} x_\alpha & x_\beta & x_\gamma \\ y_\alpha & y_\beta & y_\gamma \\ z_\alpha & z_\beta & z_\gamma \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -\psi_\alpha z & \cos\Delta\psi & -\sin\Delta\psi \\ \psi_\alpha y & \sin\Delta\psi & \cos\Delta\psi \end{vmatrix} = 1.$$

The cause of forced torsional vibrations may be the moment

$$M = M_0 \sin(\omega t) \tag{41}$$

with an amplitude M_0 that acts in a cross section with the coordinate $\alpha = L/2$ with a frequency ω that does not necessarily coincide with the proper ω_0.

Considering system (35), Equation (38) and angular velocity $\psi_t|_{\alpha=L/2} = \theta\omega\cos(\omega t)$, the moment (41) produces the power,

$$W_{\text{ext}} = M\psi_t|_{\alpha=L/2} = M_0\theta\omega\sin(\omega t)\cos(\omega t), \tag{42}$$

converted into elastic and kinetic energies of particles, for the rate of change of which (taking into account $\delta W_i = d\delta E_i/dt$) the following equations are valid:

$$\frac{\delta W_{\text{def}}}{2\kappa\delta V_0} = \psi_\alpha \psi_{\alpha t} r^2 = \pi^2 \theta_0^2 \omega (r/L)^2 \cos^2(\pi\alpha/L)\sin(\omega t)\cos(\omega t),$$
$$\frac{\delta W_{\text{kin}}}{2\kappa\delta V_0} = \frac{1}{\lambda_0^2}\psi_t \psi_{tt} r^2 = -\theta_0^2 \pi^2 \eta^2 \omega (r/L)^2 \sin^2(\pi\alpha/L)\sin(\omega t)\cos(\omega t).$$

As in the case of transverse vibrations, the kinetic energy of particles depends on the density of the material, ρ_0, so it contains a multiplier $\eta = \omega/\omega_0$, which characterizes the ratio of the frequencies of forced and natural vibrations.

Integrating the powers over the volume:

$$W_{\text{def}} = 0.5\kappa V_0 \pi^2 \theta^2 (R/L)^2 \omega \sin(\omega t)\cos(\omega t), \tag{43}$$

$$W_{\text{kin}} = -0.5\kappa V_0 \theta^2 \pi^2 \eta^2 (R/L)^2 \omega \sin(\omega t) cos(\omega t), \tag{44}$$

and using the energy identity that includes the external moment (42), as well as the internal forces (43) and (44), one finds the moment,

$$M_0 = 0.5\kappa V_0 \theta \pi^2 (R/L)^2 (1 - \eta^2). \tag{45}$$

The system (35) under the action of a moment (41) with an amplitude (45) corresponds to forced harmonic oscillations with a circular frequency of the external moment ω. If the external moment (41) ceases to act, for example, after the completion of the next cycle, the kinetic energy of the particles remains in the system, which causes its own vibrations.

Sum of capacities (43) and (44),

$$W_{\text{def}} + W_{\text{kin}} = 0.5\kappa V_0 \theta_0^2 \pi^2 (R/L)^2 (1 - \eta^2) \omega \sin(\omega t) \cos(\omega t), \tag{46}$$

in the absence of external influence characterizes the possible transitions of elastic energy to kinetic energy, and vice versa. Stationary mode occurs when $\omega/\omega_0 = 1$. If the actual frequency of vibrations is lower than the proper ω_0 determined by the elastic properties of the material (39), the positive power will lead to its increase by one and three quarters of cycles. Otherwise, the oscillation frequency will decrease, and the mode will correspond to its own oscillations. Equation (46) can be considered as a mechanism for converting forced oscillations into proper ones after the external influence ceases.

This feature is confirmed by the analysis of the total elastic and kinetic energy in the rod volume, which depends on the frequency ratio,

$$E_{\text{def}} + E_{\text{kin}} = \kappa V_0 \theta^2 \pi^2 \frac{R^2}{4L^2} \left[\sin^2(\omega t) + \eta^2 \cos^2(\omega t) \right],$$

and only if they are equal ($\eta = 1$), remains constant, as follows from the definition of the system's natural oscillations [1–5].

Resonance is possible if the free vibrations are superimposed, forced with a circular frequency of natural vibrations ω_0. Let us use the equations for natural oscillations

$$x = \alpha,\ y = \beta \cos \Delta\psi_0 - \gamma \sin \Delta\psi_0,\ z = \beta \sin \Delta\psi_0 + \gamma \cos \Delta\psi_0, \tag{47}$$

and forced fluctuations

$$x = \alpha,\ y = \beta \cos \Delta\psi_1 - \gamma \sin \Delta\psi_1,\ z = \beta \sin \Delta\psi_1 + \gamma \cos \Delta\psi_1, \tag{48}$$

where $\Delta\psi_0$ and $\Delta\psi_1$ are the angles of rotation of the sections in free and forced oscillations,

$$\Delta\psi_0 = \theta_0 \sin(\pi\alpha/L) \sin(\omega_0 t),\ \Delta\psi_1 = \theta_1 \sin(\pi\alpha/L) \sin(\omega_0 t).$$

Using the general rule of superposition of motions [9], the Lagrange variables in the equations for forced oscillations (48) are replaced with expressions for the corresponding Euler variables of natural oscillations (47):

$$\begin{gathered} y = \beta \cos \Delta\psi_1 - \gamma \sin \Delta\psi_1 = (\beta \cos \Delta\psi_0 - \gamma \sin \Delta\psi_0) \cos \Delta\psi_1 - (\beta \sin \Delta\psi_0 + \\ + \gamma \cos \Delta\psi_0) \sin \Delta\psi_1 = \beta \cos(\Delta\psi_0 + \Delta\psi_1) - \gamma \sin(\Delta\psi_0 + \Delta\psi_1) \end{gathered}$$

As a result, one obtains a system of type of system (35), in which the angle of rotation in the joint oscillation, instead of Equation (38), is equal to the sum of the angles of rotation of the forced and free oscillations:

$$\Delta\psi = (\theta_0 + \theta_1) \sin(\pi\alpha/L) \sin(\omega_0 t). \tag{49}$$

The rationale for the energy feasibility of joint oscillations in accordance with Equations (35) and (49) differs only slightly from the one given for transverse oscillations. In accordance with Equations (35) and (49), particles of the rod with volume δV_0, density ρ_0, and elastic modulus κ rotate relative to the x-axis with angular velocities

$$\psi_t(\alpha, t) = (\theta_0 + \theta_1)\omega_0 \sin(\pi\alpha/L)\cos(\omega_0 t),$$

and due to this, the system acquires kinetic energy,

$$E_{\text{kin}} = \frac{1}{4}\kappa V_0 (\pi R/L)^2 (\theta_0 + \theta_1)^2 \cos^2(\omega_0 t). \tag{50}$$

Elastic energy for system (35) with tensor (2),

$$x_{i,p} = \begin{pmatrix} 1 & 0 & 0 \\ -\psi_\alpha z & \cos\Delta\psi & -\sin\Delta\psi \\ \psi_\alpha y & \sin\Delta\psi & \cos\Delta\psi \end{pmatrix},$$

considering the invariant (11), is equal to

$$e_{\text{def}} = e_e + e_s = \psi_\alpha^2 r^2 = (\theta_0 + \theta_1)^2 \pi^2 (r/L)^2 \cos^2(\pi\alpha/L)\sin^2(\omega_0 t).$$

The elastic energy in the volume of the rod is:

$$E_{\text{def}} = \frac{1}{4}\kappa V_0 (\pi R/L)^2 (\theta_0 + \theta_1)^2 \sin^2(\omega_0 t). \tag{51}$$

Equations (46), (50), and (51) correspond to harmonic oscillations (34) with changes in angles (49). Kinetic and elastic energies provide a constant value of their sum at any moment:

$$E_{\text{kin}} + E_{\text{def}} = 0.25\kappa V_0 \pi^2 (\theta_0 + \theta_1)^2 (R/L)^2 = \text{const}. \tag{52}$$

Resonance from the point of view of the law of conservation of energy is possible and allows for a significant increase in the amplitude of the new phase of free oscillation and the energy parameters associated with the amplitude at the expense of internal forces [15].

The structure of the local kinematic parameters of the volumetric energy density (11), which depend on the mean value e and the standard deviation e_s,

$$\begin{gathered} e = 2/3 + \sqrt{1 + \psi_\alpha^2 r^2}/3,\ e_e = 3(e^2 - 1) = \tfrac{1}{3}\psi_\alpha^2 r^2 + \tfrac{4}{3}\left(\sqrt{1 + \psi_\alpha^2 r^2} - 1\right) = e_{e1} + e_{e2}, \\ e_s = \tfrac{2}{3}\psi_\alpha^2 r^2 + \tfrac{4}{3}\left(1 - \sqrt{1 + \psi_\alpha^2 r^2}\right) = e_{s1} + e_{s2}, \end{gathered} \tag{53}$$

and are the same as for transverse vibrations. Considering the series expansion of the function with a square root, one gets:

$$\frac{e_e}{e_{\text{def}}} = \frac{\psi_\alpha^2 r^2}{\psi_\alpha^2 r^2} = 1,\ \frac{e_{e1}}{e_{\text{def}}} = \frac{(1/3)\psi_\alpha^2 r^2}{\psi_\alpha^2 r^2} = \frac{1}{3},\ \frac{e_{e2}}{e_{\text{def}}} = \frac{(2/3)\psi_\alpha^2 r^2}{\psi_\alpha^2 r^2} = \frac{2}{3},$$

$$\frac{e_s}{e_{\text{def}}} = \frac{2}{3} - \frac{2}{3} = 0,\ \frac{e_{s1}}{e_{\text{def}}} = \frac{2}{3}\frac{\psi_\alpha^2 r^2}{\psi_\alpha^2 r^2} = \frac{2}{3},\ \frac{e_{s2}}{e_{\text{def}}} = \frac{(4/3)(-\psi_\alpha^2 r^2/2)}{\psi_\alpha^2 r^2} = -\frac{2}{3}.$$

As in the case of transverse vibrations, the energy fractions e_{e1} and e_{s1} change synchronously and in total provide the elastic energy necessary to fulfill the integral conservation law, which is converted into kinetic energy. The components e_{e2} and e_{s2} are two times larger than e_{e1}, but do not participate in such transformations and determine the energy consumption only for changing the shape of particles, associated with changes in the length of the edges of particles e_e and their standard deviation from the mean e_s according to the Equation (11). Changes occur with the same frequency but in opposite phases, and energy costs are compensated by the opposite type of deformation.

The analysis shows that part of the energy involved in free and combined vibrations is not associated with the energy coming from external sources, and it is not converted into kinetic energy, but rather is an integral element of resonant phenomena.

5. Longitudinal Vibrations

Using the method of the previous sections, let us consider the longitudinal oscillations with the equations

$$x(\alpha, t) = \alpha + p\sin(\pi\alpha/L)\sin(\omega t),\ y = \beta,\ z = \gamma. \tag{54}$$

The Lagrange variables α_p coincide with the initial x_p at t = 0. As in the previous sections, the equations in the system (54) correspond to a coordinate system whose origin is aligned with the left fixed end of the elastic rod, while the x-axis is directed along the axis of the rod with length L and cross section S_0.

Equation (54) with derivatives

$$\begin{gathered} x_\alpha(\alpha, t) = 1 + p\pi/L\cos(\pi\alpha/L)\sin(\omega t),\ x_{t\alpha}(\alpha, t) = p\pi\omega/L\cos(\pi\alpha/L)\cos(\omega t), \\ x_t = p\omega\sin(\pi\alpha/L)\cos(\omega t),\ x_{tt} = -p\omega^2\sin(\pi\alpha/L)\sin(\omega t) \end{gathered} \tag{55}$$

satisfy the condition (9) as well as the boundary and initial conditions

$$x(\alpha = 0, t) = 0,\ x(\alpha = L, t) = L,\ x_t(\alpha = 0, t) = x_t(\alpha = L, t) = 0,\ x_i(\alpha_p, t = 0) = \alpha_p \tag{56}$$

with natural frequency

$$\omega_0 = \pi/(\mu L). \tag{57}$$

Longitudinal vibrations differ from transverse and torsional by deformations of tension and compression, which are determined by the derivative $\partial x/\partial\alpha \equiv x_\alpha$:

$$x_{i,p} = \begin{pmatrix} x_\alpha & x_\beta & x_\gamma \\ y_\alpha & y_\beta & y_\gamma \\ z_\alpha & z_\beta & z_\gamma \end{pmatrix} = \begin{pmatrix} 1 + \pi p/L\cos(\pi\alpha/L)\sin(\omega_0 t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{58}$$

The kinematic parameter of the elastic strain energy (11) for longitudinal vibrations contains two terms, and, due to the smallness of the p/L ratio, the first term may be several orders of magnitude larger than the second term

$$e_{\text{def}} = 2(\pi p/L)\cos(\pi\alpha/L)\sin(\omega_0 t) + (\pi p/L)^2\cos^2(\pi\alpha/L)\sin^2(\omega_0 t), \tag{59}$$

but the volume integral value of E_{def} coincides in form with (20) for transverse vibrations

$$E_{\text{def}} = \int\limits_{V_0} \kappa(\Gamma_e^2 - 3)\delta V_0 = 0.5\kappa V_0(\pi p/L)^2\sin^2(\omega_0 t). \tag{60}$$

Kinetic energy of particles considering Equation (55),

$$\delta E_{\text{kin}} = 0.5\rho_0 x_t^2 \delta V_0 = \kappa \delta V_0 (\pi p/L)^2 \sin^2(\pi\alpha/L)\cos^2(\omega_0 t), \tag{61}$$

for the entire volume of the rod is

$$E_{\text{kin}} = \frac{L}{4} S_0 \rho_0 \frac{\pi^2 p^2}{L^2}\cos^2(\omega_0 t) = V_0 \kappa \frac{\pi^2 p^2}{2L^2}\cos^2(\omega_0 t). \tag{62}$$

In each section, the rate of change of elastic and kinetic energy changes in accordance with Equations (59) and (61), but for the entire volume of the rod, the sum of the energies,

$$E_{\text{def}} + E_{\text{kin}} = V_0 \kappa \frac{\pi^2 p^2}{2L^2} = \text{const}, \tag{63}$$

remains constant, coincides with the one in the system (62) at $t = 0$, and does not change in time, which indicates that free oscillations continue.

The occurrence of resonance should be facilitated by a periodic force acting in the central section along the length $\alpha = L/2$ with a frequency ω that does not necessarily coincide with Equation (57):

$$F = F_0 \sin(\omega t), \tag{64}$$

which produces the power

$$W_{\text{ext}} = F x_t|_{\alpha = L/2} = F_0 p \omega \sin(\omega t)\cos(\omega t), \tag{65}$$

which is spent on elastic deformation and kinetic energies of particles,

$$\delta W_{\text{kin}} = \rho_0 \delta V_0 x_t x_{tt},\ \delta W_{\text{def}} = \kappa \delta V_0 (e_{\text{def}})_t = \kappa \delta V_0 (x_\alpha^2 - 1)_t = 2\kappa \delta V_0 x_\alpha x_{t\alpha},$$

where κ is the elastic modulus.

Considering Equation (57), $\eta = \omega/\omega_0$, and derivatives (55), one obtains:

$$\frac{\delta W_{\text{kin}}}{2\kappa \delta V_0} = -\omega(\pi p \eta/L)^2 \sin^2(\pi\alpha/L)\sin(\omega t)\cos(\omega t).$$

The local power of elastic deformation, in comparison with Equation (19), has a summand with the first degree of amplitude

$$\frac{\delta W_{\text{def}}}{2\kappa \delta V_0} = x_\alpha x_{t\alpha} = p\omega\pi/L\cos(\pi\alpha/L)\cos(\omega t) + (\pi p/L)^2 \omega \cos^2(\pi\alpha/L)\sin(\omega t)\cos(\omega t).$$

The total power of elastic and kinetic energies also differs from Equation (19) by an additional term

$$\begin{array}{l}\frac{\delta W_{\text{def}}}{2\kappa \delta V_0} + \frac{\delta W_{\text{kin}}}{2\kappa \delta V_0} = p\pi\omega/L\cos(\pi\alpha/L)\cos(\omega t) + \\ +(\pi p/L)^2 \omega \sin(\omega t)\cos(\omega t)\left\{\cos^2(\pi\alpha/L) - \eta^2 \sin^2(\pi\alpha/L)\right\}\end{array}$$

but this does not affect the integral power value for the rod volume

$$\frac{W_{\text{def}}}{2\kappa V_0} + \frac{W_{\text{kin}}}{2\kappa V_0} = \frac{1}{4}(\pi p/L)^2 \omega (1 - \eta^2)\sin(2\omega t). \tag{66}$$

From the equality of powers (65) and (66), we find the modulus of force in Equation (64),

$$F_0 = \kappa V_0 p(1-\eta^2)(\pi/L)^2, \tag{67}$$

which provides energy to the forced vibrations of the body.

At the end of each cycle ($t = T = 2\pi/\omega$), there is no deformation over the entire volume of the rod, and only the kinetic energy of the particles is preserved at its maximum value in the volume of the rod,

$$E_{\text{kin}} = 0.5V_0\kappa(\pi p\eta/L)^2. \tag{68}$$

The system (54) under the action of force (67) performs harmonic forced oscillations with a period $T = 2\pi/\omega$ of the exciting force. If the external force ceases to act, the kinetic energy (68) of the particles remains, which causes the vibrations to continue. As in the case of transverse vibrations, in accordance with Equation (66), due to the excess energy appearing at $\omega < \omega_0$ (or the lack of energy at $\omega > \omega_0$), the system will bring the actual frequency closer to its own. Only when $\omega = \omega_0$ ($\eta = 1$) is the sum of kinetic and elastic energy in the system unchanged, which corresponds to the definition of free vibrations.

In what follows, the resonance is considered as an overlap of two independent oscillations and the kinematics (by fulfilling the boundary conditions) and energy (by fulfilling the law of conservation of energy for the system as a whole and taking into account external forces) implementation is checked.

For free oscillations, equations of the type of Equation (54) are used which are distinguished by the notation of the amplitude

$$x(\alpha, t) = \alpha + p_0 \sin(\pi\alpha/L)\sin(\omega_0 t),\ y = \beta,\ z = \gamma. \tag{69}$$

For forced oscillations with a frequency of natural ω_0, the amplitude p_1 is denoted

$$x(\alpha, t) = \alpha + p_1 \sin(\pi\alpha/L)\sin(\omega_0 t),\ y = \beta,\ z = \gamma. \tag{70}$$

Similar to the case of transverse and torsional vibrations, when using the superposition principle [9], any movement can be taken as internal and another as external. For example, replacing the Lagrange variables in Equation (70) with expressions for the corresponding Euler variables from Equation (69), one obtains

$$x(\alpha, t) = \alpha + p_0 \sin\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t) + p_1 \sin\left\{\frac{\pi}{L}\left[\alpha + p_0 \sin\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t)\right]\right\}\sin(\omega_0 t).$$

Considering the smallness of the $p/L << 1$ ratio, it can be argued that the amplitude of the joint oscillation is equal to the sum of the amplitudes of the internal free and external forced oscillations,

$$x(\alpha, t) = \alpha + (p_0 + p_1)\sin(\pi\alpha/L)\sin(\omega_0 t). \tag{71}$$

This result also confirms the traditional approach to determining joint movements by adding velocities [3,9]. Note that the boundary conditions (56) for the system (71) are fulfilled, as well as for the joint oscillations (69) and (70).

The superposition of vibrations leads to the kinetic energy of particles in joint motion,

$$\delta E_{\text{kin}} = \frac{1}{2}\rho_0 x_t^2 \delta V_0 = \kappa\delta V_0(p_0+p_1)^2\left(\frac{\pi}{L}\right)^2 \sin^2\left(\frac{\pi\alpha}{L}\right)\cos^2(\omega_0 t).$$

For the volume of the rod, the energy is

$$E_{\text{kin}} = \frac{1}{2}\kappa V_0 (p_0 + p_1)^2 (\pi/L)^2 \cos^2(\omega_0 t). \quad (72)$$

The kinematic parameter (11) of the strain energy,

$$e_{\text{def}} = 2(p_0 + p_1)\left(\frac{\pi}{L}\right)\cos\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t) + (p_0 + p_1)^2\left(\frac{\pi}{L}\right)^2\cos^2\left(\pi\frac{\alpha}{L}\right)\sin^2(\omega_0 t),$$

allows us to determine the elastic energy for the body as a whole:

$$E_{\text{def}} = \kappa \int e_{\text{def}} \delta V_0 = 0.5\kappa (p_0 + p_1)^2 V_0 (\pi/L)^2 \sin^2(\omega_0 t). \quad (73)$$

Two types (72) and (73) of energy change with the same frequency and phase shift by $\pi/2$, what ensures that the summed energy is constant and that the law of energy conservation for the rod as a whole is fulfilled:

$$E_{\text{kin}} + E_{\text{def}} = 0,5\kappa (p_0 + p_1)^2 V_0 (\pi/L)^2 = \text{const}. \quad (74)$$

Additional energy from external sources is not required to continue the oscillations determined by the superposition principle. Equation (74) confirms the possibility of resonance when using the method of converting the deformed state determined by the equations of motion into elastic energy according to Equation (73) [15]. It shows the possibility of continuing vibrations without violating the law of conservation and energy inflow from outside if the frequency of forced vibrations coincides with the frequency of their own or close to it. After the completion of each cycle, the amplitude of the joint free oscillation increases by the magnitude of the amplitude of the imposed forced oscillation, with an increase in the energy parameters of the system in accordance with the change in the derivatives (55) due to the internal energy, determined by the elastic modulus of the material.

To identify the role of internal energy sources, let us consider, as done in previous sections, the kinematic parameters of elastic energy (11) for the equations of motion (54),

$$e_{\text{def}} = 2(\pi p/L)\cos(\pi\alpha/L)\sin(\omega_0 t) + (\pi p/L)^2\cos^2(\pi\alpha/L)\sin^2(\omega_0 t).$$

The presence of the two terms in Equation (59) does not allow us to switch to relative fractions, as this was done for transverse oscillations, so only the actual relative fractions are given as follows:

$$\begin{gathered} e_e = 2(\pi p/L)\cos(\pi\alpha/L)\sin(\omega_0 t) + (1/3)[(\pi p/L)\cos(\pi\alpha/L)\sin(\omega_0 t)]^2, \\ e_{e1} = \tfrac{1}{3}\left(\tfrac{\pi p}{L}\right)^2\cos^2\left(\pi\tfrac{\alpha}{L}\right)\sin^2(\omega t),\ e_{e2} = \tfrac{2\pi p}{L}\cos\left(\pi\tfrac{\alpha}{L}\right)\sin(\omega t), \\ e_s = e_{s1} = \tfrac{2}{3}\left(\tfrac{\pi p}{L}\right)^2\cos^2\left(\pi\tfrac{\alpha}{L}\right)\sin^2(\omega t),\ e_{s2} = 0. \end{gathered} \quad (75)$$

The frequency of changes in the elastic energy fractions e_{e1} and e_{s1} is consistent with the frequency of changes in kinetic energy (61) and differs in phase by $\pi/2$, which ensures that their sum (63) is equal throughout the cycle, and which is necessary to fulfill the law of energy conservation for the body, considering external forces.

Since the volume-integral fraction of e_{e2} is 0, it cannot participate in the implementation of the energy conservation law for the system as a whole. The energy, associated with the kinematic parameter e_e, corresponds to deformations in the system due to internal sources, in particular, a decrease and increase in the volume (3) of particles in various parts of the system without changing the total volume. In this case, the ratio

$$\frac{e_{e2}}{e_{e1}} = \frac{6}{(\pi p/L)\cos(\pi\alpha/L)\sin(\omega t)} = \frac{6}{\delta V/\delta V_0 - 1}$$

can reach large values, especially when the ratio of the amplitude to the length of the rod p/L is small.

Thus, during longitudinal vibrations, most of the energy for the deformation of particles comes from internal sources, and only a small part (e_{e1} and e_{s1}), converted to kinetic energy (61), is associated with the energy acquired from outside and participates in the implementation of the energy conservation law for an elastic body as a whole.

For comparison, the equations for joint longitudinal vibrations is presented when the frequency of the forced oscillation does not coincide with the frequency of its own. Then, instead of Equation (70) we should use the system (54), and as a result of the superposition, one gets:

$$x(\alpha, t) = \alpha + p_0 \sin\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t) + p_1 \sin\left\{\frac{\pi}{L}\left[\alpha + p_0 \sin\left(\pi\frac{\alpha}{L}\right)\sin(\omega_0 t)\right]\right\}\sin(\omega t).$$

Even in the region of small ratios $p/L << 1$ system,

$$x(\alpha, t) = \alpha + [p_0 \sin(\omega_0 t) + p_1 \sin(\omega t)]\sin(\pi\alpha/L),\ y = \beta,\ z = \gamma,$$

describes vibrations with a time-varying amplitude that depends on the ratio of amplitudes and the frequencies of joint vibrations. The equations for local and integral kinetic and elastic energies will have a more complex form in comparison with Equations (60)–(63), which do not provide for the occurrence of resonance.

The main condition for resonance is the equality of the frequencies of the joint movements, where it is possible with a superposition of various types of vibrations, such as longitudinal (69) and transverse (14) ones. As a result, one gets the equations of motion,

$$x = \alpha + p\sin(\pi\alpha/L)\sin(\omega t),\ y = \beta + q\sin(\pi\alpha/L)\sin(\omega t),\ z(\gamma,\ t) = \gamma \tag{76}$$

with the tensor (2),

$$x_{i,p} = \begin{pmatrix} x_\alpha & x_\beta & x_\gamma \\ y_\alpha & y_\beta & y_\gamma \\ z_\alpha & z_\beta & z_\gamma \end{pmatrix} = \begin{pmatrix} 1 + \pi p/L\cos(\pi\alpha/L)\sin(\omega t) & 0 & 0 \\ \pi q/L\cos(\pi\alpha/L)\sin(\omega t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The system satisfies differential Equation (10) as well as initial and boundary conditions of type of Equation (56) for each of the considered and joint oscillations. In this case, added are the algebraically local,

$$e_{\text{def}} = \Gamma_e^2 - 3 = e'_{\text{def}} + e''_{\text{def}} =$$

$$= 2\tfrac{\pi p}{L}\cos\left(\tfrac{\pi\alpha}{L}\right)\sin(\omega_0 t) + \left[\tfrac{\pi p}{L}\cos\left(\tfrac{\pi\alpha}{L}\right)\sin(\omega_0 t)\right]^2 + \left[\tfrac{\pi q}{L}\cos\left(\tfrac{\pi\alpha}{L}\right)\sin(\omega_0 t)\right]^2$$

$$e_{\text{kin}} = 0.5\rho_0 v^2 = 0.5\rho_0(x_t^2 + y_t^2) = e'_{\text{kin}} + e''_{\text{kin}},$$

and integral energy characteristics (at the frequency of natural),

$$E_{\text{def}} = E'_{\text{def}} + E''_{\text{def}} = \frac{1}{2}\kappa V_0\left\{\left[\frac{\pi p}{L}\sin(\omega_0 t)\right]^2 + \left[\frac{\pi q}{L}\sin(\omega_0 t)\right]^2\right\},$$

$$E_{\text{kin}} = V_0\kappa\frac{\pi^2 p^2}{2L^2}\cos^2(\omega_0 t) + V_0\kappa\frac{\pi^2 q^2}{2L^2}\cos^2(\omega_0 t),$$

and for the energy implementation of the movement (76), no additional energy is required.

6. Discussion and Conclusions

To analyze the features of the transformation of kinetic and elastic energy in oscillating bodies, a new concept of mechanics based on the concepts of space, time, and energy, with one modulus of elasticity (8) and a new scale of average stresses which takes into account the energy of particles in the initial state, is used. In this model, the elastic deformation of particles is determined by the quadratic invariant of the tensor (2), whose components are partial derivatives of Euler variables with respect to Lagrange variables of the equations of motion (1). The increment of the invariant due to elastic deformation can be represented as the sum of two invariants e_e and e_s (11), one of which depends on the average value of the relative lengths of the edges of the particles in the form of an infinitesimal parallelepiped, while the second is equal to the standard deviation of these lengths from the average value. Equation (12) allows for the spontaneous development of deformations without the participation of external forces if the sum (e_e + e_s) remains unchanged. In classical solid mechanics, such a possibility is interpreted as the transition of the energy of shape change into the energy of volume change, or vice versa.

The features of the phase changes of these invariants during the considered oscillations do not coincide with the corresponding changes in the kinetic energy of the particles. Therefore, there must be other participants in the energy conversion process that play an important role in the oscillations, and which must be considered in the right part of Equation (12). They can be identified by subtracting from the e_e and e_s in Equation (11) the terms e_{e1} and e_{s1} corresponding to the frequency characteristics of the change in the kinetic energy of the particles for the corresponding process. In particular, for transverse oscillations (13), taking into account Equations (21) and (33), one obtains:

$$e_{e2} = e_e - e_{e1} = \frac{4}{3}\left\{\left[1 + \left(\frac{\pi q}{L}\right)^2 \cos^2\left(\frac{\pi \alpha}{L}\right) \sin^2(\omega_0 t)\right]^{1/2} - 1\right\},$$

$$e_{s2} = e_s - e_{s1} = \frac{4}{3}\left\{1 - \left[1 + \left(\frac{\pi q}{L}\right)^2 \cos^2\left(\frac{\pi \alpha}{L}\right) \sin^2(\omega_0 t)\right]^{1/2}\right\}.$$

These two types of deformation are the kinematic parameters of the elastic energy of particles, which change in antiphases and do not require energy from external sources. Similarly, Equation (53) for torsional oscillations is obtained.

In the case of longitudinal oscillations the Equation (75) for e_s contains only one term. If one uses the described algorithm, the value of e_{e2} goes to e_{s2}. In both cases, the energy, determined by the kinematic parameter e_{e2}, should be used to deform the particles. Parts of the elastic energy e_{e1} and e_{s1} in all considered oscillations turn into kinetic energy and participate in the implementation of the law of conservation of energy for the body as a whole. The other parts of e_{e2} and e_{s2} are not converted into kinetic energy but change the deformed state of the particles, in accordance with the equations of motion due to internal sources.

Thus, based on the analysis of the structure of the deformed state and the phase changes of its components, the necessity of participation in energy transitions of four fractions of elastic energy, differing in their interaction with the kinetic energy of particles and the implementation of the energy conservation law in integral and local volumes of an oscillating body, is justified: two fractions participate in mutual transformations of kinetic and elastic energy, and the other two change the deformed state of particles while remaining elastic, taking into account the peculiarities of the equations of motion.

For all the considered types of vibrations, four types of elastic energy reveal the mechanism of participation of internal types of energy in the development of vibrations with the fulfillment of the energy conservation law. In particular, part of the energy for changing the shape of particles during transverse and torsional vibrations comes from the particles themselves, with an equivalent change in the deformed state. The change in the volume of particles in the areas of tension and compression during longitudinal vibrations also does not require an influx of energy from external sources.

The dimensionless parameters e_e, e_s, e_{e1}, e_{s1}, e_{e2}, and e_{s2} are real deformations, corresponding to the equations of motion, are determined through derivatives of Euler variables by Lagrange variables, and are reflected in invariants (3), (11) and (12). They expand the possibilities of transitions of elastic energy from one form to another as allowed in the classical mechanics of deformable solids, and by analogy with the transition of the energy of shape change to the energy of volume change and vice versa.

A distinctive feature of resonance with Equations (31), (48), (49), and (71) is an increase in the amplitude of the natural oscillations when they interact with forced ones. Multiple interactions lead to a multiple increase in the amplitude and energy of the resulting resonant wave due to internal sources.

Equations (27), (46), and (66) correspond to the mechanism of transformation of the forced oscillations with a frequency as determined by external influences into their own after the termination of the driving force. This mechanism continues to operate with a superposition of free and forced oscillations, the frequency of which is close to its own but does not coincide with it.

Resonance is possible as a result of a superposition of both similar and different types of vibrations if their frequencies coincide with their own or are close to them [15]. For example, as with a superposition of longitudinal and transverse oscillations (76) at the same frequencies and coinciding with the frequencies of natural oscillations. Local and volume–integral energy characteristics have the property of additivity, the law of conservation of energy is fulfilled, and resonance is possible.

The ratios (23), (40), and (63) can be used to determine the elastic constant of a material according to experimental studies with the main forms of free oscillations [8].

The obtained results on the transformation of forced oscillations into free ones after the termination of the external force, on the implementation of the conservation law for volume integral values of kinetic and elastic energy, on participation in the resonance of internal energy sources can be considered as additional arguments for the validity of the energy model for solving various problems of mechanics.

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