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A Short Note on Weak Estimation of Sharp Function

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Abstract

In paper [1] Ahmad et al. investigated the use of sharp function, known from functional analysis, in image processing. The sharp function gives a measure of variations of a function and can be used as an edge detector [2]. We extend the classical notion of sharp function to prove the classical Lebesgue differentiation theorem and Marcinkiewicz theorem for the sublinear operator $T(x, y)$.

Keywords: Maximal function, bounded mean oscillations (BMO), sharp function, distribution, sublinear operator.

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1 Introduction

The sharp function is a well known functional analytic concept to measure the oscillatory behavior of functions. It goes back to the maximal function which was introduced in Hardy and Littlewood [3] to solve a problem in the theory of complex variable. Based on this idea John and Nirenberg [4] introduced the concept of bounded mean oscillation (BMO) functions. Fefferman and Stein [5] introduced the sharp function (denoted by $f^{\#}$) and found that a function $f \in BMO$ is equivalent with $f^{\#} \in L^{\infty}$. For more details we refer to Fefferman [6], Kurtz [7] and Wojtaszczyk [8]. The idea of applying the sharp operator to measure the oscillation and classification of images was first proposed by Ahmad and Siddiqi [1] where it was used to find a suitable compression technique.

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In this paper, we use sharp function to prove the Marcinkiewicz Interpolation theorem for the weak sublinear operator $T(x,y)$ and for generalization of the concept of distribution. This paper also includes a proof of Lebesgue differentiation theorem.

2 Maximal Function

In this section we give a short introduction to the maximal function and its background. There is a rich theory behind it, and we are going to point out some of the main results connected to it. The Hardy-Littlewood maximal function was developed to solve a problem in the theory of functions of complex variable. Further it can also be used in the proof of Lebesgue differentiation theorem, Fatou's theorem and in the theory of singular integral operators.

Definition 2.1. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $f(x)$ be a real valued measurable function on $\mathbb{R}^n$. For such a function f on $\mathbb{R}^n$ its Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \underset{Q:x\in Q}{sup}\left\{\frac{1}{\lambda(Q)}\int_Q |f(y)|dy \ : \ Q\subset\mathbb{R}^n, x\in Q\right\}, \tag{2.1}$$

where the supremum ranges over all finite cubes Q in $\mathbb{R}^n$ and $\lambda(Q)$ is the Lebesgue measure of Q.

Definition 2.2. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $f(x)$ be a real valued measurable function on $\mathbb{R}^n$. For such a function f on $\mathbb{R}^n$ its distribution is defined by the formula

$$d_f(t) = \lambda(\{x\in\mathbb{R}^n \ : \ |f(x)|>t\}), \text{ for } \ t\geq 0. \tag{2.2}$$

It is easy to find a function whose maximal function is unbounded.

Example 2.1. *For $f(x)=|x|^t$ with $t>0$, we get $Mf(x)=\infty$, for each $x\in\mathbb{R}$.*

Theorem 2.2. [Hardy-Littlewood maximal theorem] *For each function $f\in L^1(\mathbb{R}^n)$ we have*

$$\lambda(\{x: Mf(x)>t\}) \leq 6^n t^{-1}\|f\|_1, \ t>0. \tag{2.3}$$

Proof. If $\|f\|_1=\infty$, then it is trivial. Without the loss of generality, we can assume that $\|f\|_1=1$. For a fixed $t\geq 0$, let $E_t=\{x : Mf(x)>t\}$. Then for each $x\in E_t$ there is a cube Q_x such that $x\in Q_x\subset E_t$, and

$$\frac{1}{\lambda(Q_x)}\left\{\int_{Q_x}|f(y)|dy\right\} > t. \tag{2.4}$$

Thus,

$$\frac{\|f\|_1}{t}\geq\lambda(Q_x) \quad \text{or} \quad \frac{1}{t}\geq\lambda(Q_x). \tag{2.5}$$

Let $\alpha_1 = \max\{\lambda(Q_x) \ : \ x\in E_t\}$, so $\alpha_1\leq t^{-1}$. Let us fix a cube Q_x, call it Q_1 such that $\lambda(Q_1)>\frac{1}{2}\alpha_1$. Consider all cubes Q_x such that $Q_x\cap Q_1=\phi$. If there are no such cubes, we stop. Otherwise we put $\alpha_2=\max\{\lambda(Q_x) \ : \ x\in E_t \ and \ Q_x\cap Q_1=\phi\}$ and fix such a cube Q_x, call it Q_2 satisfying $\lambda(Q_2)>\frac{1}{2}\alpha_2$. Continuing in this way we get a sequence of cubes $Q_1, Q_2, ...$, possibly finite such that

(i) the cubes Q_j are disjoint,

(ii) $\lambda(Q_j)>\frac{1}{2}\max\{\lambda(Q_x) \ : \ Q_x\cap Q_s=\phi\,;$ for $s=1,2,3,...,j-1\}$,

(iii) if $Q_x\cap Q_s=\phi$ for $s=1,2,3,...,j-1$, then $\lambda(Q_x)\leq 2\lambda(Q_j)$.

From(2.4) and (i), we get

$$\lambda\left(\bigcup_{i=1}^{n} Q_i\right) = \sum_{i=1}^{n} \lambda(Q_i) \leq \frac{1}{t}\sum_{i=1}^{n}\int_{Q_i} |f(y)|dy \leq \frac{1}{t}\int_{\cup Q_i} |f(y)|dy \leq \frac{\|f\|_1}{t} \ = \ \frac{1}{t}. \tag{2.6}$$

It is important to mention here that each Q_x intersects some Q_i. If there exists a Q_x disjoint from all Q_i's then our process was infinite. So from (2.6), we see that $\lambda(Q_j) \to \infty$ which contradicts (iii). Now for a given Q_x, let Q_s be the first Q_i that intersects with Q_x. So by (iii), $\lambda(Q_x) \leq 2\lambda(Q_s)$ and hence $Q_x \subset 6 \diamond Q_s$ (by $c \diamond Q$ we mean a cube with the same center as Q whose sides are c times longer than sides of $Q, c > 0$). Thus from equation (2.6), we have

$$\begin{aligned}\lambda(\{x \ : \ Mf(x) > t\}) &= \lambda(\bigcup_x Q_x)\\ &\leq \lambda(\bigcup_i 6 \diamond Q_i)\\ &\leq \sum_i \lambda(6 \diamond Q_i)\\ &\leq 6^n \sum_i \lambda(Q_i) \leq 6^n \frac{1}{t} = 6^n \frac{\|f\|_1}{t}.\end{aligned} \tag{2.7}$$

□

An interesting application of the Maximal Theorem is a version of the Lebesgue differentiation theorem.

Theorem 2.3. *Let $f \in L^1(\mathbb{R}^n)$. For almost all $x \in \mathbb{R}^n$ and for every decreasing sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ such that $\bigcap_{j=1}^{\infty} Q_j \ = \ \{x\}$, we have*

$$\lim_{j\to\infty} \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y)dy \ = \ f(x). \tag{2.8}$$

Proof. If $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then (2.8) holds for all $x \in \mathbb{R}^n$. Given $f \in L^1(\mathbb{R}^n)$ we take ϵ such that $0 < \epsilon < 1$ and write $f = g + h$ with $g \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\|h\|_1 < \epsilon$. Then by the above observation and by the definition of Hardy-Littlewood maximal function, we have,

$$\begin{aligned}&\limsup_{j\to\infty} \left|\tfrac{1}{\lambda(Q_j)}\textstyle\int_{Q_j} f(y)dy - f(x)\right|\\ &= \limsup_{j\to\infty}\left|\frac{1}{\lambda(Q_j)}\int_{Q_j}(g+h)(y)dy - (g+h)(x)\right|\\ &= \limsup_{j\to\infty}\left|\frac{1}{\lambda(Q_j)}\int_{Q_j} g(y)dy - g(x) + \frac{1}{\lambda(Q_j)}\int_{Q_j} h(y)dy - h(x)\right|\\ &= \limsup_{j\to\infty}\left|\frac{1}{\lambda(Q_j)}\int_{Q_j} h(y)dy - h(x)\right|\\ &\leq \limsup_{j\to\infty}\left\{\frac{1}{\lambda(Q_j)}\int_{Q_j}|h(y)|dy\right\} + |h(x)|\\ &\leq |h(x)| + Mh(x).\end{aligned}$$

Now by equations (2.2) and (2.3), we get

$$\lambda(\{x \in \mathbb{R}^n \ : \ |h(x)| > \sqrt{\epsilon}\}) \leq \frac{\|h\|_1}{\sqrt{\epsilon}} \leq \sqrt{\epsilon}$$

and

$$\lambda\left(\{x : Mh(x) > \sqrt{\epsilon}\}\right) \leq 6^n \frac{\|h\|_1}{\sqrt{\epsilon}} \leq 6^n\sqrt{\epsilon}$$

respectively. This shows that outside the set $\{x \in \mathbb{R}^n : Mh(x) > \sqrt{\epsilon}\} \cup \{x \in \mathbb{R}^n : |h(x)| > \sqrt{\epsilon}\}$ which has measure at most $C\sqrt{\epsilon}$, we have

$$\limsup_{j\to\infty} \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y)dy - f(x) \right| \leq 2\sqrt{\epsilon}.$$

Since ϵ was arbitrary small, we obtain

$$\lim_{j\to\infty} \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y)dy = f(x).$$

□

The Hardy-Littlewood maximal operator M is a significant nonlinear operator used in real and harmonic analysis. It takes a locally integrable function $f : \mathbb{R}^n \to C$ and returns another function Mf that at each $x \in \mathbb{R}^n$ gives the maximal average value that f can have on cubes containing the point $x \in Q_x$. Hardy-Littlewood maximal inequality states that M is bounded as a sublinear operator from $L^p(\mathbb{R}^n)$ to itself for $p > 1$.

Definition 2.3. A measurable function f on $\mathbb{R}^n$ has bounded p-mean oscillation, $1 \leqslant p < \infty$, if

$$\|f\|_{BMO_p} = \sup_Q \left\{ \frac{1}{\lambda(Q)} \int_Q |f(x) - f_Q|^p dx \right\}^{1/p} < \infty$$

where the $\sup$ ranges over all finite cubes Q in $\mathbb{R}^n$ and $f_Q = \frac{1}{\lambda(Q)} \int_Q f(x)dx$ is the mean value of the function f on the cube Q.

The set of all functions of bounded p-mean oscillation is denoted by $BMO_p(\mathbb{R}^n)$. $\|f\|_{BMO_p}$ is "almost" a norm since it has the following properties

(i) $\|f + g\|_{BMO_p} \leqslant \|f\|_{BMO_p} + \|g\|_{BMO_p}$

(ii) $\|\alpha f\|_{BMO_p} \leqslant |\alpha|\|f\|_{BMO_p}$

(iii) $\|f\|_{BMO_p} = 0$ iff $f = constant\ a.e.$,

where f and g are the measurable functions on $\mathbb{R}^n$ and α is some scalar quantity.

Example 2.4. *The function*

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases}$$

is in $BMO_1(\mathbb{R})$.

Let $Q = [0, 1]$, then

$$\frac{1}{1-0} \int_0^1 |f(x) - f_Q|dx = \int_0^1 |f(x) - 1|dx = 0 < \infty.$$

Every bounded function has a bounded p-mean oscillations.
That is, $L^\infty(\mathbb{R}^n) \subset BMO_1(\mathbb{R}^n), 1 \leqslant p < \infty$.

Example 2.5. *For $f(x) = |x|^\alpha$ with $\alpha \in (-1, 0)$, we find maximal function $Mf(x)$ of $f(x)$ as*

$$Mf(x) = \sup_{x \in I} \left\{ \frac{1}{|I|} \int_I |f(t)| dt \right\}.$$

We know that $Mf(x) = \max\{Mf_+(x), Mf_-(x)\}$. Assume that $x > 0$, then clearly,

$$Mf(x) = Mf_-(x) = \sup_{y<x} \frac{1}{x-y} \int_y^x |f(t)| dt = \sup_{y<x} \frac{1}{x-y} \int_y^x |t|^\alpha dt.$$

Further, to obtain $\sup$ we consider $y < 0$, then we have

$$Mf(x) = \sup_{y<0} \frac{1}{x-y} \left\{ \frac{|t|^{\alpha+1}}{\alpha+1} \right\}_y^x.$$

Since $y < 0$, so that choose $y = -s$, where $s > 0$ then

$$\begin{aligned} Mf(x) &= \sup_{s>0} \frac{1}{x+s} \left\{ \frac{x^{\alpha+1} + s^{\alpha+1}}{\alpha+1} \right\} \\ &= \sup_{s>0} \frac{1}{x(1+(s/x))} \left\{ \frac{x^{\alpha+1}(1+(s/x)^{\alpha+1})}{\alpha+1} \right\} \\ &= \sup_{s>0} x^\alpha \left\{ \frac{(1+(s/x)^{\alpha+1})}{(1+(s/x))(1+\alpha)} \right\}. \end{aligned}$$

Putting $\frac{s}{x} = t > 0$, we get

$$Mf(x) = x^\alpha \sup_{t>0} \left\{ \frac{1+t^{\alpha+1}}{(1+t)(1+\alpha)} \right\} = x^\alpha C_\alpha.$$

C_α tends to 0 as t goes to infinity and the supremum is obtained.

3 Sharp Function

Sharp function is a very powerful tool that mediates between L^p spaces and the space of bounded mean oscillations, i.e., BMO_p.

Definition 3.1. For locally integrable function f on $\mathbb{R}^n$ the sharp function is given by

$$f^\#(x) = \sup_{Q: x \in Q} \left\{ \frac{1}{\lambda(Q)} \int_Q |f(y) - f_Q|^p dy \right\}^{1/p} < \infty.$$

Thus, $f \in BMO$ is identical with the statement $f^\# \in L^\infty$. The interest in $f^\#$ lies in the fact that $f^\# \in L^p, p < \infty$ implies that $f \in L^p$.

Proposition 3.1. *If $f^\#$ is a function in $L^p(\mathbb{R}^n)$, and $1 \leqslant p < \infty$, then*

$$d_{f^\#}(t) \leqslant t^{-p} \|f^\#(x)\|_p^p \tag{3.1}$$

and

$$\|f^\#(x)\|_p^p = p \int_0^\infty t^{p-1} d_{f^\#}(t) dt. \tag{3.2}$$

Proof. By the definition of the distribution

$$\begin{aligned} d_{f^\#}(t) &= \lambda(\{x \in \mathbb{R}^n : |f^\#(x)| > t\}) \\ &= \int_{\{x\in\mathbb{R}^n:|f^\#(x)|>t\}} dx \end{aligned} \tag{3.3}$$

or,

$$\begin{aligned} t^p d_{f^\#}(t) &= \int_{\{x\in\mathbb{R}^n:|f^\#(x)|>t\}} t^p dx \\ &\leqslant \int_{\mathbb{R}^n} |f^\#(x)|^p dx. \end{aligned} \tag{3.4}$$

Since $f^\#$ is integrable on $\mathbb{R}^n$, then we have

$$t^p d_{f^\#}(t) \leqslant \|f^\#(x)\|_p^p$$

$$d_{f^\#}(t) \leqslant t^{-p}\|f^\#(x)\|_p^p.$$

For the remaining part we define the set $A \subset \mathbb{R}^n \times [0,\infty)$ as $A = \{(x,s) : s < |f^\#(x)|^p\}$. Using Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |f^\#(x)|^p dx &= \int_{\mathbb{R}^n}\int_0^{|f^\#(x)|^p} \mathbf{1} ds dx \\ &= \int_{\mathbb{R}^n\times[0,\infty)} \mathbf{1_A}(x,s) ds dx \\ &= \int_0^\infty \lambda(\{x : |f^\#(x)|^p > s\}) ds. \end{aligned}$$

Putting $s = t^p \Rightarrow ds = pt^{p-1}dt$ in the above integral, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f^\#(x)|^p dx &= \int_0^\infty pt^{p-1}\lambda(\{x : |f^\#(x)|^p > t^p\}) dt \\ &= p\int_0^\infty t^{p-1}\lambda(\{x : |f^\#(x)| > t\}) dt. \end{aligned} \tag{3.5}$$

Therefore,

$$\|f^\#(x)\|_p^p = p\int_0^\infty t^{p-1} d_{f^\#}(t) dt.$$

□

Definition 3.2. An operator T defined on some class of measurable functions and mapping it into measurable functions is called sublinear if

$$|T(f+g)(x)| \leqslant |T(f)(x)| + |T(g)(x)| \;\; a.e.$$

and

$$|T(\lambda f)(x)| \leqslant |\lambda||T(f)(x)|, \;\; a.e.,$$

for all admissible functions f, g and all scalars λ.

Definition 3.3. A sublinear operator T from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ is said to be of weak type (p,q), $1 \leq p < \infty, q < \infty$ if there exists a constant C such that for each $f \in L^p(\mathbb{R}^n)$ and each $t > 0$ we have

$$\lambda(\{x : |T(f)(x)| > t\}) \leqslant \left(\frac{C\|f\|_p}{t}\right)^q.$$

It is clear that each linear operator is sublinear. It immediately follows from (3.1) that if a sublinear operator T satisfies

$$\|T(f)\|_1 \leqslant C\|f\|_1,$$

then T is of weak type (1,1).

Theorem 3.1. [Marcinkiewicz] *Suppose T is a sublinear operator defined on $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ which is of weak type (1,1) and for some C satisfies*

$$\|Tf^{\#}\|_\infty \leqslant C\|f^{\#}\|_\infty, \tag{3.6}$$

then for each $p, 1 < p < \infty$, there exists a constant $C(p)$ such that

$$\|Tf^{\#}\|_p \leqslant C(p)\|f^{\#}\|_p, \; where \; C(p) = \frac{2^p p}{(p-1)}.$$

Proof. We are given that T is a sublinear operator defined on $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and is of weak type (1,1) then by the definition of T we can say without the loss of generality that for $C = 1$ and $t > 0$,

$$\lambda(\{x : |Tf^{\#}(x)| > t\}) \leqslant \frac{\|f^{\#}\|_1}{t}.$$

And also T satisfies (3.6), i.e.,

$$\|Tf^{\#}\|_\infty \leqslant C\|f^{\#}\|_\infty$$

or,

$$\|Tf^{\#}\|_\infty \leqslant \|f^{\#}\|_\infty.$$

Now for $f^{\#} \in L^p(\mathbb{R}^n)$ and given $t > 0$, we write $f^{\#}(x) = f_t(x) + f^t(x)$, where $f_t(x) \in L^1(\mathbb{R}^n)$, $f^t(x) \in L^\infty(\mathbb{R}^n)$ and

$$f^t(x) = \begin{cases} f^{\#}(x) & x \geq t \\ 0 & x < t, \end{cases}$$

$$f_t(x) = \begin{cases} f^{\#}(x) & x \leq t \\ 0 & x > t. \end{cases}$$

Thus $f_t(x) = \mathbf{1}_{\{s \,:\, |f^{\#}(s)| \leqslant t\}} f^{\#}(x)$. Since $f^{\#}$ is a measurable function and T is a sublinear operator, therefore $Tf^{\#}$ is also a measurable function. Then by countable subadditivity, we have

$$\left\{x : |Tf^{\#}(x)| > t\right\} \subset \left\{x : |Tf_{t/2}(x)| > \frac{t}{2}\right\} \cup \left\{x : |Tf^{t/2}(x)| > \frac{t}{2}\right\}.$$

So from (3.2), we obtain

$$\begin{aligned} \|Tf^{\#}\|_p^p &= \int_{\mathbb{R}^n} |Tf^{\#}(x)|^p dx \\ &= p\int_0^\infty t^{p-1}\lambda(\{x : |Tf^{\#}(x)| > t\})dt \end{aligned} \tag{3.7}$$

or,

$$\|Tf^{\#}\|_p^p \leqslant p\int_0^\infty t^{p-1}\lambda\left(\left\{x : |Tf_{\frac{t}{2}}(x)| > \frac{t}{2}\right\}\right)dt + p\int_0^\infty t^{p-1}\lambda\left(\left\{x : |Tf^{t/2}(x)| > \frac{t}{2}\right\}\right)dt.$$

But we suppose that T satisfies (3.6) with $C = 1$, and this implies that $\|Tf_{t/2}\|_\infty \leqslant \frac{t}{2}$ and thus, the second integral is zero. Then by the definition of sublinear operator and (3.2) we get,

$$
\begin{aligned}
\|Tf^{\#}\|_p^p &\leq p\int_0^{\infty} t^{p-1}\frac{2}{t}\|f_{t/2}\|_1 dt \\
&= 2p\int_0^{\infty} t^{p-2}\int_{\{x:|f^{\#}(x)|>\frac{t}{2}\}} |f^{\#}(x)|dxdt \\
&= 2p\int_{\mathbb{R}^n} |f^{\#}(x)|\int_0^{2|f^{\#}(x)|} t^{p-2}dtdx \\
&= 2p\int_{\mathbb{R}^n} |f^{\#}(x)|\frac{1}{p-1}(2|f^{\#}(x)|)^{p-1}dx \\
&\leq \frac{2^p p}{(p-1)}\int_{\mathbb{R}^n} |f^{\#}(x)|^p dx \\
&\leq C(p)\|f^{\#}\|_p \quad 1 \leq p < \infty.
\end{aligned}
$$

□

The relationship between weak (p, q) inequalities and the almost everywhere convergence is given by the following result.

Theorem 3.2. *Let $\{T_t\}$ be a family of linear operators on $L^p(\mathbb{R}^n); 1 \leq p < \infty$ and define*

$$T^* f^{\#}(x) = \sup_t |T_t f^{\#}(x)|.$$

If T^* is weak (p, q) then the set

$$\{f^{\#} \in L^p(\mathbb{R}^n) : \lim_{t\to t_0} T_t f^{\#}(x) = f^{\#}(x) \text{ a.e.}\}$$

is closed in $L^p(\mathbb{R}^n)$.

Proof. Let $\{f_n\}$ be a sequence of functions which converges to $f^{\#}$ in $L^p(\mathbb{R}^n)$ such that $Tf^{\#}(x)$ converges to $f^{\#}(x)$ a.e.

Now, $\mu(\{f^{\#} \in L^p(\mathbb{R}^n) : \limsup_{t\to t_0}|T_t f^{\#}(x) - f^{\#}(x)| > \lambda\})$

$$
\begin{aligned}
&= \mu(\{f^{\#} \in L^p(\mathbb{R}^n) : \limsup_{t\to t_0}|T_t f^{\#}(x) - f_n(x) + f_n(x) - f^{\#}(x)| > \lambda\}) \\
&\leq \mu(\{f^{\#} \in L^p(\mathbb{R}^n) : \limsup_{t\to t_0}|T_t(f^{\#} - f_n)(x)| > \frac{\lambda}{2}\}) \\
&\qquad +\mu(\{f^{\#} \in L^p(\mathbb{R}^n) : \limsup_{t\to t_0}|(f^{\#} - f_n)(x)| > \frac{\lambda}{2}\}) \\
&\leq \mu(\{f^{\#} \in L^p(\mathbb{R}^n) : |T^*(f^{\#} - f_n)(x)| > \frac{\lambda}{2}\}) \\
&\qquad +\mu(\{f^{\#} \in L^p(\mathbb{R}^n) : |f^{\#}(x) - f_n(x)| > \frac{\lambda}{2}\}) \\
&\leq \left(\frac{2C(p)}{\lambda}\|f^{\#}(x) - f_n(x)\|_p\right)^q + \left(\frac{2}{\lambda}\|f^{\#}(x) - f_n(x)\|_p\right)^p.
\end{aligned}
$$

Since T^* is weak (p, q) and $f_n \to f^{\#}$ as $n \to \infty$, therefore

$$
\begin{aligned}
&\mu(\{f^{\#} \in L^p(\mathbb{R}^n) : \limsup_{t\to t_0}|T_t f^{\#}(x) - f^{\#}(x)| > 0\}) \\
&\qquad \leq \sum_{k=1}^{\infty}\mu(\{f^{\#} \in L^p(\mathbb{R}^n) : \limsup_{t\to t_0}|T_t f^{\#}(x) - f^{\#}(x)| > \frac{1}{k}\}) \\
&\qquad = 0.
\end{aligned}
$$

4 Conclusions

The present study aimed at the generalization of the distribution function and the relation of distribution function with the norm of sharp function. Also, we have proved an interpolation theorem for sharp function and discussed the almost everywhere convergence of sharp function by means of the weak (p, q) operator.

□

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Competing Interests

The authors declare that no competing interests exist.

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