Hindawi
Abstract and Applied Analysis
Volume 2023, Article ID 1749302, 7 pages
https://doi.org/10.1155/2023/1749302

Research Article

On the $\mathscr{E}$-Hyperstability of the Inhomogeneous σ-Jensen's Functional Equation on Semigroups

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Received 16 September 2022; Revised 29 May 2023; Accepted 5 June 2023; Published 17 July 2023

Academic Editor: Victor Kovtunenko

In this paper, we study the hyperstability problem for the well-known σ-Jensen's functional equation $f(xy) + f(x\sigma(y)) = 2f(x)$ for all $x, y \in S$, where S is a semigroup and σ is an involution of S. We present sufficient conditions on $\mathscr{E} \subset (\mathbb{R}_+)^{S^2}$ so that the inhomogeneous form of σ-Jensen's functional equation $f(xy) + f(x\sigma(y)) = 2f(x) + \varphi(x, y)$ for all $x, y \in S$, where the inhomogeneity φ is given, can be $\mathscr{E}$-hyperstable on S.

1. Introduction

Throughout this paper, we will denote by $\mathbb{N} \coloneqq \{1, 2, \cdots\}$ the set of positive integers, $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$. We let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ \coloneqq [0,\infty)$ be the set of nonnegative real numbers, and $\mathbb{C}$ be the set of complex numbers. We write $\mathscr{B}^{\mathscr{A}}$ to mean "the family of all functions mapping from a nonempty set $\mathscr{A}$ into a nonempty set $\mathscr{B}$".

The stability problem of functional equations originated from a question of Ulam [1], posed in 1940 before the Mathematics Club of the University of Wisconsin in which he suggested the following stability problem, well-known as the Ulam stability problem:

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given a real number $\varepsilon > 0$, does there exist a real number $\delta > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(x \cdot y), f(x) * f(y)) \leq \delta$ for all $x, y \in G_1$, then there is a homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) \leq \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, then we call that the equation $f(x \cdot y) = f(x) * f(y)$ is stable in the sense of Ulam.

In 1941, Hyers [2] gave the first affirmative answer to Ulam's stability problem for the Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. Găvruja [5] provided a further generalization of the Rassias theorem by using a general control function. During the last decades, the stability problems of several functional equations have been investigated by a number of authors (see [6–8]).

In 2001, Maksa and Páles [9] studied the stability problem of a class of linear functional equation

$$\frac{1}{n}\sum_{i=1}^{n} f(x\varphi_i(y)) = f(x) + f(y), \tag{1}$$

for all $x, y \in S$, where f is a real-valued mapping defined on an arbitrary semigroup $(S, \cdot)$ and where $\varphi_1, \cdots, \varphi_n : S \longrightarrow S, n \geq 2$ are pairwise distinct automorphisms of S. More precisely, they proved that if the error bound between the two sides of equation (1) satisfies a specific asymptotic property, then the two sides must be equal. This occurrence is referred to as hyperstability of the functional equation (1) on S. The terms hyperstability and stability were coined in these historical contexts and have since been applied to various functional equations. To learn more about this, one can refer to [10–17]. Moreover, Brzdęk and Ciepliński introduced a definition in their paper [14] that describes the main ideas of hyperstability for functional equations in several variables.

Definition 1 (see [14], Definition 7). Let $n \in \mathbb{N}$, X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in (\mathbb{R}_+)^{X^n}$, and $\mathcal{F}_1$ and $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the functional equation

$$\mathcal{F}_1\varphi(x_1, \cdots, x_n) = \mathcal{F}_2\varphi(x_1, \cdots, x_n), \tag{2}$$

for all $x_1, \cdots, x_n \in X$ is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d(\mathcal{F}_1\varphi_0(x_1, \cdots, x_n), \mathcal{F}_2\varphi_0(x_1, \cdots, x_n)) \leq \varepsilon(x_1, \cdots, x_n), \tag{3}$$

for all $x_1, \cdots, x_n \in X$ fulfills equation (2).

Note that in Definition 1, if we consider $\mathcal{E}$ the family of all functions $\varepsilon \in (\mathbb{R}_+)^{X^n}$ which yields the ε-hyperstability of (2), we can formulate the following definition of what we call $\mathcal{E}$-hyperstability of functional equations (see [17]).

Definition 2. Let $n \in \mathbb{N}$, X be a nonempty set, $P \subset X^n$ be nonempty, (Y, d) be a metric space, $\mathcal{E} \subset (\mathbb{R}_+)^P$ be a nonempty subset, $\mathcal{F}_1$ and $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^P, and $\mathcal{L} \subset \mathcal{D}$ be nonempty. Suppose that the conditional functional equation

$$\mathcal{F}_1\varphi(x_1, \cdots, x_n) = \mathcal{F}_2\varphi(x_1, \cdots, x_n), \tag{4}$$

for all $(x_1, \cdots, x_n) \in P$ admits a solution $f_0 \in \mathcal{L}$. Then, we say that the conditional equation (4) is $\mathcal{E}$-hyperstable in $\mathcal{L}$ provided for any $\varepsilon \in \mathcal{E}$, if for each function $\varphi_0 \in \mathcal{L}$ satisfying the inequality

$$d(\mathcal{F}_1\varphi_0(x_1, \cdots, x_n), \mathcal{F}_2\varphi_0(x_1, \cdots, x_n)) \leq \varepsilon(x_1, \cdots, x_n), \tag{5}$$

for all $(x_1, \cdots, x_n) \in P$, then φ_0 is the solution of (4).

If $\mathcal{L} = \mathcal{D}$, then we omit the part "in $\mathcal{L}$" and simply say "$\mathcal{E}$-hyperstable."

Example 1. One of the most classical results concerning the hyperstability problem of the additive Cauchy equation is as follows:

$$f(x + y) = f(x) + f(y). \tag{6}$$

Let E_1 and E_2 be normed spaces and $c \geq 0$ and p be fixed real numbers. Assume also that $f : E_1 \longrightarrow E_2$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq c\left(\|x\|^p + \|y\|^p\right), \tag{7}$$

for all $x, y \in E_1 \setminus \{0\}$. If $p < 0$, then f is additive [11]. That is, the functional equation (6) is ε-hyperstable on $E_1 \setminus \{0\}$, where $\varepsilon(x, y) \coloneqq c(\|x\|^p + \|y\|^p)$.

By taking $\mathcal{E}$, the set of all functions $\varepsilon : (E_1 \setminus \{0\})^2 \longrightarrow \mathbb{R}_+$ such that

$$\varepsilon(x, y) \coloneqq c\left(\|x\|^p + \|y\|^p\right), \tag{8}$$

for all $x, y \in E_1 \setminus \{0\}$, for some fixed real numbers $c \geq 0$ and $p < 0$; then, from Definition 2, equation (6) is $\mathcal{E}$-hyperstable on $E_1 \setminus \{0\}$.

A common and significant variation of the additive Cauchy functional equation (6) is known as Jensen's functional equation. This equation is particularly notable for its simplicity and importance. In the case of real numbers, Jensen's functional equation can be expressed as follows:

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}, \tag{9}$$

for all $x, y \in \mathbb{R}$. On a multiplicative group $G = (G, \cdot)$, Ng has proposed two extensions to Jensen's functional equation. The first extension is given by the equation

$$f(x \cdot y) + f\left(x \cdot y^{-1}\right) = 2f(x), \tag{10}$$

for all $x, y \in G$, where f is a function from the group G with neutral element e into an abelian additive group [18–20]. The second extension is expressed as follows:

$$f(x \cdot y) + f\left(y^{-1} \cdot x\right) = 2f(x), \tag{11}$$

for all $x, y \in G$. Every solution of the Jensen functional equation is referred to as a Jensen function. It is easy to observe that equation (10) under the normalization condition $f(e) = 0$ is equivalent to the additive Cauchy functional equation (6) on G (see [19, 21–23]).

Combining the concept of Jensen's function defined on semigroups introduced by H. Stetkær in [24] and the research conducted by M. Almahalebi on the hyperstability of the σ-Drygas equation [25], which is defined as

$$f(x \cdot y) + f(x \cdot \sigma(y)) = 2f(x) + \mathrm{f}(y) + 2f(\sigma(y)), \tag{12}$$

for all $x, y \in S$, where $(S, \cdot)$ denotes a semigroup, and acknowledging that equation (10) cannot be applied to a semigroup S due to the general absence of y^{-1}, we can introduce the following definition.

Definition 3. Let $(S, \cdot)$ be a semigroup and let σ be an involution on S satisfying $\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. The σ-Jensen functional equation on the semigroup S refers to the functional equation of the form

$$f(x \cdot y) + f(x \cdot \sigma(y)) = 2f(x), \tag{13}$$

for all $x, y \in S$, where f denotes functions mapping from S to an abelian group.

The functional equation (13) takes the form

$$f(x\cdot y)+f\left(x\cdot y^{-1}\right)=2f(x), \tag{14}$$

for all $x, y \in S$ when $\sigma(x)=x^{-1}$ and S is a group. Sinopoulos [26] determined that the general solution of the functional equation (13) on commutative semigroups is represented by a summation of the additive Cauchy function and a constant. An inhomogeneous form of σ-Jensen's functional equation can be written

$$f(x\cdot y)+f(x\cdot\sigma(y))=2f(x)+\varphi(x,y), \tag{15}$$

for all $x, y \in S$, where S is a semigroup, σ is an involution of S, and φ are given.

Initially studied by Kominek [27], the stability problem of Jensen's functional equation has been explored by several mathematicians, including Jung [28], Faziev and Sahoo [29], Mihet [30], Ciepliński [31], and Almahalebi et al. [32]. Various authors have studied the stability of the σ-Jensen functional equation (13) on abelian groups or vector spaces [33–35]. In their paper [36], a novel stability approach for the Pexider functional equation with involution σ in a normed space E was introduced by Bouikhalene et al., which is given by

$$f(x+y)+g(x+\sigma(y))=h(x)+l(x), \tag{16}$$

for all $x, y \in E$.

The hyperstability question of Jensen's functional equation (10) on abelian groups or vector spaces has been studied by various authors. Bahyrycz and Piszczek in [37] studied the hyperstability of Jensen's functional equation of the form

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}, \tag{17}$$

in the class of functions from a nonempty subset of a normed space into a normed space and by Piszczek in [38] and by Bahyrycz and Olko in [39].

In [40], Brzdęk et al. established the stability of a general functional equation given by

$$Af(px*ry)+Bf(qx*sy)=Cf(x)+Df(y), \tag{18}$$

for all $x, y \in G$ which covers various specific cases including the additive Cauchy equation (6), the Jensen equation (10), the quadratic equation, and equation (13). The functional equation is defined for functions that map a groupoid $(X,*)$ into a Banach space Y, where the parameters p, q, r, and s are endomorphisms of the groupoid X and A, B, C, and D are fixed scalars.

El-Fassi and Brzdęk in [41] presented and solved a functional equation of the form

$$f(x\cdot y)+f(x\cdot\sigma(y))=2f(x)+h(x), \tag{19}$$

for all $x, y \in G$, where f and h are functions defined on a semigroup $(G,\cdot)$ and take values in a commutative semigroup $(E,+)$. In this equation, σ is an endomorphism of G such that $\sigma(\sigma(x))=x$ for all $x\in G$. Equation (13) is a special case of (19). However, the authors focused their discussion on two specific cases and did not provide a general analysis of the hyperstability problem for the equation. Specifically, they studied the following two cases:

$$f(x\cdot y)+f(x\cdot\sigma(y))=2f(x)+2f(y), \tag{20}$$

for all $x, y \in G$ and

$$f(x\cdot y)+f(x\cdot\sigma(y))+2f(e)=2f(x)+f(y)+f(\sigma(y)), \tag{21}$$

for all $x, y \in G$ without addressing the issue of hyperstability of equation (13) in a general context.

In their recent work, El Ghali and Kabbaj investigated the question of hyperstability in the context of non-Archimedean 2-Banach spaces, focusing specifically on σ-Jensen's equation (13). The details of their study can be found in their paper [42].

This paper establishes the $\mathscr{E}$-hyperstability of the σ-Jensen's functional equation (13) under certain asymptotic properties of the control function $\varepsilon\in\mathscr{E}$. Particularly, the inhomogeneous forms of this equation are proven to be hyperstable in the class of functions mapping a semigroup into a normed space.

2. Auxiliary Results

Before proceeding to the main results, we will state the following theorem (Theorem 4), which is relevant to our objective and can be regarded as a specific case of Theorem 2.2 in [13].

In the following, let $n\in\mathbb{N}$, X be a nonempty set, $P\subset X^n$ be nonempty, $(Y,+)$ be a group with a translation invariant metric d (i.e., $d(a+c,b+c)=d(a,b)$ for all a, b, and c in Y), and $(Y^X,+)$ be a group, where $+$ is a binary operation in Y^X (as usual, $(f+g)(x)=f(x)+g(x)$, and $(-f)(x)=-f(x)$ for $f,g\in Y^X$, $x\in X$). We say that a function $\mathscr{F}$ from a subgroup $(\mathscr{D},+)$ of the group $(Y^X,+)$ into Y^P is additive if

$$\mathscr{F}(f+g)(x_1,\cdots,x_n)=\mathscr{F}f(x_1,\cdots,x_n)+\mathscr{F}g(x_1,\cdots,x_n), \tag{22}$$

for all $f,g\in\mathscr{D}$, $(x_1,\cdots,x_n)\in P$.

Theorem 4. *Let $n\in\mathbb{N}$, X be a nonempty set, $P\subset X^n$ be nonempty, and the triple $(Y,+,d)$ be a translation invariant metric group. Consider the nonempty family $\mathscr{E}$ of functions from P to $\mathbb{R}_+$. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two additive functions from a subgroup $\mathscr{D}$ of the group $(Y^X,+)$ into Y^P and $\mathscr{U}$ be a subgroup of the group $(\mathscr{D},+)$, and $\mu: P\longrightarrow Y$. Suppose that the equation*

$$\mathscr{F}_1\varphi_0(x_1,\cdots,x_n)=\mathscr{F}_2\varphi_0(x_1,\cdots,x_n)+\mu(x_1,\cdots,x_n), \tag{23}$$

for all $(x_1,\cdots,x_n)\in P$ admits a solution $f_0\in\mathscr{U}$. Then, the equation

$$\mathcal{F}_1\varphi_0(x_1,\cdots,x_n)=\mathcal{F}_2\varphi_0(x_1,\cdots,x_n), \tag{24}$$

for all $(x_1,\cdots,x_n)\in P$ *is* $\mathcal{E}$*-hyperstable in* $\mathcal{U}$ *if and only if (23) is* $\mathcal{E}$*-hyperstable.*

Proof. Assume that equation (23) is $\mathcal{E}$-hyperstable in $\mathcal{U}$. Let $\varepsilon\in\mathcal{E}$ and let $f\in\mathcal{U}$ satisfy the inequality

$$d(\mathcal{F}_1 f(x_1,\cdots,x_n),\mathcal{F}_2 f(x_1,\cdots,x_n))\le\varepsilon(x_1,\cdots,x_n), \tag{25}$$

for all $(x_1,\cdots,x_n)\in P$. Write $g:=f+f_0$. Then, $g\in\mathcal{U}$ and

$$\begin{aligned}&d(\mathcal{F}_1 g(x_1,\cdots,x_n),\mathcal{F}_2 g(x_1,\cdots,x_n)+\mu(x_1,\cdots,x_n))\\&\quad=d(\mathcal{F}_1(f+f_0)(x_1,\cdots,x_n),\mathcal{F}_2(f+f_0)(x_1,\cdots,x_n)\\&\qquad+\mu(x_1,\cdots,x_n))\le d(\mathcal{F}_1 f(x_1,\cdots,x_n),\mathcal{F}_2 f(x_1,\cdots,x_n))\\&\qquad+d(\mathcal{F}_1 f_0(x_1,\cdots,x_n),\mathcal{F}_2 f_0(x_1,\cdots,x_n)+\mu(x_1,\cdots,x_n))\\&\quad\le d(\mathcal{F}_1 f(x_1,\cdots,x_n),\mathcal{F}_2 f(x_1,\cdots,x_n))\le\varepsilon(x_1,\cdots,x_n),\end{aligned} \tag{26}$$

for all $(x_1,\cdots,x_n)\in P$. Since (23) is $\mathcal{E}$-hyperstable in $\mathcal{U}$, g is a solution of (23). That is,

$$\mathcal{F}_1 g(x_1,\cdots,x_n)=\mathcal{F}_2 g(x_1,\cdots,x_n)+\mu(x_1,\cdots,x_n), \tag{27}$$

for all $(x_1,\cdots,x_n)\in P$. Moreover,

$$\begin{aligned}&d(\mathcal{F}_1 f(x_1,\cdots,x_n),\mathcal{F}_2 f(x_1,\cdots,x_n))\\&\quad=d(\mathcal{F}_1(g-f_0)(x_1,\cdots,x_n),\mathcal{F}_2(g-f_0)(x_1,\cdots,x_n))\\&\quad=d(\mathcal{F}_1 g(x_1,\cdots,x_n)-\mathcal{F}_1 f_0(x_1,\cdots,x_n),\mathcal{F}_2 g(x_1,\cdots,x_n)\\&\qquad-\mathcal{F}_2 f_0(x_1,\cdots,x_n))=d(\mathcal{F}_1 g(x_1,\cdots,x_n)\\&\qquad-\mathcal{F}_1 f_0(x_1,\cdots,x_n),\mathcal{F}_2 g(x_1,\cdots,x_n)-\mathcal{F}_1 f_0(x_1,\cdots,x_n)\\&\qquad+\mu(x_1,\cdots,x_n))=d(\mathcal{F}_1 g(x_1,\cdots,x_n),\mathcal{F}_2 g(x_1,\cdots,x_n)\\&\qquad+\mu(x_1,\cdots,x_n))=0,\end{aligned} \tag{28}$$

for all $(x_1,\cdots,x_n)\in P$. Evidently, f is a solution to (24). Therefore, (24) is $\mathcal{E}$-hyperstable in $\mathcal{U}$.

The converse implication is analogous. □

3. Hyperstability of σ-Jensen's Equation

In this section, let $S:=(S,\cdot)$ be a semigroup and $(E,\|.\|)$ be an arbitrary normed vector space over $\mathbb{K}$ ($\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$).

A motivating idea used by Maksa and Páles in [9] was the basis of the proof method for the main results. This method relies on a lemma that establishes an identity for the two variable functions obtained by taking the difference of the left- and right-hand sides of (13).

Lemma 5. *Let E be a normed vector space over $\mathbb{K}$ and $(S,\cdot)$ be a semigroup and let σ be an involution of S. Let $f:S\longrightarrow E$ be an arbitrary function. Then, the function $D_f:S\times S\longrightarrow E$ defined by*

$$D_f(x,y)=2f(x)-f(x\cdot y)-f(x\cdot\sigma(y)), \tag{29}$$

for all $x,y\in S$ satisfies the following functional equation:

$$\begin{aligned}&2D_f(x,y)+D_f(x\cdot y,z)+D_f(x\cdot\sigma(y),z)\\&\quad=D_f(x,y\cdot z)+D_f(x,y\cdot\sigma(z)),\end{aligned} \tag{30}$$

for all $x,y,z\in S$.

Proof. Let $f:S\longrightarrow E$ be an arbitrary function and let $D_f:S\times S\longrightarrow E$ be given by (29). Evaluating the left- and the right-hand side of (30), we get

$$\begin{aligned}&2D_f(x,y)+D_f(x\cdot y,z)+D_f(x\cdot\sigma(y),z)\\&\quad=4f(x)-f(x\cdot y\cdot z)-f(x\cdot y\cdot\sigma(z))\\&\qquad-f(x\cdot\sigma(y)\cdot z)-f(x\cdot\sigma(y\cdot z)),\end{aligned} \tag{31}$$

for all $x,y,z\in S$, and

$$\begin{aligned}&D_f(x,y\cdot z)+D_f(x,y\cdot\sigma(z))=4f(x)-f(x\cdot y\cdot z)\\&\quad-f(x\cdot y\cdot\sigma(z))-f(x\cdot\sigma(y)\cdot z)-f(x\cdot\sigma(y\cdot z)),\end{aligned} \tag{32}$$

for all $x,y,z\in S$. Thus, (30) is valid. □

The following theorem presents an $\mathcal{E}$-hyperstability result for equation (13). Namely, we show that, under some asymptotic properties of control functions $\varepsilon\in\mathcal{E}$, the functional equation (13) is $\mathcal{E}$-hyperstable in the class of functions f from an arbitrary semigroup S into a normed vector space E.

Theorem 6. *Let E be a normed vector space, S be a semigroup, and σ be an involution of S. Let $\mathcal{E}$ be a nonempty family of all functions ε whose domain is contained in S^2 and range is contained in $\mathbb{R}_+$ such that there exists a sequence $(s_n)_n$ of elements of S satisfying conditions*

$$\begin{cases}\lim\limits_{n\longrightarrow\infty}\varepsilon(x,y\cdot s_n)=0,\\ \lim\limits_{n\longrightarrow\infty}\varepsilon(x,y\cdot\sigma(s_n))=0,\end{cases} \tag{33}$$

for all $x,y\in S$. Then, equation (13) is $\mathcal{E}$-hyperstable.

Proof. Let $\varepsilon\in\mathcal{E}$. Assume that $f:S\longrightarrow E$ satisfies the inequatilty

$$\|f(x\cdot y)+f(x\cdot\sigma(y))-2f(x)\|\le\varepsilon(x,y), \tag{34}$$

for all $x,y\in S$. Let $D_f:S\times S\longrightarrow E$ be the function given by (29). Then, (34) becomes

$$\|D_f(x,y)\|\le\varepsilon(x,y), \tag{35}$$

for all $x,y\in S$. Using Lemma 5, then D_f satisfies the functional equation

$$2D_f(x,y)+D_f(x\cdot y,z)+D_f(x\cdot\sigma(y),z)\\=D_f(x,y\cdot z)+D_f(x,y\cdot\sigma(z)),\tag{36}$$

for all $x,y,z\in S$. □

Suppose that there exists a sequence $(s_n)_n$ of elements of S satisfying conditions (33). Then, by replacing y with $y\cdot s_n$ in (34), we get

$$\left\|D_f(x,y\cdot s_n)\right\|\le\varepsilon(x,y\cdot s_n),\tag{37}$$

for all $x,y\in S$ and all $n\in\mathbb{N}$. Thus, by (33), we have

$$\lim_{n\longrightarrow\infty}D_f(x,y\cdot s_n)=0,\tag{38}$$

for all $x,y\in S$. Replacing y with $y\cdot\sigma(s_n)$ in (34), we get

$$\left\|D_f(x,y\cdot\sigma(s_n))\right\|\le\varepsilon(x,y\cdot s_n),\tag{39}$$

for all $x,y\in S$ and all $n\in\mathbb{N}$. Thus, by (33), we have

$$\lim_{n\longrightarrow\infty}D_f(x,y\cdot\sigma(s_n))=0,\tag{40}$$

for all $x,y\in S$. Let $x,y,t\in S$ be fixed. By replacing z by $t\cdot s_n$ in (36), we get

$$2D_f(x,y)+D_f(x\cdot y,t\cdot s_n)+D_f(x\cdot\sigma(y),t\cdot s_n)\\=D_f(x,y\cdot t\cdot s_n)+D_f(x,y\cdot\sigma(t\cdot s_n)).\tag{41}$$

Taking the limit as $n\longrightarrow\infty$ and by (37), we deduce from (38) and (40) that

$$D_f(x,y)=0,\quad x,y\in S,\tag{42}$$

that is, f is a solution of (13). Therefore, equation (13) is $\mathscr{E}$-hyperstable.

Corollary 7. *Let $\varepsilon:S\times S\longrightarrow\mathbb{R}_+$ be a function such that there exists $u\in S$, $0\le r<1$, and $0\le q<1$ such that*

$$\begin{cases}\varepsilon(x,u\cdot y)\le r\varepsilon(x,y),\\ \varepsilon(x,\sigma(u)\cdot y)\le q\varepsilon(x,y),\end{cases}\tag{43}$$

for all $x,y\in S$. Assume that $f:S\longrightarrow E$ satisfies the inequality

$$\|f(x\cdot y)+f(x\cdot\sigma(y))-2f(x)\|\le\varepsilon(x,y),\quad x,y\in S.\tag{44}$$

Then, f is a solution of (13).

Proof. By induction, it is easy to show that

$$\begin{cases}\varepsilon\left(x,u^k\cdot y\right)\le r^k\varepsilon(x,y),\\ \varepsilon\left(x,\sigma\left(u^k\right)\cdot y\right)\le q^k\varepsilon(x,y),\end{cases}\tag{45}$$

for all $x,y\in S$ and all $k\in\mathbb{N}_0$. Therefore, the asymptotic properties (33) of the control function ε are satisfied with the sequence $s_k=u^k$, and the statement follows from Theorem 6. □

From the above results (Theorem 4 and Theorem 6), we can derive a $\mathscr{E}$-hyperstability result for the inhomogeneous form of σ-Jensen's functional equation

$$f(x\cdot y)+f(x\cdot\sigma(y))=2f(x)+\varphi(x,y),\quad x,y\in S.\tag{46}$$

Theorem 8. *Let E be a real vector space, S be a semigroup, and σ be an involution of S and an inhomogeneity $\varphi:S\times S\longrightarrow E$. Let $\mathscr{E}$ be a nonempty family of all functions ε whose domain is contained in S^2, and range is contained in $\mathbb{R}_+$ such that there exists a sequence $(s_n)_n$ of elements of S satisfying conditions*

$$\begin{cases}\lim_{n\longrightarrow\infty}\varepsilon(x,y\cdot s_n)=0,\\ \lim_{n\longrightarrow\infty}\varepsilon(x,y\cdot\sigma(s_n))=0,\end{cases}\tag{47}$$

for all $x,y\in S$. Suppose that the equation

$$f(x\cdot y)+f(x\cdot\sigma(y))=2f(x)+\varphi(x,y),\tag{48}$$

for all $x,y\in S$ admits a solution $f_0:S\longrightarrow E$. Then, this equation is $\mathscr{E}$-hyperstable.

Remark 9. The assumption in Theorem 8 that equation (46) admits a solution is quite natural because it seems that it makes sense to study stability or hyperstability of an equation only if it has solutions. However, we can still ask if such equation (without solutions) admits functions that satisfy it approximately in a certain way. This problem has been investigated in [43] for a quite general functional equation, a particular case of which is the inhomogeneous version of the Jensen functional equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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