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Integer Solutions of the Diophantine Equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}$

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we mainly find all solutions of the diophantine equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}$ in integer variables (x, y, z, l) .

Keywords: Diophantine equation; integer solution.

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1 Introduction

The so-called indefinite equation is a type of equation, characterized by fewer equations than the number of variables, and its solution is subject to certain limitations (such as rational numbers, integers, or positive integers, etc.) [1][2].

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In the early 3rd century, the ancient Greek mathematician Diophantus, who had systematically studied this type of equation, was known as the ancestor of the indefinite equation, hence the indefinite equation is also called the Diophantine equation[1][3]. Ko Zhao was one of the founders and pioneers of modern number theory in China. He devoted himself to the study of indefinite equations and made important contributions in many aspects of this field, especially in the study of the famous Catalan conjecture, which obtained an important result known as KOch's theorem[3][4].

Indefinite equations are an important branch of number theory and one of the most active mathematical fields in history, and there are also important research topics in present[5][6][7][8]. Indefinite equations have been fully demonstrated in mathematical competitions around the world. For solving indefinite equations, people must be require to creatively solve problems through the ideas, methods, and techniques of elementary number theory[9]. In this paper, we mainly discuss the integer solutions for the equation[1]

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}, \tag{1.1}$$

which is widely used in estimating system performance, solving physics problems etc.

2 Positive Integer Solutions of the Equation (1.1)

For the symmetry of x, y, z in the indeterminate equation(1.1), we assume that $x \geq y \geq z$ in this section. Obviously, z and l are not equal to 1, thus, $x \geq y \geq z \geq 2$, therefore,

$$\frac{1}{l}=(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) \geq \left(\frac{1}{2}\right)^3=\frac{1}{8},$$

so must be $l \leq 8$.

Considering $l \geq 2$, if the equation (1.1) has positive integer solution, there must be $2 \leq l \leq 8$. We will discuss these seven cases sequently.

Case 1. $l = 2$

At this point, the equation (1.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{2} \tag{2.1}$$

(1) If $z \geq 5$, and $x \geq y \geq 5$, thus

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) \geq \left(\frac{4}{5}\right)^3 > \frac{1}{2}.$$

so, the equation (2.1) has no integer solution in this case.

(2) If $z = 4$, the equation (2.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y}) \times \frac{3}{4}=\frac{1}{2} \tag{2.2}$$

namely,

$$(x-3)(y-3)=6.$$

For the assumption $x \geq y$, we get

$$\begin{cases} x &= 9 \\ y &= 4 \end{cases}, \quad \begin{cases} x &= 6 \\ y &= 5 \end{cases}.$$

That is, we can obtain two integer solutions of the equation (1.1) , such as

$$(x, y, z, l) = \quad (9, 4, 4, 2), \quad (6, 5, 4, 2).$$

(3) If $z = 3$, the equation (2.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y}) \times \frac{2}{3} = \frac{1}{2} \tag{2.3}$$

namely,

$$(x - 4)(y - 4) = 12 \tag{2.4}$$

For the assumption $x \geq y$, we get

$$\begin{cases} x &= 16 \\ y &= 5 \end{cases}, \quad \begin{cases} x &= 10 \\ y &= 6 \end{cases}, \quad \begin{cases} x &= 8 \\ y &= 7 \end{cases}.$$

There are three integer solutions of the equation (1.1) , namely

$$(x, y, z, l) = \quad (16, 6, 3, 2), \quad (10, 6, 3, 2), \quad (8, 7, 3, 2)$$

(4) If $z = 2$, the equation (2.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y}) = 1 \tag{2.5}$$

There is not hold for the positive integer number x, y at the above equation, so the equation(1.1) has no positive integer solution for $l = 2, z = 2$.

Case 2. $l = 3$

At this point, the equation (1.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{3} \tag{2.6}$$

(1) If $z \geq 4$, and $x \geq y \geq 4$, thus

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) \geq \left(\frac{3}{4}\right)^3 > \frac{1}{3}.$$

Obviously, the equation (2.6) has no integer solution in this case.

(2) If $z = 3$, the equation (2.6) can be transformed into

$$(x - 2)(y - 2) = 2 \tag{2.7}$$

that is $l = 3, z = 3$, and

$$\begin{cases} x &= 4 \\ y &= 3 \end{cases}$$

namely, in this case, we have

$$(x, y, z, l) = \quad (4, 3, 3, 3).$$

(3) If $z = 2$, the equation (2.6) can be changed to

$$(x - 3)(y - 3) = 6 \tag{2.8}$$

we obtain that $z = 2, l = 3$,

$$\begin{cases} x-3 & = & 6 \\ y-3 & = & 1 \end{cases}, \quad \begin{cases} x-3 & = & 3 \\ y-3 & = & 2 \end{cases}$$

namely,

$$(x, y, z, l) = \quad (9, 4, 2, 3), \quad (6, 5, 2, 3)$$

are the solutions of the equation (1.1).

Case 3. $l = 4$

The equation (1.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{4} \tag{2.9}$$

(1) If $z \geq 3$, then $x \geq y \geq z \geq 3$, we obtain

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) \geq \left(\frac{2}{3}\right)^3 > \frac{1}{4}$$

The equation (2.9) has no solutions.

(2) If $z = 2$, the equation (2.9) can be transformed into

$$(x-2)(y-2) = 2$$

then $l = 4, z = 2$. and

$$(x, y, z, l) = (4, 3, 2, 4)$$

is solution of the equation (1.1).

Case 4. $l = 5$

The equation (1.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{5} \tag{2.10}$$

(1) If $z \geq 3$, then $x \geq y \geq z \geq 3$, we obtain

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) \geq \left(\frac{2}{3}\right)^3 > \frac{1}{5}$$

The equation (2.10) has no solutions.

(2) If $z = 2$, the equation (2.10) can be simplified to

$$(3x-5)(3y-5) = 10. \tag{2.11}$$

then $l = 5, z = 2$, and we get a solution

$$(x, y, z, l) = (5, 2, 2, 5).$$

Case 5. $l = 6$

The equation (1.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{6} \tag{2.12}$$

For z must be less than 3, that is $z=2$, the equation (2.12) can be simplified to

$$(2x-3)(2y-3)=3 \tag{2.13}$$

Then $l=6, z=2$, we have a solution

$$(x,y,z,l)=(3,2,2,6).$$

Case 6. $l=7$

Samely, the equation (1.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{7}, \tag{2.14}$$

and $z=2$, then the equation (2.14) can be simplified to

$$(5x-7)(5y-7)=14 \tag{2.15}$$

we can see that no positive integer solutions for the above equation.

Case 7. $l=8$

In this case, the equation (1.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{8} \tag{2.16}$$

and $z=2$, then the equation (2.16) can be simplified to

$$(3x-4)(3y-4)=4. \tag{2.17}$$

We can obtain a group of positive integer solution, namely, $l=8, z=2$, that is

$$(x,y,z,l)=(2,2,2,8).$$

To summarize, we have finished solving all the positive solutions for the equation (1.1) in this section. Since division by 0 is not allowed, all variables x, y, z, l are not equal to 0. We will discuss negative integer solutions of the equation (1.1) in the next section.

3 Negative Integer Solutions of the Equation (1.1)

In this section, we try to discuss the case where x, y and z can be negative integers. Hence the item

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})>0,$$

whether x, y and z are positive numbers or negative numbers, thus l is a positive integer.

Case 1. If x, y and z are negative integer number,

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})>1,$$

and $\frac{1}{l} \leq 1$, we can see that the equation (1.1) has no solution.
Case 2. If there are two negative integer in x, y and z , there is no harm in suppose that x, y are negative, then

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > 1-\frac{1}{z} \geq 1-\frac{1}{2} = \frac{1}{2}$$

that is l=1 only.

We transforming the equation (1.1) into

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = 1,$$

that is

$$(x+z-1)(y+z-1) = z(z-1).$$

Suppose $z = t$, ($t \geq 2$), Let d is a factor of $t(t-1)$, we obtain

$$\begin{cases} x + t-1 = d \\ y + t-1 = \frac{t(t-1)}{d} \end{cases}$$

so

$$\begin{cases} x = 1 - t + d \\ y = 1 - t + \frac{t(t-1)}{d} \end{cases}$$

where $d < t-1, \frac{t}{d} < 1$, the condition must be $d < 0$. So let $d = -d$, we obtain a group of negative solution of the equation (1.1) is

$$(x, y, z, l) = \left(1-t-d, 1-t-\frac{t(t-1)}{d}, t, 1\right),$$

where d is a positive factor of $t(t-1), t \geq 2$.

Case 3. If there is only one negative integer in x, y and z , suppose that x is negative, for $y \geq z \geq 2$, we have

$$\frac{1}{l} = (1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

so $l < 4$. We will discuss the cases for $l = 1, 2, 3$ at the following.

(1) If $l = 1$, the equation (1.1) is equal to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = 1$$

Following the discussion above, we get a group of negative solution of the equation (1.1) is

$$(x, y, z, l) = \left(1-t+d, 1-t+\frac{t(t-1)}{d}, t, 1\right),$$

where d is a positive factor of $t(t-1)$, $t \geq 2$.

(2) If $l = 2$, the equation (1.1) is equal to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = \frac{1}{2} \tag{3.1}$$

1^o. $z \geq 4$, and $y \geq z$, the above equation is equal to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > (1-\frac{1}{y})(1-\frac{1}{z}) \geq \left(\frac{3}{4}\right)^2 > \frac{1}{2}.$$

We can obtain that the equation (1.1) has no solution if $z \geq 4, l = 2$.

2^o. $z = 2$, we have

$$(1 - \frac{1}{x})(1 - \frac{1}{y}) \times \frac{1}{2} = \frac{1}{2}$$

namely, $x + y = 1$.

Let $y = t \geq 2$, than a solution of the equation (1.1) is

$$(x, y, z, l) = (1 - t, t, 2, 2)\,,$$

where $t \geq 2$.

3^o. $z = 3$, we have

$$(1 - \frac{1}{x})(1 - \frac{1}{y}) \times \frac{2}{3} = \frac{1}{2}$$

namely,

$$(x - 4)(y - 4) = 12.$$

Considering $x < 0$, so

$$\begin{cases} x &= -2 \\ y &= 2 \end{cases}, \qquad \begin{cases} x &= -8 \\ y &= 3 \end{cases}.$$

That is, we have two negative integer solutions of equation(1.1), such as

$$(x, y, z, l) = (-2, 2, 3, 2)\,, \quad (-8, 3, 3, 2)\,.$$

(3) If $l = 3$, the equation (1.1) can be simplified to

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{3}$$

Suppose $y \geq z \geq 3$, we know

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) > \left(\frac{2}{3}\right)^2 > \frac{1}{3}.$$

That is the equation (1.1) has no solution in this case.

So, $z = 2$, the equation (1.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y}) \times \frac{1}{2} = \frac{1}{3}.$$

It can be simplified to

$$(x - 3)(y - 3) = 6.$$

So

$$\begin{cases} x &= -3 \\ y &= 2 \end{cases},$$

That is one more negative solutions, such as

$$(x, y, z, l) = (-3, 2, 2, 3)\,.$$

Now we have finished to solve the negative integer solutions for the equation (1.1) .

4 Conclusions

Basing on the discussion in the above two sections, we obtain all the integer solutions of the equation(1.1).

There are twelve groups of positive integer solutions (suppose $x \geq y \geq z$) of the equation (1.1), namely, (x, y, z, l) equals to $(16, 5, 3, 2)$, $(10, 6, 3, 2)$, $(8, 7, 3, 2)$, $(9, 4, 4, 2)$, $(6, 5, 4, 2)$, $(9, 4, 2, 3)$, $(6, 5, 2, 3)$, $(4, 3, 3, 3)$, $(4, 3, 2, 4)$, $(5, 2, 2, 5)$, $(3, 2, 2, 6)$, and $(2, 2, 2, 8)$.

The negative integer solutions of the equation (1.1) are: (x, y, z, l) equals to $\left(1-t-d, 1-t-\frac{t(t-1)}{d}, t, 1\right)$, $\left(1-t+d, 1-t+\frac{t(t-1)}{d}, t, 1\right)$, $(1-t, t, 2, 2)$, (where d is a positive factor of $t(t-1), \forall t \geq 2$), and $(-2, 2, 3, 2)$, $(-8, 8, 3, 2)$, $(-3, 2, 2, 3)$.

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Competing Interests

Authors have declared that no competing interests exist.

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