**British Journal of Mathematics & Computer Science**
**4(11): 1551-1566, 2014**

**SCIENCEDOMAIN** *international*
www.sciencedomain.org

# Analytical Solutions to the Fractional Fisher Equation by Applying the Fractional Sub-equation Method

**H. Yépez-Martínez[1*], J. M. Reyes[1] and I. O. Sosa[1]**

[1]*College of Science and Technology, University Autonomous of Mexico City, San Isidro extension 151, Col. San Lorenzo Tezonco, Del. Iztapalapa, PO Box 09790 Mexico DF, Mexico.*

***Original Research Article***

*Received: 29 January 2014*
*Accepted: 23 March 2014*
*Published: 29 March 2014*

## Abstract

The fractional sub-equation method is proposed to construct analytical solutions of nonlinear fractional partial differential equations (FPDEs), involving Jumarie's modified Riemann-Liouville derivative. The fractional sub-equation method is applied to the fractional Fisher equation. The analytical solutions show that the fractional sub-equation method is very effective for the analytical solutions of the Fisher equation. The fractional sub-equation method introduces a promising tool for solving many fractional partial differential equations.

Keywords: Fractional sub-equation method, analytical solutions, fractional fisher equation.

2010 Mathematics Subject Classification: 26A33; 35R11; 35G20

## 1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, nonlinear fractional differential equations (FDEs) have gained considerable interest. It is caused by the development of the theory of fractional calculus itself but also by their applications in various sciences such as physics, engineering, biology and others areas [1–7]. Among the investigations for fractional differential equations, research for seeking exact solutions is an important topic as well as applying them to practical problems [8–13]. Many powerful and efficient methods have been proposed to obtain numerical and exact solutions of FDEs. For example, the finite difference method [14], the finite element method [15, 16], the differential transform method [17,18], the adomian decomposition method (ADM) [19–21], the variational iteration method [22–24], the homotopy perturbation method [25]. The optimal homotopy asymptotic method (OHAM) has been applied to construct new approximate solutions of the Falkner-Skan equation with heat transfer and the coupled Drinfel'd-Sokolov-Wilson equations [26,27]. The modified ($G'/G$) expansion method has been applied to construct analytical travelling wave solutions to the space-time fractional order nonlinear Burgers' equation and the coupled

**Corresponding author: huitzilin.yepez.martinez@uacm.edu.mx;*

Burgers' equation [28,29], the first integral method has been applied to construct analytical travelling wave solutions for some space-time fractional order equations [30].

By taking into account the results obtained in Ref. [31], Zhang and Zhang [32] proposed a new algebraic method named the fractional sub-equation method to look for travelling wave solutions of nonlinear FPDEs. The fractional sub-equation method is a very strong technique to obtain exact solutions for nonlinear FPDEs. The method is based on the homogeneous balance principle [33] and the Jumarie's modified Riemann-Liouville derivative of fractional order [34,35]. With the help of this method, Zhang et al. have successfully obtained travelling wave solutions of nonlinear time fractional biological population model and (4+1)-dimensional space-time fractional Fokas equation [32].

Biological population problem are widely investigated in many papers [36,37]. The spatial diffusion of biological populations was considered in Ref. [38] and obtained its corresponding numerical solution using the variational iteration method for the biological population model:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + h(u) \quad , \tag{1}$$

where $u$ denotes the population density and $h(u)$ represents the population supply. The explicit solutions of travelling waves for the Fisher equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \varepsilon u\left(1 - u^\beta\right) \quad , \tag{2}$$

were considered in Ref. [39], for the special case where $\beta=1$. A kink-like travelling wave solution of Eq. (2) describes a constant-velocity front of transition from one homogeneous state to another. The Fisher equation has been applied in the fields of logistic population growth [40], neurophysiology [41], autocatalytic chemical reaction [42] and Brownian motion processes [43].

A representative Fisher equation, $u_t = u_{xx}+u(1-u)$, was first introduced by Fisher as a model for the propagation of a mutant gene [40], where $u(x,t)$ denotes the population density and $u(1-u)$ represents the population supply due to births and deaths. Several studies in the literature, employing a large variety of methods, have been conducted to derive explicit solutions for the Fisher equation and for the generalized Fisher equation(2). For more details about these investigations see references [44–50] and references therein. Recently, Wazwaz and Gorguis [51] studied the Fisher equation subject to initial conditions by using ADM.

The present work is committed to study the fractional Fisher equation within the sub-equation method [32]. The fractional Fisher equation as a nonlinear model for a physical system involving nonlinear growth takes the form [52]:

$$D_t^\alpha u(x,t) = D_x^{2\alpha} u(x,t) + \varepsilon u\left(1 - u^\beta\right) \quad , \; t > 0\,, 0 < \alpha \le 1, \tag{3}$$

where $D^\alpha_x$ and $D^\alpha_t$ are the partial fractional derivatives and $\alpha$ is a parameter describing the order of the fractional derivatives.

The outline of this work is as follows: in section 2, the sub-equation method is presented. Section 3 contains the application of the method to solve the fractional Fisher equation for two special cases ε=6, β=1 and ε=1, β=6. In section 4, the analytical solution for the generalized fractional Fisher equation is obtained by applying the sub-equation method. Finally in section 5 some conclusions are presented.

## 2 Description of the Fractional Sub-equation Method and Its Applications to the Space-time Fractional Differential Equations

In this section we present the main ideas of the fractional sub-equation method. This method considers the Jumarie's modified Riemann-Liouville fractional derivative of order $\alpha$ [34,35]:

$$D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} \left[ f(\xi) - f(0) \right] d\xi, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} \left[ f(\xi) - f(0) \right] d\xi & 0 < \alpha < 1 \\ \left[ f^{(\alpha-n)}(x) \right]^{(n)} & n \le \alpha \le n+1,\ n \ge 1. \end{cases} \tag{4}$$

Some properties for the proposed modified Riemann–Liouville derivative are:

$$\begin{gathered} D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha} \\ D_x^{\alpha}\, c = 0, \quad \alpha \ge 0 \ \ c = const. \\ D_x^{\alpha}\left( cf(x) \right) = c\left( D_x^{\alpha} f(x) \right), \quad \alpha \ge 0 \ \ c = const. \\ D_x^{\alpha}\left( f(x) g(x) \right) = g(x)\left( D_x^{\alpha} f(x) \right) + f(x)\left( D_x^{\alpha} g(x) \right) \\ D_x^{\alpha} f\left[ g(x) \right] = f_g'\left[ g(x) \right] D_x^{\alpha} g(x) = \left( D_g^{\alpha} f\left[ g(x) \right] \right) \left( g'(x) \right)^{\alpha} \quad . \end{gathered} \tag{5}$$

The above properties play an important role in the fractional sub-equation method. The main steps of this method are described as follows [32]:

**Step 1:** Suppose that a nonlinear FPDE, say in two independent variables, is given by:

$$P\left( u, u_x, u_t, D_x^{\alpha} u, D_t^{\alpha} u, \ldots \right) = 0 \quad 0 < \alpha \le 1 \quad , \tag{6}$$

where $D^{\alpha}_{x} u$ and $D^{\alpha}_{t} u$ are the Jumarie's modified Riemann-Liouville derivatives of $u$, $u = u(x,t)$ is an unknown function, $P$ is a polynomial in $u$ and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

**Step 2:** By using the travelling wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{7}$$

where $k$ and $c$ are constants to be determined later, the FPDE (6) is reduced to the following nonlinear fractional ordinary differential equation for $u(x,t) = u(\xi)$:

$$P\left(u, ku', cu', k^{\alpha} D_{\xi}^{\alpha} u, c^{\alpha} D_{\xi}^{\alpha} u, ...\right) = 0 \quad . \tag{8}$$

**Step 3:** We suppose that Eq. (8) has the following solution:

$$u\left(\xi\right) = \sum_{i=0}^{n} a_i \phi^i \quad , \tag{9}$$

where $a_i$ ($i$=0,1,2,…,$n$) are constants to be determined later, $n$ is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (8), and $\phi = \phi(\xi)$ satisfies the following fractional Riccati equation:

$$D_{\xi}^{\alpha} \phi = \sigma + \phi^2 \quad , \tag{10}$$

where $\sigma$ is a constant. By using the generalized exp-function method via Mittag-Leffler function, Zhang et al. [31], obtained the following solutions of fractional Riccati equation (10):

$$\phi\left(\xi\right) = \begin{cases} -\sqrt{-\sigma} \tanh_{\alpha}\left(\sqrt{-\sigma}\xi\right) & \sigma < 0 \\ -\sqrt{-\sigma} \coth_{\alpha}\left(\sqrt{-\sigma}\xi\right) & \sigma < 0 \\ \sqrt{\sigma} \tan_{\alpha}\left(\sqrt{\sigma}\xi\right) & \sigma > 0 \\ \sqrt{\sigma} \cot_{\alpha}\left(\sqrt{\sigma}\xi\right) & \sigma > 0 \\ -\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+\omega}, \quad \omega = const. & \sigma = 0 , \end{cases} \tag{11}$$

where the generalized hyperbolic and trigonometric functions are defined as:

$$\begin{aligned} &\sinh_{\alpha}\left(x\right) = \frac{E_{\alpha}(x^{\alpha}) - E_{\alpha}(-x^{\alpha})}{2}, \quad \cosh_{\alpha}\left(x\right) = \frac{E_{\alpha}(x^{\alpha}) + E_{\alpha}(-x^{\alpha})}{2}, \\ &\tanh_{\alpha}\left(x\right) = \frac{\sinh_{\alpha}\left(x\right)}{\cosh_{\alpha}\left(x\right)}, \qquad \coth_{\alpha}\left(x\right) = \frac{\cosh_{\alpha}\left(x\right)}{\sinh_{\alpha}\left(x\right)}, \\ &\sin_{\alpha}\left(x\right) = \frac{E_{\alpha}(ix^{\alpha}) - E_{\alpha}(-ix^{\alpha})}{2}, \quad \cos_{\alpha}\left(x\right) = \frac{E_{\alpha}(ix^{\alpha}) + E_{\alpha}(-ix^{\alpha})}{2}, \\ &\tan_{\alpha}\left(x\right) = \frac{\sin_{\alpha}\left(x\right)}{\cos_{\alpha}\left(x\right)}, \qquad \cot_{\alpha}\left(x\right) = \frac{\cos_{\alpha}\left(x\right)}{\sin_{\alpha}\left(x\right)}, \end{aligned} \tag{12}$$

and $E_\alpha(z)$ is the Mittag-Leffler function, given as:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}. \tag{13}$$

**Step 4:** Substituting Eq. (9) into Eq. (8) taking into account Eq. (10) and the properties of the Jumarie's modified Riemann-Liouville derivative, Eq. (8) is converted to a polynomial in $\phi^i(\xi)$ ($i$=0,1,2,…). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i$ ($i$=0,1,2,…), $k$ and $c$.

**Step 5:** Solving the equations system in step 4, and by using the solutions Eq. (11), we can construct a variety of exact solutions of Eq. (8).

**Remark:** If $\alpha\to 1$, the Riccati equation becomes $\phi'(\xi)=\sigma+\phi^2(\xi)$. The method can be used to solve integer order differential equations. In this sense the sub-equation method includes the existing tanh-function method as special case.

# 3 Fractional Sub-equation Method Applied to the Fisher Equation

In this section we apply the fractional sub-equation method to construct the exact solutions for space-time fractional Fisher equation (3), for two particular cases: I) $\varepsilon=6$, $\beta=1$ and II) $\varepsilon=1$, $\beta=6$, considered previously for the integer order Fisher equation.

## 3.1 Case I: the fractional Fisher equation with ε=6, β=1.

If we analyze the fractional Fisher equation (3) for the special case $\varepsilon=6$, $\beta=1$, i.e:

$$D_t^\alpha u(x,t)=D_x^{2\alpha}u(x,t)+6u(1-u) \quad . \tag{14}$$

By considering the travelling wave transformation Eq. (7), the Eq. (14) can be reduced to the following nonlinear fractional ordinary differential equation:

$$c^\alpha D_\xi^\alpha u-k^{2\alpha}D_\xi^{2\alpha}u-6u(1-u)=0. \tag{15}$$

We suppose that Eq.(15) has the following formal solution:

$$u(\xi)=\sum_{i=0}^{m_{\max}}a_i\phi^i, \tag{16}$$

where $\phi(\xi)$ satisfies Eq. (10). Balancing the highest order derivatives, linear terms and nonlinear terms in Eq. (15), it is possible to determine the value of $m_{\max}$. Putting together Eq. (16) along with Eq. (10) into Eq. (15), we obtain the following ansätz:

$$u(\xi) = a_0 + a_1\phi + a_2\phi^2 \quad , \tag{17}$$

substituting Eq. (17) into Eq. (15), we obtain the following set of algebraic equations:

$$\begin{aligned}
\phi^0 &: \; a_1\sigma c^\alpha = 2a_2\sigma^2 k^{2\alpha} + 6\left(a_0 - a_0^2\right) \\
\phi^1 &: \; 2a_2\sigma c^\alpha = 2a_1\sigma k^{2\alpha} + 6a_1 - 12a_1a_0 \\
\phi^2 &: \; c^\alpha a_1 = 8a_2\sigma k^{2\alpha} + 6a_2 - 6a_1^2 - 12a_2a_0 \\
\phi^3 &: \; 2a_2c^\alpha = 2a_1k^{2\alpha} - 12a_1a_2 \\
\phi^4 &: \; 0 = 6a_2k^{2\alpha} - 6a_2^2 \quad .
\end{aligned} \tag{18}$$

From these equations we obtain for $a_i$ (i=0,1,2), $k$ and $c$:

$$\begin{aligned}
a_2 &= k^{2\alpha} \\
a_1 &= -\frac{c^\alpha}{5} \\
a_0 &= \frac{1}{2} + \sigma k^{2\alpha} .
\end{aligned} \tag{19}$$

If we considered the well known solution of the Fisher equation (14), with $\alpha$=1 [53]:

$$u\left(x,t\right) = \frac{\left(1 - \tanh\left(\frac{x}{2} - \frac{5}{2}t\right)\right)^2}{4} \quad . \tag{20}$$

We can compare this solution with Eq. (11) for the tanh-type solution and obtain the following values for the coefficients $a_i$, the parameters $k$, $c$ and the constant σ:

$$\begin{aligned}
k &= 1 \\
-\sigma a_2 &= -\sigma k^{2\alpha} = \frac{1}{4} \\
\sqrt{-\sigma}a_1 &= -\sqrt{-\sigma}\frac{c^\alpha}{5} = \frac{1}{2} \\
a_0 &= \frac{1}{2} + \sigma k^{2\alpha} = \frac{1}{4} \quad ,
\end{aligned} \tag{21}$$

therefore, we finally obtain for the coefficients $a_i$ (i=0,1,2), $k$, $c$ and σ:

$$
\begin{aligned}
a_0 &= \frac{1}{4} \\
a_1 &= 1, \\
a_2 &= 1, \\
\sigma &= -\frac{1}{4} \\
k &= 1 \\
c^{\alpha} &= -5 \quad ,
\end{aligned}
\tag{22}
$$

and the solution to the nonlinear fractional Fisher equation (14) is given by:

$$u(x,t) = \frac{\left(1-\tanh_{\alpha}\left[\frac{1}{2}\left(x+(-5)^{1/\alpha}t\right)^{\alpha}\right]\right)^{2}}{4} \quad . \tag{23}$$

We can observe that this solution reproduces the result obtained in reference [53], for the special case where $\alpha$ =1, when the homotopy perturbation method was applied. It should be noted that the integer order analytical exact solution has been an important reference in the comparison when some approximated methods, like the adomian decomposition method and the homotopy perturbation method, were applied for solving the Fisher equation (2) with ε=6, β=1. The results of the approximated methods are in good agreement with the exact analytical solution (23) with $\alpha$ =1, and the absolute errors values were less than 0.1 % [54,55].

### 3.2 Case II: the generalized Fisher equation ε=1, β=6.

If we analyze the fractional Fisher equation (3) for the special case ε=1, β=6, i.e:

$$D_t^{\alpha}u(x,t) = D_x^{2\alpha}u(x,t) + u\left(1-u^6\right) \quad . \tag{24}$$

We consider the following transformation:

$$u^3 = y \quad , \tag{25}$$

introduced by Wang [48]. With this transformation the Eq. (24) can be written as:

$$yD_t^{\alpha}y = yD_x^{2\alpha}y - \frac{2}{3}\left(D_x^{\alpha}y\right)^2 + 3y^2\left(1-y^2\right) \quad . \tag{26}$$

By considering the travelling wave transformation Eq. (7), the Eq. (26) can be reduced to the following nonlinear fractional ordinary differential equation:

$$c^{\alpha}yD_{\xi}^{\alpha}y - k^{2\alpha}\left(yD_{\xi}^{2\alpha}y - \frac{2}{3}\left(D_{\xi}^{\alpha}y\right)^2\right) - 3y^2\left(1-y^2\right) = 0 \quad . \tag{27}$$

We suppose that Eq. (27) has the following formal solution:

$$y(\xi)=\sum_{i=0}^{m_{\max}} a_i\phi^i\ , \tag{28}$$

where $\phi(\xi)$ satisfies Eq. (10). Balancing the highest order derivatives, linear terms and nonlinear terms in Eq. (27), it is possible to determine the value of $m_{\max}$. Putting together Eq. (28) along with Eq. (10) into Eq. (27), we obtain the following ansätz:

$$y(\xi)=a_0+a_1\phi\quad , \tag{29}$$

substituting Eq. (29) into Eq. (27) we obtain the following set of algebraic equations:

$$\begin{aligned}
\phi^0 &: \ a_0a_1\sigma c^\alpha=-\frac{2}{3}a_1^2\sigma^2k^{2\alpha}+3\left(a_0^2-a_0^4\right)\\
\phi^1 &: \ c^\alpha\sigma a_1^2=2\ k^{2\alpha}\sigma a_0a_1+6\ a_0a_1-12\ a_1a_0^3\\
\phi^2 &: \ c^\alpha a_0a_1=3a_1^2-18a_0^2a_1^2+\frac{2}{3}k^{2\alpha}\sigma a_1^2\\
\phi^3 &: \ c^\alpha a_1^2=2\ k^{2\alpha}a_0a_1-12a_0a_1^3\\
\phi^4 &: \ 0=-\frac{4}{3}k^{2\alpha}a_1^2+3a_1^4\quad .
\end{aligned} \tag{30}$$

From these equations we obtain for $a_i$ (i=0,1), $k$ and $c$:

$$\begin{aligned}
a_1&=\frac{2}{3}k^\alpha\\
a_0&=-\frac{c^\alpha}{5k^\alpha}\\
\sigma&=-\frac{9}{16k^{2\alpha}}\\
\frac{c^\alpha}{k^\alpha}&=-\frac{5}{2}\quad .
\end{aligned} \tag{31}$$

If we considered the well known solution of the Fisher equation (24), with $\alpha$=1 [53]:

$$\begin{aligned}
y(x,t)&=\frac{1-\tanh\left[\frac{3}{4}\left(x-\frac{5}{2}t\right)\right]}{2}\\
&\Rightarrow\\
u(x,t)&=\left\{\frac{1-\tanh\left[\frac{3}{4}\left(x-\frac{5}{2}t\right)\right]}{2}\right\}^{1/3}\quad .
\end{aligned} \tag{32}$$

We can compare this solution with Eq. (11) for the tanh-type solution and obtain the following values for the coefficients $a_i$, the parameters $k$, $c$ and the constant σ:

$$\begin{aligned} \sqrt{-\sigma}k^{\alpha} &= \frac{3}{4} \\ -\sqrt{-\sigma}a_1 &= -\frac{2\sqrt{-\sigma}k^{\alpha}}{3} = -\frac{1}{2} \\ a_0 &= -\frac{c^{\alpha}}{5k^{\alpha}} = \frac{1}{2} \quad , \end{aligned} \tag{33}$$

and finally, we obtain for the coefficients $a_i$ (i=0,1), $k$, $c$ and σ:

$$\begin{aligned} a_0 &= \frac{1}{2} \\ a_1 &= 1, \\ \sigma &= -\frac{1}{4} \\ k^{\alpha} &= \frac{3}{2} \\ c^{\alpha} &= -\frac{15}{4}, \end{aligned} \tag{34}$$

and the solution to the nonlinear fractional Fisher equation (24) is given by:

$$\begin{aligned} y(x,t) &= \frac{1-\tanh_{\alpha}\left[\frac{1}{2}\left(\left(\frac{3}{2}\right)^{1/\alpha}x+\left(-\frac{15}{4}\right)^{\frac{1}{\alpha}}t\right)^{\alpha}\right]}{2} \\ &\Rightarrow \\ u(x,t) &= \left\{\frac{1-\tanh_{\alpha}\left[\frac{1}{2}\left(\left(\frac{3}{2}\right)^{1/\alpha}x+\left(-\frac{15}{4}\right)^{\frac{1}{\alpha}}t\right)^{\alpha}\right]}{2}\right\}^{1/3} \quad . \end{aligned} \tag{35}$$

Again this solution reproduces the result obtained in reference [53], for the special case where $\alpha$=1.

# 4 Fractional Sub-equation Method Applied to the Generalized Fisher Equation

In this section we apply the fractional sub-equation method to construct the exact solutions for space-time fractional Fisher equation, for the general case where ε and β are any arbitrary positive numbers in the Eq. (3), i.e:

$$D_t^{\alpha}u\left(x,t\right)=D_x^{2\alpha}u\left(x,t\right)+\varepsilon u\left(1-u^{\beta}\right) \quad . \tag{36}$$

We consider the following transformation:

$$u^{2/\beta}=y \quad , \tag{37}$$

introduced by Wang [48]. With this transformation, Eq. (36) can be written as:

$$yD_t^{\alpha}y=yD_x^{2\alpha}y-\frac{2}{\beta}\left(\frac{\beta}{2}-1\right)\left(D_x^{\alpha}y\right)^2+\frac{\varepsilon\beta}{2}y^2\left(1-y^2\right) \quad . \tag{38}$$

By considering the travelling wave transformation Eq. (7), the Eq. (38) can be reduced to the following nonlinear fractional ordinary differential equation:

$$c^{\alpha}yD_{\xi}^{\alpha}y-k^{2\alpha}\left(yD_{\xi}^{2\alpha}y-\frac{2}{\beta}\left(\frac{\beta}{2}-1\right)\left(D_{\xi}^{\alpha}y\right)^2\right)-\frac{\varepsilon\beta}{2}y^2\left(1-y^2\right)=0 \quad . \tag{39}$$

We suppose that Eq. (39) has the following formal solution:

$$y(\xi)=\sum_{i=0}^{m_{\max}}a_i\phi^i , \tag{40}$$

where $\phi(\xi)$ satisfies Eq. (10). Balancing the highest order derivatives, linear terms and nonlinear terms in Eq. (39), it is possible to determine the value of $m_{\max}$. Putting together Eq. (40) along with Eq. (10) into Eq. (39), we obtain the following ansätz:

$$y(\xi)=a_0+a_1\phi \quad , \tag{41}$$

substituting Eq. (41) into Eq. (39) we obtain the following set of algebraic equations:

$$
\begin{aligned}
&\phi^0 \;:\; c^\alpha \sigma a_0 a_1 = -\frac{(-2+\beta)\sigma^2 a_1^2 k^{2\alpha}}{\beta} + \frac{1}{2}\varepsilon\beta\left(a_0^2 - a_0^4\right)\\
&\phi^1 \;:\; c^\alpha \sigma a_1^2 = 2\,k^{2\alpha}\sigma a_0 a_1 + \varepsilon\beta a_1 a_0 - 2\beta\varepsilon a_1 a_0^3\\
&\phi^2 \;:\; c^\alpha a_0 a_1 = \frac{\varepsilon\beta a_1^2}{2} + 2\,k^{2\alpha}\sigma a_1^2 - \frac{2\,k^{2\alpha}(-2+\beta)\sigma a_1^2}{\beta} - 3\varepsilon\beta\; a_0^2 a_1^2\\
&\phi^3 \;:\; c^\alpha a_1^2 = 2\,k^{2\alpha} a_0 a_1 - 2\varepsilon\beta a_0 a_1^3\\
&\phi^4 \;:\; 0 = -2k^{2\alpha} a_1^2 + \frac{k^{2\alpha}\left(-2+\beta\right)a_1^2}{\beta} + \frac{\beta\varepsilon}{2}a_1^4 \;.
\end{aligned}
\qquad (42)
$$

From these equations, we obtain for the coefficients $a_i$ (i=0,1), $k$, $c$ and σ:

$$
\begin{aligned}
\sigma &= -\frac{1}{4}\\
k^\alpha &= \sqrt{\frac{\varepsilon}{4+2\beta}}\beta\\
a_1 &= \frac{\sqrt{2\left(\beta+2\right)}}{\sqrt{\varepsilon\beta}}k^\alpha\\
a_0 &= -\frac{\left(2+\beta\right)}{\sqrt{\varepsilon}\sqrt{4+2\beta}\left(4+\beta\right)}\frac{c^\alpha}{k^\alpha}\\
\frac{c^\alpha}{k^\alpha} &= -\frac{\left(4+\beta\right)\sqrt{\varepsilon}}{\sqrt{4+2\beta}} \;,
\end{aligned}
\qquad (43)
$$

and the solution to the non linear fractional Fisher equation Eq. (36) is given by:

$$
\begin{aligned}
y\left(x,t\right) &= \frac{1-\tanh_\alpha\left[\frac{1}{2}\left(\left(\sqrt{\frac{\varepsilon}{4+2\beta}}\beta\right)^{1/\alpha} x + \left(-\frac{\varepsilon\beta(4+\beta)}{4+2\beta}\right)^{\frac{1}{\alpha}} t\right)^\alpha\right]}{2}\\
&\Rightarrow\\
u\left(x,t\right) &= \left\{\frac{1-\tanh_\alpha\left[\frac{1}{2}\left(\left(\sqrt{\frac{\varepsilon}{4+2\beta}}\beta\right)^{1/\alpha} x + \left(-\frac{\varepsilon\beta(4+\beta)}{4+2\beta}\right)^{\frac{1}{\alpha}} t\right)^\alpha\right]}{2}\right\}^{2/\beta} .
\end{aligned}
\qquad (44)
$$

We can observe that this solution is in good agreement with the results obtained previously in references [48,53], for the special case where $\alpha$=1. Also we notice that for the values ε=6, β=1, we recover the result of Eq. (23), and for the values ε=1, β=6, we recover the result of Eq. (35).

# 5 Conclusion

In this paper we have investigated the exact travelling wave solutions to the space-time fractional Fisher equation to illustrate the reliability of the sub-equation method. We found new exact solutions for the fractional order Fisher equation that in the limit of integer order derivatives reduce to the solutions obtained before [48,53]. These new exact solutions can be very useful as a starting point of comparison when some approximate methods are applied to the fractional Fisher equation, see for example [54,55] where the comparison between approximated solutions and the exact solutions has to be extended to consider not only the case of time-fractional derivatives but also space fractional derivatives.

# Acknowledgement

We gratefully acknowledge to the Universidad Autónoma de la Ciudad de México and the ADI of the Universidad Autónoma de la Ciudad de México for supporting this research work.

# Competing Interests

Authors have declared that no competing interests exist.

# References

[1] Oldham KB, Spanier J. The fractional calculus. Academic Press, New York, NY, USA; 1974.

[2] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives. Gordon and Breach Science, Yverdon, Switzerland; 1993.

[3] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. John Wiley and Sons, New York, NY, USA; 1993.

[4] Podlubny I. Fractional differential equations. Academic Press, San Diego, Calif, USA; 1999.

[5] Hilfer R. Applications of fractional calculus in physics. World Scientific Publishing, River Edge, NJ, USA; 2000.

[6] West BJ, Bologna M, Grigolini P. Physics of fractal operators. Springer, New York, NY, USA; 2003.

[7] Kilbas A, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. North-Holland Mathematics Studies, Elsevier Science, Amsterdam, the Netherlands. 2006;204.

[8] Kiryakova V. Generalized fractional calculus and applications. Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Harlow, UK. 1994;301.

[9] Sabatier J, Agrawal OP, Machado JAT. Advances in fractional calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, New York, NY, USA; 2007.

[10] Mainardi F. Fractional calculus and waves in linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, London, UK; 2010.

[11] Yang XJ. Local fractional functional analysis and its applications. Asian Academic Publisher, Hong Kong ; 2011.

[12] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. Fractional calculus: Models and numerical methods. Series on Complexity, Nonlinearity and Chaos, World Scientific Publishing, Boston, Mass, USA. 2012;3.

[13] Yang XJ. Advanced local fractional calculus and its applications. World Science Publisher, New York, NY, USA; 2012.

[14] Li C, Chen A, Ye J. Numerical approaches to fractional calculus and fractional ordinary differential equation. J. Comput. Phys. 2011;230(9):3352–3368.

[15] Gao GH, Sun ZZ, Zhang YN. A finite difference scheme for fractional Sub-diffusion equations on an unbounded domain using artificial boundary conditions. J. Comput. Phys. 2012;231(7):2865-2879.

[16] Deng W. Finite element method for the space and time fractional Fokker–Planck equation. SIAM Journal on Numerical Analysis. 2008;47(1):204–226.

[17] Momani S, Odibat Z, Erturk VS. Generalized differential transform method for solving a space and Time-fractional diffusion-wave equation. Phys. Lett. A. 2007;370(5-6):379-387.

[18] Odibat Z, Momani S. A generalized differential transform method for linear partial differential equations of fractional order. Appl. Math. Lett. 2008;21(2):194-199.

[19] Hu Y, Luo Y, Lu Z. Analytical solution of the linear fractional differential equation by Adomian decomposition method. J. Comput. Appl. Math. 2008;215(1):220-229.

[20] El-Sayed AMA, Gaber M. The adomian decomposition method for solving partial differential equations of fractal order in finite domains. Phys. Lett. A. 2006;359(20):175-182.

[21] El-Sayed AMA, Behiry SH, Raslan WE. Adomian's decomposition method for solving an intermediate fractional advection–dispersion equation. Comput. Math. Appl. 2010;59(5):1759-1765.

[22] Odibat Z, Momani S. The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics. Comput. Math. Appl. 2009;58(11-12):2199-2208.

[23] Inc M. The approximate and exact solutions of the space- and Time-fractional Burgers equations with initial conditions by variational iteration method. J. Math. Anal. Appl. 2008;345(1):476-484.

[24] Wu GC, Lee EWM. Fractional variational iteration method and its application. Phys. Lett. A. 2010;374(25):2506-2509.

[25] He JH. Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. 1999;178(3-4):257-262.

[26] Ullah H, Islam S, Idrees M, Nawaz R. Optimal homotopy asymptotic method to doubly wave solutions of the coupled Drinfel'd-Sokolov-Wilson equations. Math. Prob. Engg. 2013;2013: Arti. ID 362816.

[27] Ullah H, Islam S, Idrees M, Arif, M. Solution of boundary layer problems with heat transfer by optimal homotopy asymptotic method. Abst. App. Anal. 2013;2013: Arti. ID 324869.

[28] Younis M, Zafar A. Exact solution to nonlinear differential equations of fractional order via (G′/G)-expansion method. Applied Mathematics. 2014;5(1):1-6.

[29] Younis M, Zafar A, Ul-Haq K, Rahman M. Travelling wave solutions of fractional order coupled Burgers' equations by (G'/G)-expansion method. American Journal of Computational and Applied Mathematics. 2013;3(2):81-85.

[30] Younis M. The first integral method for time-space fractional differential equations. Journal of Advanced Physics. 2013;2(3):220-223.

[31] Zhang S, Zong QA, Liu D, Gao Q. A generalized exp-function method for fractional Riccati differential equations. Comm. Fract. Calc. 2010;1(1):48-52.

[32] Zhang S, Zhang HQ. Fractional Sub-equation method and its applications to nonlinear fractional PDEs. Phys. Lett. A. 2011;375(7):1069-1073.

[33] Wang M. Solitary wave solutions for variant Boussinesq equations. Phys. Lett. A. 1995;199(3-4):169-172.

[34] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of non differentiable functions further results. Comput. Math. Appl. 2006;51(9-10):1367-1376.

[35] Jumarie G. Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution. J. Appl. Math. Comput. 2007;24(1-2):31-48.

[36] El-Sayed AMA, Rida SZ, Arafa AAM. Exact solutions of fractional-order biological population model. Commun. Theor. Phys. 2009;52(6): 992-996.

[37] Liu Y. General solutions of space fractional Fisher's nonlinear diffusion equation. Journal of Fractional Calculus and Applications. 2011;1(2):1-8.

[38] Shakeri F, Dehghan M. Numerical solution of a biological population model using He's variational iteration method. Comput. And Maths. With Appl. 2007;54(7-8):1197-1209.

[39] Tan Y, Xu H, Liao SJ. Explicit series solution of travelling waves with a front of Fisher equation. Chaos Solitions and Fractals. 2007;31(2):462-472.

[40] Fisher RA. The wave of advance of advantageous genes. Ann. Eugenics. 1937;7(4):355-369.

[41] Tuckwell HC. Introduction to Theoretical Neurobiology, Cambridge University Press, Cambridge, UK; 1988.

[42] Aronson DG, Weinberger HF. Nonlinear diffusion in population genetics, combustion and nerve propagation. Proc. Tulane Program in Partial Differential Equations and Related Topics, Lecture Notes in Math. Springer, Berlin, Germany. 1975;446.

[43] Bramson MD. Maximal displacement of branching Brownian motion. Commun. Pure Appl. Math. 1978;31(5):531-581.

[44] Kawahara T, Tanaka M. Interactions of traveling fronts: An exact solution of a nonlinear diffusion equation. Phys. Lett. A. 1983;97(8):311-314.

[45] Sheratt J. On the transition from initial data traveling waves in the Fisher-KPP equation. Dynamics and Stability of Systems. 1998;13(2):167-174.

[46] Brazhnik PK, Tyson JJ. On traveling wave solutions of Fisher's equation in two spatial dimensions. SIAM J. App. Math. 1999;60(2):371-391.

[47] Malfliet W. Solitary wave solutions of nonlinear wave equations. Am. J. Phys. 1992;60(7):650-654.

[48] Wang XY. Exact and explicit solitary wave solutions for the generalized Fisher equation. Phys. Lett. A. 1998;131(4-5):277-279.

[49] Ablowitz MJ, Zepetella A. Explicit solutions of Fisher's equation for a special wave speed. Bulletin of Mathematical Biology. 1979;41(6):835-840.

[50] Jone DS, Sleeman BD. Differential Equations and Mathematical Biology. Chapman and Hall/CRC, New York, USA; 2003.

[51] Wazwaz AM, Gorguis A. An analytic study of Fisher's equation by using adomian decomposition method. Applied Mathematics and Computation. 2004;154(3):609-620.

[52] Jafari H, Tajadodi H, Baleanu D. Application of a homogeneous balance method to exact solutions of nonlinear fractional evolution equations. J. Comput. Nonlinear Dynam; 2013. doi:10.1115/1.4025770.

[53] Ağirseven D, Özis T. An analytical study for Fisher type equations by using homotopy perturbation method. Computers & Mathematics with Applications. 2010;60(3):602-609.

[54] Khan NA, Ayaz M, Jin L, Yildirim A. On approximate solutions for the Time-fractional reaction-diffusion equation of Fisher type. International Journal of Physical Sciences. 2011;6(10):2483-2496.

[55] Ali U, Kamal R, Mohyud-Din ST. On nonlinear fractional differential equations. International Journal of Modern Mathematical Sciences. 2012;(3):116-124.



***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*www.sciencedomain.org/review-history.php?iid=477&id=6&aid=4170*