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A cubically convergent class of root finding iterative methods

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In this paper, we propose a new two-parameter class of iterative methods to solve a nonlinear equation. It is proved that any method in this class is cubically convergent if and only if the parameters sum up to one. Some of the existing third-order methods, by suitable selection of parameters, can be put in this class. Every iteration of the class requires an evaluation of the function, three of the first derivative, and none of the second derivative. Hence, its efficiency index is $3^{1/4} = 1.316$ that is worse than all other cubically convergent methods considered. However, numerical experiments show that a special method in our class is comparable to those in terms of iterations number.

Key words: Nonlinear equations, root finding, iterative method, third-order convergence.

INTRODUCTION

One of the most basic problems in numerical analysis (and of the oldest numerical approximation problems) is finding values of the variable x, say α, which satisfy $f(x) = 0$ for a given function f. There are numerous methods for solving the nonlinear equation $f(x) = 0$, $f: I \mapsto R$ is a scalar function and I is an open interval containing the root α. Most of these methods have fixed point style: they transform the equation $f(x) = 0$ to the $x = \varphi(x)$ in such a way that α is a fixed point of φ, namely $\alpha = \varphi(\alpha)$. With an initial approximation x_0 to the α, these methods generate the sequence $\{x_n\}$, in which $x_{n+1} = \varphi(x_n)$, $n \geq 0$. It is obvious that if φ is continuous and the sequence $\{x_n\}$ is convergent, then $x_n \longrightarrow \alpha$.

The Newton method is the most popular method for solving such equations. Some historical points about this method can be found in Yamamoto (2000).

In recent years, a number of authors have considered methods to solve the nonlinear equations. These techniques calculate the new approximation to a zero of the given function by sampling per iteration of the function and possibly its derivatives for a number of values of the independent variables. All these techniques are variants of Newton's method and the main practical difficulty associated with these techniques is that they fail miserably if at any stage of computation, the derivative of the function is either zero or very small in the vicinity of the required root, [For example, Ujević et al. (2007) and references therein]. It is known that some of these methods can be obtained using Taylor or interpolation polynomials.

In this paper, we focus on the third-order convergence methods that do not use any second derivatives. We propose a class of such methods, containing two parameters, and show that some of existing methods can be put in our class.

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BASIC FACTS

Definition 1

Let $f(x)$ be a real function with a simple root α and let $\{x_n\}_{n\geq 0}$ be a sequence of real numbers that converges towards α. Then, we say that the order of convergence of the sequence is p, if there exist a real $p \geq 1$ such that:

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C$$

for some $C \neq 0$. C is known as the asymptotic error constant.

If $p = 1$, 2, or 3, the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Definition 2

Let $e_n = x_n - \alpha$ be the error in the n-th iteration. We call the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1})$$

as the error equation.

If we can obtain the error equation for any iterative method, then the value of p is its order of convergence.

Definition 3

Let r be the number of new pieces of information required by a method. A "piece of information" typically is any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index (Gautschi, 1997) and is defined by

$$\rho = p^{1/r}$$

Where p is the order of the method.

As mentioned before, most of the methods used to solve $f(x) = 0$ have fixed point structure, say $x_{n+1} = \varphi(x_n)$, $n \geq 0$. Using the p-th order Taylor series of $x_{n+1} - \alpha = \varphi(x_n) - \varphi(\alpha)$about α, it is easy to prove the following theorem.

Theorem 4

Let sequence $x_{n+1} = \varphi(x_n), n \geq 0$, be convergent to the fixed point α of φ. If

$$\varphi'(\alpha) = \varphi''(\alpha) = \cdots = \varphi^{(p-1)}(\alpha) = 0, \qquad \varphi^{(p)}(\alpha) \neq 0,$$

then the sequence $\{x_n\}$ is convergent of order p with asymptotic error constant $C = |\varphi^{(p)}(\alpha)|/p!$ (Gautschi, 1997).

THE NEW CLASS

Recently, there have been some developed new modifications for Newton method with third-order convergence (Chun, 2005, 2006, 2007, 2008; Chun and Kim 2010; Forntini and Sormani, 2003a, b; Homeier, 2003, 2005; Jian, 2007; Jisheng et al., 2007; Kou et al., 2006; Özban, 2004; Potra and Pták, 1984; Ujević et al., 2007; Weerakoon and Fernando, 2000), almost all of which are based on the computation of the integral

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt, \tag{1}$$

arising from Newton's theorem, using different quadrature formulae. For example, Weerakoon and Fernando (2000) re-derived the Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

by approximating the integral by the rectangular rule according to

$$\int_{x_n}^{x} f'(t)dt \approx (x - x_n)f'(x_n)$$

and using $f(x) = 0$. It is well known that the Newton method is quadratically convergent with error equation

$$e_{n+1} = c_2 e_n^2 + O(e_n^3),$$

In which

$$c_k = \frac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}.$$

When they used the trapezoidal approximation

$$\int_{x_n}^{x} f'(t)dt \approx (x - x_n)(f'(x_n) + f'(x))/2$$

In combination with the approximation $f'(x) \approx f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$ and $f(x) = 0$, they arrived at the modified Newton-type iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}, \tag{3}$$

and proved that this scheme converges cubically in some neighborhood of α. Its error equation is

$$e_{n+1} = (6c_2^2 + 3c_3)e_n^3 + O(e_n^4).$$

Frontini and Sormani (2003a, b) considered the midpoint rule

$$\int_{x_n}^{x} f'(t)dt \approx (x - x_n) f'\left(\frac{x + x_n}{2}\right)$$

and arrived analogously at a further modified Newton-type iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)}. \tag{4}$$

This scheme has also been derived in Homeier (2003 2005) by requiring that the iteration function

$$\Phi_f(x) = x - \frac{f(x)}{f'(x + h(x)f(x))}$$

satisfies $\Phi_f(\alpha) = \alpha$, $\Phi_f'(\alpha) = \Phi_f''(\alpha) = 0$. Hence, $h(\alpha) = -\left(2f'(\alpha)\right)^{-1}$ follows. This is satisfied for $h(x) = -(2f'(x))^{-1} + g(x)f(x)$. The Scheme (4) is obtained for the special case $h(x) = -(2f'(x))^{-1}$ and using $x_{n+1} = \Phi_f(x_n)$. As the modified Newton-type method of Weerakoon and Fernando (Equation 3), Scheme (4) converges cubically in some neighborhood of α. Its error equation is

$$e_{n+1} = (6c_2^2 - 1.5c_3)e_n^3 + O(e_n^4).$$

Frontini and Sormani (2003b) also proved that every interpolatory quadrature formula of order higher than zero give a cubically convergent modification of Newton method.

Paying attention to methods of Equations 2, 3, and 4, we consider the class of iterative methods

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'\left(x_n - a\frac{f(x_n)}{f'(x_n)}\right) + f'\left(x_n - b\frac{f(x_n)}{f'(x_n)}\right)}, \tag{5}$$

in which a and b are parameters.

It is noticed that the choice $a = b = 0$ give the Newton method (2), $a = 0$ and $b = 1$ gives the Weerakoon and Fernando method of Equation 3, whereas $a = b = 1/2$ gives the Frontini and Sormani method of Equation 4. Also, whenever $a + b = 1$, for the special case $f(x) = x^2 - s$, after some calculations, we have

$$x_{n+1} = \frac{x_n(x_n^2 + 3s)}{3x_n^2 + s}$$

That is Halley method for computing $\alpha = \sqrt{s}$.

We notice that every iteration of the method of Equation 5 requires one evaluation of the function and three of the first derivative f'. Hence, its efficiency index is $3^{1/4} = 1.316$. On the other hand, the efficiency index of Newton method of Equation 2 is $2^{1/2} = 1.414$, and that of the methods of Equations 3 and 4 is $3^{1/3} = 1.442$. Although the index efficiency of our method is worse than that of all methods of Equations 2, 3, and 4, numerical experiments show that it is comparable to those in terms of the number of iterations.

In the following theorem, we prove that the method (5) has third-order convergence.

Theorem 5

Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f: I \mapsto R$ for an open interval I. If x_0 is sufficiently close to α, then the method defined by Equation 5 has third-order convergence if and only if $a + b = 1$, satisfying the error equation

$$e_{n+1} = (c_2^2 + (1.5(a^2 + b^2) - 1)c_3)e_n^3 + O(e_n^4),$$

where $e_n = x_n - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$.

Proof

Using Taylor expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)) \tag{6}$$

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)\right). \tag{7}$$

Dividing Equation 6 by Equation 7 gives

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{8}$$

Let

$$z_n = x_n - a\frac{f(x_n)}{f'(x_n)}, \qquad w_n = x_n - b\frac{f(x_n)}{f'(x_n)}.$$

Using Equation 8 and some simplifications, we get

$$z_n = \alpha + b e_n + a c_2 e_n^2 - 2a(c_2^2 - c_3)e_n^3 + O(e_n^4) \tag{9}$$

$$w_n = \alpha + a e_n + b c_2 e_n^2 - 2b(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{10}$$

Now, using Equation 9 in the Taylor expansion of $f'(z_n)$ about α, we have that

$$f'(z_n) = f'(\alpha)\left(1 + 2c_2(z_n - \alpha) + 3c_3(z_n - \alpha)^2 + 4c_4(z_n - \alpha)^3 + O(e_n^4)\right)$$

$$= f'(\alpha)\left(1 + 2bc_2 e_n + (2ac_2^2 + 3b^2 c_3)e_n^2 + \left(6abc_2c_3 - 4ac_2(c_2^2 - c_3)\right)e_n^3 + O(e_n^4)\right).$$

In a similar way,

$$f'(w_n) = f'(\alpha)\left(1 + 2ac_2 e_n + (2bc_2^2 + 3a^2 c_3)e_n^2 + \left(6abc_2c_3 - 4bc_2(c_2^2 - c_3)\right)e_n^3 + O(e_n^4)\right).$$

Hence,

$$f'(z_n) + f'(w_n) = 2f'(\alpha)\left(1 + c_2 e_n + i e_n^2 + j e_n^3 + O(e_n^4)\right),$$

in which

$$i = c_2^2 + 1.5(a^2 + b^2)c_3, \qquad j = 6abc_2c_3 - 2c_2(c_2^2 - c_3).$$

Therefore,

$$\frac{2f'(\alpha)}{f'(z_n) + f'(w_n)} = 1 - c_2 e_n + (c_2^2 - i)e_n^2 - (j - 2ic_2 + c_2^3)e_n^3 + O(e_n^4),$$

and, by Equation 6,

$$\frac{2f(x_n)}{f'(z_n) + f'(w_n)} = e_n + (c_3 - i)e_n^3 + O(e_n^4). \tag{11}$$

In summary, Equations 11 and 5 result in

$$e_{n+1} = -(c_3 - i)e_n^3 + O(e_n^4)$$

$$= (c_2^2 + (1.5(a^2 + b^2) - 1)c_3)e_n^3 + O(e_n^4). \qquad \blacksquare$$

It has been shown that the Maple package can be successfully employed to re-derive error equations of iterative methods, that is, to find their order of convergence (Chun, 2005, 2006) for details). The method of Equation 5 in this case is found to be third-order convergent as shown in the following:

Let α be a simple zero of f. Consider the iteration function F defined by

$$F(x) = x - \frac{2f(x)}{f'\left(x - a\frac{f(x)}{f'(x)}\right) + f'\left(x - b\frac{f(x)}{f'(x)}\right)}.$$

According to Theorem 4, it is sufficient to show that

$$F(\alpha) = \alpha, \qquad F'(\alpha) = 0, \qquad F''(\alpha) = 0,$$
$$F'''(\alpha) = (1.5(a^2 + b^2) - 1)\frac{f'''(\alpha)}{f'(\alpha)} + 1.5\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2.$$

The computations of the above derivatives can be performed using mathematical software package Maple, one of computer algebra systems. To do that, we run the following Maple statements consecutively:

$> z := \mathrm{x} -> \mathrm{x} - \mathrm{af(x)} / \mathrm{D(f)(x)};$

$$z := x -> x - \frac{af(x)}{D(f)(x)}$$

$> w := \mathrm{x} -> \mathrm{x} - \mathrm{bf(x)} / \mathrm{D(f)(x)};$

$$w := x -> x - \frac{bf(x)}{D(f)(x)}$$

$> \mathrm{F} := \mathrm{x} -> \mathrm{x} - 2\mathrm{f(x)} / ((\mathrm{D(f)@z)(x)} + (\mathrm{D(f)@w)(x)};$

$$F := x -> x - \frac{2f(x)}{(D(f)@z)(x) + (D(f)@w)(x)}$$

$> algsubs(\mathrm{f}(\alpha) = 0, \mathrm{F}(\alpha));$

$$\alpha$$

$> algsubs(\mathrm{f}(\alpha) = 0, \mathrm{D(F)}(\alpha));$

$1 - a - b$

$> algsubs(\mathrm{f}(\alpha) = 0, (\mathrm{D}@@2)(\mathrm{F})(\alpha));$

0

$> algsubs(\mathrm{f}(\alpha) = 0, (\mathrm{D}@@3)(\mathrm{F})(\alpha));$

$$(1.5(a^2 + b^2) - 1)\frac{(D^{(3)})(f)(\alpha)}{D(f)(\alpha)} + 1.5\frac{(D^{(2)})(f)(\alpha)^2}{D(f)(\alpha)^2}$$

One special choice of parameters is $a = \frac{3+\sqrt{3}}{6}$ and $b = \frac{3-\sqrt{3}}{6}$. In this case, $1.5(a^2 + b^2) - 1 = 0$ and we obtain the following new modification of Newton method:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'\left(x_n - \frac{3+\sqrt{3}}{6}\frac{f(x_n)}{f'(x_n)}\right) + f'\left(x_n - \frac{3-\sqrt{3}}{6}\frac{f(x_n)}{f'(x_n)}\right)}. \quad (12)$$

This modification is cubically convergent and has the error equation

$$e_{n+1} = c_2^2 e_n^3 + O(e_n^4).$$

The method of Equation 12 is corresponding to the two-point Gauss-Legendre quadrature formula to approximate the right integral of of Equation 1. Comparing the error equations of the third-order methods of Equations 3, 4, 5, and 12, we can see that if $|f^{(k)}(x)| \le M_k$, $k = 2,3$, for all x in an neighborhood of the root α, then the method of Equation 12 is the best. Numerical methods show this fact, too.

NUMERICAL EXPERIMENTS

All computations were done using MATLAB 6.5 with format of long floating point arithmetics. We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criteria for computer programs: $|x_{n+1} - x_n| < \varepsilon$. So, when the stopping criterion is satisfied, $x^* := x_{n+1}$ is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\varepsilon = 10^{-15}$.

We present some numerical test results for various cubically convergent iterative schemes in Table 1. Compared were the methods of Newton (Equation 2), Weerakoon and Fernando (Equation 3), Frontini and Sormani (Equation 4), and our method (Equation 12) introduced in the present contribution. We used the following test functions and displayed the approximate zeros x^* found up to the 20 digits.

$$f_1(x) = x^3 + 4x^2 - 10 \qquad f_2(x) = \sin^2 x - x^2 + 1$$
$$f_3(x) = x^2 - e^x - 3x + 2 \qquad f_4(x) = \cos x - x$$
$$f_5(x) = (x-1)^3 - 1 \qquad f_6(x) = \sin x - x/2$$
$$f_7(x) = xe^{x^2}x - \sin^2 x + 3\cos x + 5 \qquad f_8(x) = e^{x^2+7x-30} - 1$$
$$f_9(x) = x^4 + 9x^3 + 11x^2 + 19x - 41$$
$$f_{10}(x) = \frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6$$

As convergence criterion, it is required that the distance of two consecutive approximation for the zero is less than 10^{-15}. Also, displayed is the number of iterations to approximate the zero (IT) and the value of $|f(x^*)|$.

The test results in Table 1 show that for most of the functions we tested, our method (Equation 12) has at least equal performance compared to the other third-order methods, and can also compete with Newton method (Equation 2).

CONCLUSIONS

In this paper, we proposed a new cubically convergent class of modifications for Newton method. It is shown that, by suitable selection of parameters, some methods can be obtained from this class. Every iteration of the class requires one evaluation of the function f, three of the derivative f', no evaluation of the second derivative f''. Hence, its efficiency index is $3^{1/4} = 1.316$ that is

Table 1. Comparison of various cubically convergent methods and the Newton's method.

	Method	x^*	IT	$\|f(x^*)\|$
$f_1, x_0 = -0.3$	(2)	1.3652300134140968879	53	0.70e-15
	(3)	1.3652300134140968879	6	0.70e-15
	(4)	1.3652300134140968879	18	0.70e-15
	(12)	1.3652300134140968879	4	0.70e-15
$f_2, x_0 = 3.5$	(2)	1.4044916482153411152	6	0.28e-15
	(3)	1.4044916482153411152	4	0.28e-15
	(4)	1.4044916482153413373	4	0.28e-15
	(12)	1.4044916482153413373	4	0.28e-15
$f_3, x_0 = -1.0$	(2)	0.25753028543986078436	5	0.90e-16
	(3)	0.25753028543986078436	3	0.90e-16
	(4)	0.25753028543986072885	3	0.12e-15
	(12)	0.25753028543986072885	3	0.12e-15
$f_4, x_0 = 3.5$	(2)	0.73908513321516067229	247	0.51e-16
	(3)	0.73908513321516067229	8	0.51e-16
	(4)	0.73908513321516067229	5	0.51e-16
	(12)	0.73908513321516067229	5	0.51e-16
$f_5, x_0 = 0.5$	(2)	2	1	0
	(3)	2	65	0
	(4)	2	7	0
	(12)	2	1	0
$f_6, x_0 = 2.5$	(2)	1.8954942670339809396	5	0.62e-17
	(3)	1.8954942670339809396	3	0.62e-17
	(4)	1.8954942670339809396	3	0.62e-17
	(12)	1.8954942670339809396	3	0.62e-17
$f_7, x_0 = -2.0$	(2)	-1.2076478271309187829	8	0.29e-14
	(3)	-1.2076478271309187829	6	0.29e-14
	(4)	-1.2076478271309187829	5	0.29e-14
	(12)	-1.2076478271309187829	5	0.29e-14
$f_8, x_0 = 5.0$	(2)	3	35	0
	(3)	3	24	0
	(4)	3	21	0
	(12)	3	22	0
$f_9, x_0 = 0$	(2)	1.0137725000771651285	7	0.45e-14
	(3)	1.0137725000771651285	5	0.45e-14
	(4)	1.0137725000771651285	5	0.45e-14
	(12)	1.0137725000771651285	4	0.45e-14
$f_{10}, x_0 = -0.4$	(2)	-0.13161801809960649301	8	0
	(3)	-0.13161801809960649301	7	0
	(4)	-0.13161801809960646525	5	0.89e-15
	(12)	-0.13161801809960649301	5	0

worse than all other cubically convergent methods considered. However, numerical experiments show that a special method in the class is comparable to those in terms of iterations number.

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