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Powers of Octonions

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Abstract

As it is known, Binomial expansion, De Moivre's formula, and Euler's formula are suitable methods for computing the powers of a complex number, but to compute the powers of an octonion number in easy way, we need to derive suitable formulas from these methods. In this paper, we present a novel way to compute the powers of an octonion number using formulas derived from the binomial expansion.

Keywords

Octonion, Matrix Algebra

1. Introduction

An octonion number a can be expressed as:

$$a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 + a_4 i_4 + a_5 i_5 + a_6 i_6 + a_7 i_7 \quad (1)$$

where $a_0, \cdots, a_7$ are real numbers and $i_1, \cdots, i_7$ are imaginary units. Their multiplication is given in Table 1. Octonion algebra is an eight-dimensional, non-commutative, non-associative and normed division algebra.

Table 1. Imaginary units multiplication.

	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	-1	i_3	$-i_2$	i_5	$-i_4$	$-i_7$	i_6
i_2	$-i_3$	-1	i_1	i_6	i_7	$-i_4$	$-i_5$
i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	$-i_4$
i_4	$-i_5$	$-i_6$	$-i_7$	-1	i_1	i_2	i_3
i_5	i_4	$-i_7$	i_6	$-i_1$	-1	$-i_3$	i_2
i_6	i_7	i_4	$-i_5$	$-i_2$	i_3	-1	$-i_1$
i_7	$-i_6$	i_5	i_4	$-i_3$	$-i_2$	i_1	-1

Octonions have been used in many fields of mathematics, and they have many applications, about octonions and their applications see [1] [2] [3], and [4] [5] to take a historical overview.

In the matrix representation, an octonion a can be represented by 8×8 real matrices. One form of these matrices is a matrix A (the left matrix representation) [6]:

$$A=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

The problem of computing the n^{th} power of an octonion is still interesting to many researchers. Some methods are used, such as binomial expansion, De Moivre's formula, and Euler's formula. To solve this problem, we use a new technique to construct formulas computing the powers of an octonion.

2. The Methodology

For a complex number $b=b_0+b_1 i$ where b_0, b_1 are real numbers, and i is the imaginary unit satisfies $i^2=-1$,

$$b^n=\left(b_0+b_1 i\right)^n=\sum_{j=0}^{n}\binom{n}{n-j}\left(b_0^{n-j}\right)\left(b_1 i\right)^j,\ n \text{ is a positive integer number} \tag{2}$$

If n is an even number then there will be $\frac{n}{2}+1$ real terms and $\frac{n}{2}$ imaginary terms, to simplify, we define $\operatorname{Im}[b]=b_1 i$, $\operatorname{Re}[b]=b_0$ and, let b_0, b_1 be nonzero real numbers, we can write (2) as:

$$\begin{aligned} \operatorname{Re}\left[b^n\right]&=\sum_{j=0}^{n/2}\binom{n}{n-2j} b_0^{n-2j}\left(-b_1^2\right)^j \\ \operatorname{Im}\left[b^n\right]&=\left(b-b_0\right)\sum_{j=0}^{n/2-1}\binom{n}{n-2j-1} b_0^{n-2j-1}\left(-b_1^2\right)^j \end{aligned} \tag{3}$$

If n is an odd number then there will be $\frac{n+1}{2}$ real terms and $\frac{n+1}{2}$ imaginary terms, we can write (2) as:

$$\begin{aligned} \operatorname{Re}\left[b^n\right]&=\sum_{j=0}^{(n-1)/2}\binom{n}{n-2j} b_0^{n-2j}\left(-b_1^2\right)^j \\ \operatorname{Im}\left[b^n\right]&=\left(b-b_0\right)\sum_{j=0}^{(n-1)/2}\binom{n}{n-2j-1} b_0^{n-2j-1}\left(-b_1^2\right)^j \end{aligned} \tag{4}$$

To prove (3), let us write b in a matrix representation form:

$$B = \begin{bmatrix} b_0 & -b_1 \\ b_1 & b_0 \end{bmatrix}$$

For $n = 2$, b^2 can be computed from:

$$\begin{bmatrix} b_0 & -b_1 \\ b_1 & b_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} b_0^2 - b_1^2 \\ 2b_0 b_1 \end{bmatrix},$$

so

$$\begin{aligned} \operatorname{Re}\left[b^2\right] &= b_0^2 - b_1^2 = \sum_{j=0}^{1} \binom{2}{2-2j} b_0^{2-2j} \left(-b_1^2\right)^j \\ \operatorname{Im}\left[b^2\right] &= 2b_0 b_1 = \left(b - b_0\right) \sum_{j=0}^{0} \binom{2}{1-2j} b_0^{1-2j} \left(-b_1^2\right)^j \end{aligned} \tag{5}$$

Assume that (3) is true for $n = 2k$, where k is a positive integer, therefore:

$$\begin{aligned} \operatorname{Re}\left[b^{2k}\right] &= \sum_{j=0}^{k} \binom{2k}{2k-2j} b_0^{2k-2j} \left(-b_1^2\right)^j \\ \operatorname{Im}\left[b^{2k}\right] &= \left(b - b_0\right) \sum_{j=0}^{k-1} \binom{2k}{2k-2j-1} b_0^{2k-2j-1} \left(-b_1^2\right)^j \end{aligned} \tag{6}$$

We can compute $b^{2k+2} = b^2 b^{2k}$ by using the matrix:

$$B^2 = \begin{bmatrix} b_0^2 - b_1^2 & -2b_0 b_1 \\ 2b_0 b_1 & b_0^2 - b_1^2 \end{bmatrix}$$

Multiplying the first row of B^2 by $\left(b^{2k}\right)^{\mathrm{T}}$ (the column matrix representing $\left(b^{2k}\right)$) gives $\operatorname{Re}\left[b^{2k+2}\right]$, and multiplying the second row of B^2 by $\left(b^{2k}\right)^{\mathrm{T}}$ gives $\operatorname{Im}\left[b^{2k+2}\right]$.

$$\begin{aligned} \operatorname{Re}\left[b^{2k+2}\right] &= \left(b_0^2 - b_1^2\right) \sum_{j=0}^{k} \binom{2k}{2k-2j} b_0^{2k-2j} \left(-b_1^2\right)^j \\ &\quad - 2b_0 b_1 \sum_{j=0}^{k-1} \binom{2k}{2k-2j-1} b_0^{2k-2j-1} \left(-b_1^2\right)^j \\ &= \sum_{j=0}^{k} \binom{2k}{2k-2j} b_0^{2k-2j+2} \left(-b_1^2\right)^j + \sum_{j=0}^{k} \binom{2k}{2k-2j} b_0^{2k-2j} \left(-b_1^2\right)^{j+1} \\ &\quad + 2\sum_{j=0}^{k-1} \binom{2k}{2k-2j-1} b_0^{2k-2j} \left(-b_1^2\right)^{j+1} \\ &= b_0^{2k+2} - \binom{2k}{2k-2} b_0^{2k} \left(b_1^2\right) + \binom{2k}{2k-4} b_0^{2k-2} \left(b_1^2\right)^2 + \cdots \\ &\quad + \binom{2k}{2k-2r+2} b_0^{2k-2r+4} \left(-b_1^2\right)^{r-1} + \binom{2k}{k-2r} b_0^{2k-2r+2} \left(-b_1^2\right)^r + \cdots \\ &\quad + \binom{2k}{2} b_0^4 \left(-b_1^2\right)^{k-1} + b_0^2 \left(-b_1^2\right)^k - b_0^{2k} \left(b_1^2\right) + \binom{2k}{2k-2} b_0^{2k-2} \left(b_1^2\right)^2 + \cdots \\ &\quad + \binom{2k}{2k-2r+2} b_0^{2k-2r+2} \left(-b_1^2\right)^r + \binom{2k}{2k-2r} b_0^{2k-2r} \left(-b_1^2\right)^{r+1} + \cdots \\ &\quad + \binom{2k}{2} b_0^2 \left(-b_1^2\right)^k + \left(-b_1^2\right)^{k+1} + 2\left[-\binom{2k}{2k-1} b_0^{2k} \left(b_1^2\right) + \binom{2k}{2k-3} b_0^{2k-2} \left(b_1^2\right)^2\right. \\ &\quad + \cdots + \binom{2k}{2k-2r+1} b_0^{2k-2r+2} \left(-b_1^2\right)^r + \binom{2k}{2k-2r-1} b_0^{2k-2r} \left(-b_1^2\right)^{r+1} + \cdots \end{aligned}$$

$$+\binom{2k}{3}b_0^4\left(-b_1^2\right)^{k-1}+\binom{2k}{1}b_0^2\left(-b_1^2\right)^k\Bigg]$$

$$=b_0^{2k+2}-\left[\binom{2k}{2k-2}+1+2\binom{2k}{2k-1}\right]b_0^{2k}\left(b_1^2\right)$$

$$+\left[\binom{2k}{2k-4}+\binom{2k}{2k-2}+2\binom{2k}{2k-3}\right]b_0^{2k-2}\left(b_1^2\right)^2+\cdots$$

$$+\left[\binom{2k}{2k-2r}+\binom{2k}{k-2r+2}+2\binom{2k}{2k-2r+1}\right]b_0^{2k-2r+2}\left(-b_1^2\right)^r+\cdots$$

$$+\left[\binom{2k}{2}+1+2\binom{2k}{1}\right]b_0^2\left(-b_1^2\right)^k+\left(-b_1^2\right)^{k+1}$$

$$=b_0^{2k+2}-\binom{2k+2}{2k}b_0^{2k}\left(b_1^2\right)+\binom{2k+2}{2k-2}b_0^{2k-2}\left(b_1^2\right)^2+\cdots$$

$$+\binom{2k+2}{2k-2r+2}b_0^{2k-2r+2}\left(-b_1^2\right)^r+\cdots+\binom{2k+2}{2}b_0^2\left(-b_1^2\right)^k+\left(-b_1^2\right)^{k+1}$$

$$=\sum_{j=0}^{k+1}\binom{2k+2}{2k+2-2j}b_0^{2k+2-2j}\left[-b_1^2\right]^j$$

$$\operatorname{Im}\left[b^{2k+2}\right]=i\Bigg[2b_0b_1\sum_{j=0}^{k}\binom{2k}{2k-2j}b_0^{2k-2j}\left(-b^2\right)^j$$

$$+b_1\left(b_0^2-b_1^2\right)\sum_{j=0}^{k-1}\binom{2k}{2k-2j-1}b_0^{2k-2j-1}\left(-b_1^2\right)^j\Bigg]$$

$$=b_1i\Bigg[2\sum_{j=0}^{k}\binom{2k}{2k-2j}b_0^{2k-2j+1}\left(-b_1^2\right)^j+\sum_{j=0}^{k-1}\binom{2k}{2k-2j-1}b_0^{2k-2j+1}\left(-b_1^2\right)^j$$

$$+\sum_{j=0}^{k-1}\binom{2k}{2k-2j-1}b_0^{2k-2j-1}\left(-b_1^2\right)^{j+1}\Bigg]$$

$$=b_1i\Bigg[2b_0^{2k+1}-2\binom{2k}{2k-2}b_0^{2k-1}\left(b_1^2\right)+2\binom{2k}{2k-4}b_0^{2k-3}\left(b_1^2\right)^2+\cdots$$

$$+2\binom{2k}{2k-2r+2}b_0^{2k-2r+3}\left(-b_1^2\right)^{r-1}+2\binom{2k}{2k-2r}b_0^{2k-2r+1}\left(-b_1^2\right)^r+\cdots$$

$$+2\binom{2k}{2}b_0^3\left(-b_1^2\right)^{k-1}+2b_0\left(-b_1^2\right)^k$$

$$+\binom{2k}{2k-1}b_0^{2k+1}-\binom{2k}{2k-3}b_0^{2k-1}\left(b_1^2\right)+\binom{2k}{2k-5}b_0^{2k-3}\left(b_1^2\right)^2+\cdots$$

$$+\binom{2k}{2k-2r+1}b_0^{2k-2r+3}\left(-b_1^2\right)^{r-1}+\binom{2k}{2k-2r-1}b_0^{2k-2r+1}\left(-b_1^2\right)^r+\cdots$$

$$+\binom{2k}{3}b_0^5\left(-b_1^2\right)^{k-2}+\binom{2k}{1}b_0^3\left(-b_1^2\right)^{k-1}$$

$$-\binom{2k}{2k-1}b_0^{2k-1}\left(b_1^2\right)+\binom{2k}{2k-3}b_0^{2k-3}\left(b_1^2\right)^2-\binom{2k}{2k-5}b_0^{2k-5}\left(b_1^2\right)^3+\cdots$$

$$+\binom{2k}{2k-2r+1}b_0^{2k-2r+1}\left(-b_1^2\right)^r+\binom{2k}{2k-2r-1}b_0^{2k-2r-1}\left(-b_1^2\right)^{r+1}+\cdots$$

$$+\binom{2k}{3}b_0^3\left(-b_1^2\right)^{k-1}+\binom{2k}{1}b_0\left(-b_1^2\right)^k\Bigg]$$

$$
\begin{aligned}
&= b_1 i\Bigg[b_0^{2k+1}\left[2+\binom{2k}{2k-1}\right]-b_0^{2k-1}\left(b_1^2\right)\left[2\binom{2k}{2k-2}+\binom{2k}{2k-3}+\binom{2k}{2k-1}\right] \\
&\quad +b_0^{2k-3}\left(b_1^2\right)^2\left[2\binom{2k}{2k-4}+\binom{2k}{2k-5}+\binom{2k}{2k-3}\right]+\cdots \\
&\quad +b_0^{2k-2r+1}\left(-b_1^2\right)^r\left[2\binom{2k}{2k-2r}+\binom{2k}{2k-2r-1}+\binom{2k}{2k-2r+1}\right]+\cdots \\
&\quad +b_0^3\left(-b_1^2\right)^{k-1}\left[2\binom{2k}{2}+\binom{2k}{1}+\binom{2k}{3}\right]+b_0\left(-b_1^2\right)^k\left[2+\binom{2k}{1}\right]\Bigg] \\
&= b_1 i\Bigg[b_0^{2k+1}\binom{2k+2}{2k+1}-b_0^{2k-1}\left(b_1^2\right)\binom{2k+2}{2k-1}+b_0^{2k-3}\left(b_1^2\right)^2\binom{2k+2}{2k-3}+\cdots \\
&\quad +b_0^{2k-2r+1}\left(-b_1^2\right)^r\binom{2k+2}{2k-2r+1}+\cdots \\
&\quad +b_0^3\left(-b_1^2\right)^{k-1}\binom{2k+2}{3}+b_0\left(-b_1^2\right)^k\binom{2k+2}{1}\Bigg] \\
&= b_1 i\sum_{j=0}^{k}\binom{2k+2}{2k-2j+1}b_0^{2k-2j+1}\left(-b_1^2\right)^j
\end{aligned}
$$

By the similar way, we can prove (4).

3. Results

We can use (3) and (4) to compute the powers of a quaternion number and the powers of an octonion number.

To compute the power of a, replace b_0^2 by a_0^2 and b_1^2 by $a_1^2+a_2^2+\cdots+a_7^2$ in (3) and (4),

$$
\begin{aligned}
\operatorname{Re}\left[a^n\right] &= \sum_{j=0}^{n/2}\binom{n}{n-2j}a_0^{n-2j}\left[-\left(a_1^2+\cdots+a_7^2\right)\right]^j \\
\operatorname{Im}\left[a^n\right] &= \left(a-a_0\right)\sum_{j=0}^{n/2-1}\binom{n}{n-2j-1}a_0^{n-2j-1}\left[-\left(a_1^2+\cdots+a_7^2\right)\right]^j
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\operatorname{Re}\left[a^n\right] &= \sum_{j=0}^{(n-1)/2}\binom{n}{n-2j}a_0^{n-2j}\left[-\left(a_1^2+\cdots+a_7^2\right)\right]^j \\
\operatorname{Im}\left[a^n\right] &= \left(a-a_0\right)\sum_{j=0}^{(n-1)/2}\binom{n}{n-2j-1}a_0^{n-2j-1}\left[-\left(a_1^2+\cdots+a_7^2\right)\right]^j
\end{aligned} \tag{8}
$$

(7) and (8) give $\left(a_0+a_1i_1+a_2i_2+\cdots+a_7i_7\right)^n$ when n is an even number and n is an odd number respectively.

Of course the proof is similar to that one we used to prove the powers formulas for a complex number, but to make it clearer, we will write A as:

$$A=\omega(a)+\upsilon(a) \tag{9}$$

where

$$
\omega(a)=\begin{bmatrix}
a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\
a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_2 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 \\
a_3 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 \\
a_4 & 0 & 0 & 0 & a_0 & 0 & 0 & 0 \\
a_5 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 \\
a_6 & 0 & 0 & 0 & 0 & 0 & a_0 & 0 \\
a_7 & 0 & 0 & 0 & 0 & 0 & 0 & a_0
\end{bmatrix},
$$

$$\upsilon(a)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ 0 & a_3 & 0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ 0 & -a_2 & a_1 & 0 & -a_7 & a_6 & -a_5 & a_4 \\ 0 & a_5 & a_6 & a_7 & 0 & -a_1 & -a_2 & -a_3 \\ 0 & -a_4 & a_7 & -a_6 & a_1 & 0 & a_3 & -a_2 \\ 0 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & 0 & a_1 \\ 0 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & 0 \end{bmatrix}$$

Since $\omega(a)$ will play an important role in our proof, we call it the fundamental matrix. The following proposition presents the main properties of the fundamental matrix $\omega(a)$.

3.1. Proposition

Let $a^{\mathrm{T}}=[a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7]^{\mathrm{T}}$, $\bar{a}$ be the conjugate of a, λ be a real number, and I_8 be the identity matrix. Then

(a) $A^2=\omega(a)a^{\mathrm{T}}$
(b) $\omega(a^n)=\omega^n(a)$
(c) $\omega(\bar{a})=\omega^{\mathrm{T}}(a)$
(d) $\omega(\lambda a)=\lambda\omega(a)$
(e) $\omega(1)=I_8$

Proof:

(a) $a^2=[\omega(a)+\upsilon(a)]a^{\mathrm{T}}=\omega(a)a^{\mathrm{T}}$, (since $\upsilon(a)a^{\mathrm{T}}=0$).
In general $a^{n+1}=aa^n$, $a^{n+1}=\omega(a)(a^n)^{\mathrm{T}}$, ($n$ is a real number).
(c) $a\bar{a}=\omega(a)\bar{a}^{\mathrm{T}}=a_0^2+a_1^2+\cdots+a_7^2=\|a\|^2$
The verification of remaining propositions is straightforward.
Form proposition (a), a^2 can be computed from:

$$\omega(a)a^{\mathrm{T}}=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & a_0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & a_0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

$$=\begin{bmatrix} a_0^2-(a_1^2+\cdots+a_7^2) \\ 2a_0a_1 \\ 2a_0a_2 \\ 2a_0a_3 \\ 2a_0a_4 \\ 2a_0a_5 \\ 2a_0a_6 \\ 2a_0a_7 \end{bmatrix}$$

$$a^2 = a_0^2 - \left(a_1^2 + \cdots + a_7^2\right) + 2a_0\left(a_1 i_1 + \cdots + a_7 i_7\right)$$

So, multiplying the first row of $\omega(a)$ by $(a)^{\mathrm{T}}$ gives $\mathrm{Re}\left[a^2\right]$, and to obtain $\mathrm{Im}\left[a^2\right]$ we just multiply any row (except the first row) of $\omega(a)$ by a^{T} because the multiplication result will be similar for all other rows, therefore $\mathrm{Im}\left[a^2\right]$ will be summation of these results.

In general, since $a^{2k+2} = a^2 a^{2k}$, we will obtain $\mathrm{Re}\left[a^{2k+2}\right]$ by multiply the first row of $\omega\left(a^2\right)$ by $\left(a^{2k}\right)^{\mathrm{T}}$, and to obtain $\mathrm{Im}\left[a^{2k+2}\right]$ we multiply any row (except the first row) of $\omega\left(a^2\right)$ by $\left(a^{2k}\right)^{\mathrm{T}}$.

In case $\mathrm{Re}[a] = 0$ (a is a pure octonion) and $\mathrm{Im}(a) = a_1 i_1 + a_2 i_2 + a_3 i_3 + a_4 i_4 + a_5 i_5 + a_6 i_6 + a_7 i_7 \neq 0$, if n is an even number then:

$$\begin{aligned} \mathrm{Re}\left[a^n\right] &= \left[-\left(a_1^2 + \cdots + a_7^2\right)\right]^{n/2} \\ \mathrm{Im}\left[a^n\right] &= 0 \end{aligned} \tag{10}$$

And if n is an odd number then:

$$\begin{aligned} \mathrm{Re}\left[a^n\right] &= 0 \\ \mathrm{Im}\left[a^n\right] &= a\left[-\left(a_1^2 + \cdots + a_7^2\right)\right]^{(n-1)/2} \end{aligned} \tag{11}$$

3.2. Example

Let $a = i_2 + 2i_3 - 3i_6 + i_7$

$$A = \begin{bmatrix} 0 & 0 & -1 & -2 & 0 & 0 & 3 & -1 \\ 0 & 0 & -2 & 1 & 0 & 0 & 1 & 3 \\ 1 & 2 & 0 & 0 & 3 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & -1 & -3 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 & -1 \\ -3 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & -3 & 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix}$$

From (10),

$$\mathrm{Re}\left[a^6\right] = \left[-(15)\right]^3 = -3375$$

$$\mathrm{Im}\left[a^6\right] = 0$$

So, $a^6 = -3375$

$$A^6 = \begin{bmatrix} -3375 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3375 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3375 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3375 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3375 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3375 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3375 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3375 \end{bmatrix}$$

From (11),

$$\mathrm{Re}\left[a^5\right]=0$$

$$\mathrm{Im}\left[a^5\right]=\left(i_2+2i_3-3i_6+i_7\right)\left[-(15)\right]^2=225\left(i_2+2i_3-3i_6+i_7\right)$$

$$A^5=\begin{bmatrix} 0 & 0 & -225 & -450 & 0 & 0 & 675 & -225 \\ 0 & 0 & -450 & 225 & 0 & 0 & 225 & 675 \\ 225 & 450 & 0 & 0 & 675 & -225 & 0 & 0 \\ 450 & -225 & 0 & 0 & -225 & -675 & 0 & 0 \\ 0 & 0 & -675 & 225 & 0 & 0 & -225 & -450 \\ 0 & 0 & 225 & 675 & 0 & 0 & 450 & -225 \\ -675 & -225 & 0 & 0 & 225 & -450 & 0 & 0 \\ 225 & -675 & 0 & 0 & 450 & 225 & 0 & 0 \end{bmatrix}$$

3.3. Example

To compare our formulas with the De Moivre's formula and Euler's formula that were used in [7] to find a^7 for $a=1+i_2+i_4+i_6$, take

$$A=\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

From (8),

$$\mathrm{Re}\left[a^7\right]=\sum_{j=0}^{3}\binom{7}{7-2j}\left[-(3)\right]^j=1-\binom{7}{5}(3)+\binom{7}{3}(9)-\binom{7}{1}(27)=64$$

$$\mathrm{Im}\left[a^7\right]=\left(i_2+i_4+i_6\right)\sum_{j=0}^{3}\binom{7}{6-2j}\left[-(3)\right]^j$$

$$=\left(i_2+i_4+i_6\right)\left[\binom{7}{6}-\binom{7}{4}(3)+\binom{7}{2}(9)-(27)\right]$$

$$=64\left(i_2+i_4+i_6\right)$$

So, $a^7=64\left(1+i_2+i_4+i_6\right)$, and

$$A^7=\begin{bmatrix} 64 & 0 & -64 & 0 & -64 & 0 & -64 & 0 \\ 0 & 64 & 0 & 64 & 0 & 64 & 0 & -64 \\ 64 & 0 & 64 & 0 & -64 & 0 & 64 & 0 \\ 0 & -64 & 0 & 64 & 0 & 64 & 0 & 64 \\ 64 & 0 & 64 & 0 & 64 & 0 & -64 & 0 \\ 0 & -64 & 0 & -64 & 0 & 64 & 0 & -64 \\ 64 & 0 & -64 & 0 & 64 & 0 & 64 & 0 \\ 0 & 64 & 0 & -64 & 0 & 64 & 0 & 64 \end{bmatrix}$$

Also let us compute a^8.

From (7),

$$\operatorname{Re}\left[a^8\right]=\sum_{j=0}^{4}\binom{8}{8-2j}\left[-(3)\right]^j=1-\binom{8}{6}(3)+\binom{8}{4}(9)-\binom{8}{2}(27)+(81)=-128$$

$$\begin{aligned}\operatorname{Im}\left[a^8\right]&=\left(i_2+i_4+i_6\right)\sum_{j=0}^{3}\binom{8}{7-2j}\left[-(3)\right]^j\\&=\left(i_2+i_4+i_6\right)\left[8-\binom{8}{5}(3)+\binom{8}{3}(9)-(8)(27)\right]\\&=128\left(i_2+i_4+i_6\right)\end{aligned}$$

So, $a^8=128\left(-1+i_2+i_4+i_6\right)$, and

$$A^8=\begin{bmatrix}-128 & 0 & -128 & 0 & -128 & 0 & -128 & 0\\ 0 & -128 & 0 & 128 & 0 & 128 & 0 & -128\\ 128 & 0 & -128 & 0 & -128 & 0 & 128 & 0\\ 0 & -128 & 0 & -128 & 0 & 128 & 0 & 128\\ 128 & 0 & 128 & 0 & -128 & 0 & -128 & 0\\ 0 & -128 & 0 & -128 & 0 & -128 & 0 & -128\\ 128 & 0 & -128 & 0 & 128 & 0 & -128 & 0\\ 0 & 128 & 0 & -128 & 0 & 128 & 0 & -128\end{bmatrix}$$

4. Conclusion

The formulas presented in this work are more suitable for computing the powers of an octonion number (the powers of matrices representing an octonion number). These formulas which are derived from binomial expansion also can be used to compute the power of a quaternion number (the powers of matrices representing a quaternion number), and a complex number (the powers of matrices representing a complex number).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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