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On the Simulation of Higher Order Linear Block Algorithm for Modelling Fourth Order Initial Value Problems

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The introduction of new linear block method for the direct simulation of fourth order IVPs has been developed in this article. The reason for adopting direct simulation of fourth order initial value problems is to address some setbacks in reduction method. When developing the method, we adopted the linear block approach through a one step method. We have validated the accuracy of the method on some fourth order initial value problems without reduction process, and the results are better than the conventional method. The numerical experiments were given and the results obtained were found to be better in accuracy than the existing methods in literature.

Keywords: Linear block Approach; IVPs; fourth order; simulation; numerical experiment.

1 Introduction

Many experimental works and studies in science, engineering, medicine and management are coded in numbers and mathematical symbols to form equations in order to have meaning, construct and application. One of such equations is known as differential equation. This equation may evolve from many physical systems in sciences, technology, economics, and other areas that involve rate of change of a given variable in the structure (system) with respect to another. These equations came to the limelight with the independent invention of infinitesimal

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calculus by English Mathematician, Isaac Newton (1642-1727) and a German Mathematician, Gottfried Wilhelm Leibniz (1646-1716) in the late 17th century.

This work proposes the derivation and implementation of a new numerical method for direct approximation initial value problems (IVPs) of the fourth order ordinary differential equations (ODEs) of the form,

$$y^{iv}(x)= f(x, y, y', y'', y'''), y(x_0)=\tau_1, y'(x_0)=\tau_2, y''(x_0)=\tau_3, y'''(x_0)=\tau_4 \quad (1)$$

Traditionally, the numerical approximation of (1) is to reduce it to a system of first order ODEs and then solved with the adequate methods in existence. Block methods for the solution of higher order ordinary differential equations (1) have become a general approach in existence due to the shortcomings experienced by the initial numerical approaches for solving higher order ODEs [1].

These initial approaches include the approach of reducing the higher order ODEs to a system of first order ODEs and then suitable numerical methods such as Simpsons method, Euler's method, Trapezoidal rule, Runge Kutta methods, amongst others are then applied to solve the system (1), [2–5]. The process of reduction is able to obtain the required solution, but, there is too much rigor involved in the process of reduction, wastage of human effort and also the time constraint. Other initial numerical approaches that bypassed the reduction stress were a direct approximation of the higher order ODEs using Taylor series expansions or predictor-corrector methods [6, 7].

Numerically, solving to fourth order ODEs around initial value problems (IVPs) or boundary value problems (BVPs) have been considered by authors in literature. These includes the works of [1] adopted the use of power series to developed the method for direct solution to fourth order IVPs. [8], proposed a numerical solution for fourth-order initial value problems using lucas polynomial with application in ship dynamics. The numerical approximations of (1) have been considered in literature by authors such as [9-11].

However, what motivation us to introduce the methods with improved accuracy informs the development of the implicit block method in this article, to develop the essential block method, a fresh approach coined as the linear block approach (LBA) is adopted. A detail of the algorithm of the LBA and its adoption to develop the required implicit block method is discussed in the next section. Hence, the aim of this article is to simulate higher order linear block algorithm on fourth order IVPs using a LBA.

2 Mathematical Algorithm of the Methodology

This section describes the derivation of the implicit block method using LBA. The following algorithm shows the steps involved in developing the specific block method. LBA follows the concept of considering the general form of the block method while implementing a step-by-step wise mode to obtain the expected block method for solving fourth order ODEs (1.1) [4].
The first algorithm

Obtain the block method from the given expression

$$y_{n+\xi} = \sum_{j=0}^{3} \frac{(\xi h)^j}{j!} y_n^{(j)} + \sum_{j=0}^{5} (\psi_{i\xi} f_{n+j}), \quad \xi = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \quad (2)$$

The second algorithm

Obtain the first, second and third derivative schemes of the block method from

$$y_{n+\xi}^{(q)} = \sum_{j=0}^{3-q} \frac{(\xi h)^j}{j!} y_n^{(j+q)} + \sum_{i=0}^{6} \beta_{\xi jq} f_{n+j}, q = 1_{\left(\xi=\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},1\right)}, q = 2_{\left(\xi=\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},1\right)}, q = 3_{\left(\xi=\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},1\right)} \quad (3)$$

$\psi_{\xi j} = A^{-1}M$ and $\beta_{\xi jq} = A^{-1}N$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{h}{6} & \frac{h}{3} & \frac{h}{2} & \frac{2h}{3} & \frac{5h}{6} & h \\ 0 & \frac{\left(\frac{h}{6}\right)^2}{2!} & \frac{\left(\frac{h}{3}\right)^2}{2!} & \frac{\left(\frac{h}{2}\right)^2}{2!} & \frac{\left(\frac{2h}{3}\right)^2}{2!} & \frac{\left(\frac{5h}{6}\right)^2}{2!} & \frac{(h)^2}{2!} \\ 0 & \frac{\left(\frac{h}{6}\right)^3}{3!} & \frac{\left(\frac{h}{3}\right)^3}{3!} & \frac{\left(\frac{h}{2}\right)^3}{3!} & \frac{\left(\frac{2h}{3}\right)^3}{3!} & \frac{\left(\frac{5h}{6}\right)^3}{3!} & \frac{(h)^3}{3!} \\ 0 & \frac{\left(\frac{h}{6}\right)^4}{4!} & \frac{\left(\frac{h}{3}\right)^4}{4!} & \frac{\left(\frac{h}{2}\right)^4}{4!} & \frac{\left(\frac{2h}{3}\right)^4}{4!} & \frac{\left(\frac{5h}{6}\right)^4}{4!} & \frac{(h)^4}{4!} \\ 0 & \frac{\left(\frac{h}{6}\right)^5}{5!} & \frac{\left(\frac{h}{3}\right)^5}{5!} & \frac{\left(\frac{h}{2}\right)^5}{5!} & \frac{\left(\frac{2h}{3}\right)^5}{5!} & \frac{\left(\frac{5h}{6}\right)^5}{5!} & \frac{(h)^5}{5!} \\ 0 & \frac{\left(\frac{h}{6}\right)^6}{6!} & \frac{\left(\frac{h}{3}\right)^6}{6!} & \frac{\left(\frac{h}{2}\right)^6}{6!} & \frac{\left(\frac{2h}{3}\right)^6}{6!} & \frac{\left(\frac{5h}{6}\right)^6}{6!} & \frac{(h)^6}{6!} \end{pmatrix}, M = \begin{pmatrix} \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \\ \frac{(\xi h)^{10}}{10!} \end{pmatrix}, N = \begin{pmatrix} \frac{(\xi h)^{4-q}}{(4-q)!} \\ \frac{(\xi h)^{5-q}}{(5-q)!} \\ \frac{(\xi h)^{6-q}}{(6-q)!} \\ \frac{(\xi h)^{7-q}}{(7-q)!} \\ \frac{(\xi h)^{8-q}}{(8-q)!} \\ \frac{(\xi h)^{9-q}}{(9-q)!} \\ \frac{(\xi h)^{10-q}}{(10-q)!} \end{pmatrix}$$

To implement the first Algorithm, (2) takes the form

$$\left.\begin{aligned}
y_{n+\frac{1}{6}} &= y_n + \frac{h}{6}y'_n + \frac{\left(\frac{h}{6}\right)^2}{2!}y''_n + \frac{\left(\frac{h}{6}\right)^3}{3!}y'''_n + h^4\left(\psi_{10}f_n + \psi_{11}f_{n+\frac{1}{6}} + \psi_{12}f_{n+\frac{1}{3}} + \psi_{13}f_{n+\frac{1}{2}} + \psi_{14}f_{n+\frac{2}{3}} + \psi_{15}f_{n+\frac{5}{6}} + \psi_{16}f_{n+1}\right) \\
y_{n+\frac{1}{3}} &= y_n + \frac{h}{3}y'_n + \frac{\left(\frac{h}{3}\right)^2}{2!}y''_n + \frac{\left(\frac{h}{3}\right)^3}{3!}y'''_n + h^4\left(\psi_{20}f_n + \psi_{21}f_{n+\frac{1}{6}} + \psi_{22}f_{n+\frac{1}{3}} + \psi_{23}f_{n+\frac{1}{2}} + \psi_{24}f_{n+\frac{2}{3}} + \psi_{25}f_{n+\frac{5}{6}} + \psi_{26}f_{n+1}\right) \\
y_{n+\frac{1}{2}} &= y_n + \frac{h}{2}y'_n + \frac{\left(\frac{h}{2}\right)^2}{2!}y''_n + \frac{\left(\frac{h}{2}\right)^3}{3!}y'''_n + h^4\left(\psi_{30}f_n + \psi_{31}f_{n+\frac{1}{6}} + \psi_{32}f_{n+\frac{1}{3}} + \psi_{33}f_{n+\frac{1}{2}} + \psi_{34}f_{n+\frac{2}{3}} + \psi_{35}f_{n+\frac{5}{6}} + \psi_{36}f_{n+1}\right) \\
y_{n+\frac{2}{3}} &= y_n + \frac{2h}{3}y'_n + \frac{\left(\frac{2h}{3}\right)^2}{2!}y''_n + \frac{\left(\frac{2h}{3}\right)^3}{3!}y'''_n + h^4\left(\psi_{40}f_n + \psi_{41}f_{n+\frac{1}{6}} + \psi_{42}f_{n+\frac{1}{3}} + \psi_{43}f_{n+\frac{1}{2}} + \psi_{44}f_{n+\frac{2}{3}} + \psi_{45}f_{n+\frac{5}{6}} + \psi_{46}f_{n+1}\right) \\
y_{n+\frac{5}{6}} &= y_n + \frac{5h}{6}y'_n + \frac{\left(\frac{5h}{6}\right)^2}{2!}y''_n + \frac{\left(\frac{5h}{6}\right)^3}{3!}y'''_n + h^4\left(\psi_{50}f_n + \psi_{51}f_{n+\frac{1}{6}} + \psi_{52}f_{n+\frac{1}{3}} + \psi_{53}f_{n+\frac{1}{2}} + \psi_{54}f_{n+\frac{2}{3}} + \psi_{55}f_{n+\frac{5}{6}} + \psi_{56}f_{n+1}\right) \\
y_{n+1} &= y_n + hy'_n + \frac{(h)^2}{2!}y''_n + \frac{(h)^3}{3!}y'''_n + h^4\left(\psi_{60}f_n + \psi_{61}f_{n+\frac{1}{6}} + \psi_{62}f_{n+\frac{1}{3}} + \psi_{63}f_{n+\frac{1}{2}} + \psi_{64}f_{n+\frac{2}{3}} + \psi_{65}f_{n+\frac{5}{6}} + \psi_{66}f_{n+1}\right)
\end{aligned}\right\} \quad (4)$$

Now consider the second Algorithm, (3) takes the form

$$\left.\begin{aligned}
y'_{n+\frac{1}{6}} &= y'_n + \frac{h}{6}y''_n + \frac{\left(\frac{h}{6}\right)^2}{2!}y'''_n + h^3\left(\beta_{101}f_n + \beta_{111}f_{n+\frac{1}{6}} + \beta_{121}f_{n+\frac{1}{3}} + \beta_{131}f_{n+\frac{1}{2}} + \beta_{141}f_{n+\frac{2}{3}} + \beta_{151}f_{n+\frac{5}{6}} + \beta_{161}f_{n+1}\right)\\
y'_{n+\frac{1}{3}} &= y'_n + \frac{h}{3}y''_n + \frac{\left(\frac{h}{3}\right)^2}{2!}y'''_n + h^3\left(\beta_{201}f_n + \beta_{211}f_{n+\frac{1}{6}} + \beta_{221}f_{n+\frac{1}{3}} + \beta_{231}f_{n+\frac{1}{2}} + \beta_{241}f_{n+\frac{2}{3}} + \beta_{251}f_{n+\frac{5}{6}} + \beta_{261}f_{n+1}\right)\\
y'_{n+\frac{1}{2}} &= y'_n + \frac{h}{2}y''_n + \frac{\left(\frac{h}{2}\right)^2}{2!}y'''_n + h^3\left(\beta_{301}f_n + \beta_{311}f_{n+\frac{1}{6}} + \beta_{321}f_{n+\frac{1}{3}} + \beta_{331}f_{n+\frac{1}{2}} + \beta_{341}f_{n+\frac{2}{3}} + \beta_{351}f_{n+\frac{5}{6}} + \beta_{361}f_{n+1}\right)\\
y'_{n+\frac{2}{3}} &= y'_n + \frac{2h}{3}y''_n + \frac{\left(\frac{2h}{3}\right)^2}{2!}y'''_n + h^3\left(\beta_{401}f_n + \beta_{411}f_{n+\frac{1}{6}} + \beta_{421}f_{n+\frac{1}{3}} + \beta_{431}f_{n+\frac{1}{2}} + \beta_{441}f_{n+\frac{2}{3}} + \beta_{451}f_{n+\frac{5}{6}} + \beta_{461}f_{n+1}\right)\\
y'_{n+\frac{5}{6}} &= y'_n + \frac{5h}{6}y''_n + \frac{\left(\frac{5h}{6}\right)^2}{2!}y'''_n + h^3\left(\beta_{501}f_n + \beta_{511}f_{n+\frac{1}{6}} + \beta_{521}f_{n+\frac{1}{3}} + \beta_{531}f_{n+\frac{1}{2}} + \beta_{541}f_{n+\frac{2}{3}} + \beta_{551}f_{n+\frac{5}{6}} + \beta_{561}f_{n+1}\right)\\
y'_{n+1} &= y'_n + h\,y''_n + \frac{(h)^2}{2!}y'''_n + h^3\left(\beta_{601}f_n + \beta_{611}f_{n+\frac{1}{6}} + \beta_{621}f_{n+\frac{1}{3}} + \beta_{631}f_{n+\frac{1}{2}} + \beta_{641}f_{n+\frac{2}{3}} + \beta_{651}f_{n+\frac{5}{6}} + \beta_{661}f_{n+1}\right)
\end{aligned}\right\} \tag{5}$$

And

$$\left.\begin{aligned}
y''_{n+\frac{1}{6}} &= y''_n + \frac{h}{6}y'''_n + h^2\left(\beta_{102}f_n + \beta_{112}f_{n+\frac{1}{6}} + \beta_{122}f_{n+\frac{1}{3}} + \beta_{132}f_{n+\frac{1}{2}} + \beta_{142}f_{n+\frac{2}{3}} + \beta_{152}f_{n+\frac{5}{6}} + \beta_{162}f_{n+1}\right)\\
y''_{n+\frac{1}{3}} &= y''_n + \frac{h}{3}y'''_n + h^2\left(\beta_{202}f_n + \beta_{212}f_{n+\frac{1}{6}} + \beta_{222}f_{n+\frac{1}{3}} + \beta_{232}f_{n+\frac{1}{2}} + \beta_{242}f_{n+\frac{2}{3}} + \beta_{252}f_{n+\frac{5}{6}} + \beta_{262}f_{n+1}\right)\\
y''_{n+\frac{1}{2}} &= y''_n + \frac{h}{2}y'''_n + h^2\left(\beta_{302}f_n + \beta_{312}f_{n+\frac{1}{6}} + \beta_{322}f_{n+\frac{1}{3}} + \beta_{332}f_{n+\frac{1}{2}} + \beta_{342}f_{n+\frac{2}{3}} + \beta_{352}f_{n+\frac{5}{6}} + \beta_{362}f_{n+1}\right)\\
y''_{n+\frac{2}{3}} &= y''_n + \frac{2h}{3}y'''_n + h^2\left(\beta_{402}f_n + \beta_{412}f_{n+\frac{1}{6}} + \beta_{422}f_{n+\frac{1}{3}} + \beta_{432}f_{n+\frac{1}{2}} + \beta_{442}f_{n+\frac{2}{3}} + \beta_{452}f_{n+\frac{5}{6}} + \beta_{462}f_{n+1}\right)\\
y''_{n+\frac{5}{6}} &= y''_n + \frac{5h}{6}y'''_n + h^2\left(\beta_{502}f_n + \beta_{512}f_{n+\frac{1}{6}} + \beta_{522}f_{n+\frac{1}{3}} + \beta_{532}f_{n+\frac{1}{2}} + \beta_{542}f_{n+\frac{2}{3}} + \beta_{552}f_{n+\frac{5}{6}} + \beta_{562}f_{n+1}\right)\\
y''_{n+1} &= y''_n + h\,y'''_n + h^2\left(\beta_{602}f_n + \beta_{612}f_{n+\frac{1}{6}} + \beta_{622}f_{n+\frac{1}{3}} + \beta_{632}f_{n+\frac{1}{2}} + \beta_{642}f_{n+\frac{2}{3}} + \beta_{652}f_{n+\frac{5}{6}} + \beta_{662}f_{n+1}\right)
\end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned}
y'''_{n+\frac{1}{6}} &= y'''_n + h\left(\beta_{103}f_n + \beta_{113}f_{n+\frac{1}{6}} + \beta_{123}f_{n+\frac{1}{3}} + \beta_{133}f_{n+\frac{1}{2}} + \beta_{143}f_{n+\frac{2}{3}} + \beta_{153}f_{n+\frac{5}{6}} + \beta_{163}f_{n+1}\right)\\
y'''_{n+\frac{1}{3}} &= y'''_n + h\left(\beta_{203}f_n + \beta_{213}f_{n+\frac{1}{6}} + \beta_{223}f_{n+\frac{1}{3}} + \beta_{233}f_{n+\frac{1}{2}} + \beta_{243}f_{n+\frac{2}{3}} + \beta_{253}f_{n+\frac{5}{6}} + \beta_{263}f_{n+1}\right)\\
y'''_{n+\frac{1}{2}} &= y'''_n + h\left(\beta_{303}f_n + \beta_{313}f_{n+\frac{1}{6}} + \beta_{323}f_{n+\frac{1}{3}} + \beta_{333}f_{n+\frac{1}{2}} + \beta_{343}f_{n+\frac{2}{3}} + \beta_{353}f_{n+\frac{5}{6}} + \beta_{363}f_{n+1}\right)\\
y'''_{n+\frac{2}{3}} &= y'''_n + h\left(\beta_{403}f_n + \beta_{413}f_{n+\frac{1}{6}} + \beta_{423}f_{n+\frac{1}{3}} + \beta_{433}f_{n+\frac{1}{2}} + \beta_{443}f_{n+\frac{2}{3}} + \beta_{453}f_{n+\frac{5}{6}} + \beta_{463}f_{n+1}\right)\\
y'''_{n+\frac{5}{6}} &= y'''_n + h\left(\beta_{503}f_n + \beta_{513}f_{n+\frac{1}{6}} + \beta_{523}f_{n+\frac{1}{3}} + \beta_{533}f_{n+\frac{1}{2}} + \beta_{543}f_{n+\frac{2}{3}} + \beta_{553}f_{n+\frac{5}{6}} + \beta_{563}f_{n+1}\right)\\
y'''_{n+1} &= y'''_n + h\left(\beta_{603}f_n + \beta_{613}f_{n+\frac{1}{6}} + \beta_{623}f_{n+\frac{1}{3}} + \beta_{633}f_{n+\frac{1}{2}} + \beta_{643}f_{n+\frac{2}{3}} + \beta_{653}f_{n+\frac{5}{6}} + \beta_{663}f_{n+1}\right)
\end{aligned}\right\} \tag{7}$$

To obtain the unknown coefficients ψ, it is defined that $\psi_{\xi j} = A^{-1}M$ where A and M are given above. Therefore, using the first Algorithm with reference to (4)

$$\left.\begin{array}{l}
(\psi_{10},\psi_{11},\psi_{12},\psi_{13},\psi_{14},\psi_{15},\psi_{16})^T = \\
\left(\frac{95929}{4702924800},\frac{4001}{167961600},-\frac{23033}{940584960},\frac{811}{39191040},-\frac{10693}{940584960},\frac{4219}{1175731200},-\frac{2323}{4702924800}\right)^T \\
(\psi_{20},\psi_{21},\psi_{22},\psi_{23},\psi_{24},\psi_{25},\psi_{26})^T = \\
\left(\frac{4127}{18370800},\frac{4319}{9185400},-\frac{199}{524880},\frac{97}{306180},-\frac{127}{734832},\frac{499}{9185400},-\frac{137}{18370800}\right)^T \\
(\psi_{30},\psi_{31},\psi_{32},\psi_{33},\psi_{34},\psi_{35},\psi_{36})^T = \\
\left(\frac{5471}{6451200},\frac{423}{179200},-\frac{39}{28672},\frac{29}{23040},-\frac{99}{143360},\frac{39}{179200},-\frac{193}{6451200}\right)^T \\
(\psi_{40},\psi_{41},\psi_{42},\psi_{43},\psi_{44},\psi_{45},\psi_{46})^T = \\
\left(\frac{488}{229635},\frac{7808}{1148175},-\frac{632}{229635},\frac{256}{76545},-\frac{58}{32805},\frac{128}{229635},-\frac{88}{1148175}\right)^T \\
(\psi_{50},\psi_{51},\psi_{52},\psi_{53},\psi_{54},\psi_{55},\psi_{56})^T = \\
\left(\frac{807125}{188116992},\frac{701875}{47029248},-\frac{790625}{188116992},\frac{59375}{7838208},-\frac{653125}{188116992},\frac{7625}{6718464},-\frac{29375}{188116992}\right)^T \\
(\psi_{60},\psi_{61},\psi_{62},\psi_{63},\psi_{64},\psi_{65},\psi_{66})^T = \\
\left(\frac{191}{25200},\frac{39}{1400},-\frac{3}{560},\frac{19}{1260},-\frac{3}{560},\frac{3}{1400},-\frac{1}{3600}\right)^T
\end{array}\right\} \quad (8)$$

Similarly, to obtain the unknown coefficients β, it is defined that $\beta_{\xi jq} = A^{-1}N$ where A and M are given above. Therefore, using the second Algorithm with reference to (5), (6) and (7)

$$\left.\begin{array}{l}
(\beta_{101},\beta_{111},\beta_{121},\beta_{131},\beta_{141},\beta_{151},\beta_{161}) = \\
\left(\frac{343801}{783820800},\frac{6031}{9331200},-\frac{32981}{52254720},\frac{6177}{9797760},-\frac{15107}{52254720},\frac{5947}{65318400},-\frac{9809}{783820800}\right)^T \\
(\beta_{201},\beta_{211},\beta_{221},\beta_{231},\beta_{241},\beta_{251},\beta_{261})^T = \\
\left(\frac{6887}{3061800},\frac{1499}{255150},-\frac{233}{58320},\frac{52}{15309},-\frac{379}{204120},\frac{149}{255150},-\frac{491}{6123600}\right)^T \\
(\beta_{301},\beta_{311},\beta_{321},\beta_{331},\beta_{341},\beta_{351},\beta_{361})^T = \\
\left(\frac{1959}{358400},\frac{1599}{89600},-\frac{537}{71680},\frac{1}{120},-\frac{327}{71680},\frac{129}{89600},-\frac{71}{358400}\right)^T \\
(\beta_{401},\beta_{411},\beta_{421},\beta_{431},\beta_{441},\beta_{451},\beta_{461})^T = \\
\left(\frac{3863}{382725},\frac{4664}{127575},-\frac{226}{25515},\frac{272}{15309},-\frac{31}{3645},\frac{344}{127575},-\frac{142}{382725}\right)^T \\
(\beta_{501},\beta_{511},\beta_{521},\beta_{531},\beta_{541},\beta_{551},\beta_{561})^T = \\
\left(\frac{505625}{31352832},\frac{162125}{2612736},-\frac{85625}{10450944},\frac{66875}{1959552},-\frac{119375}{10450944},\frac{1625}{373248},-\frac{18625}{31352832}\right)^T \\
(\beta_{601},\beta_{611},\beta_{621},\beta_{631},\beta_{641},\beta_{651},\beta_{661})^T = \\
\left(\frac{33}{1400},\frac{33}{350},-\frac{3}{560},\frac{2}{35},-\frac{3}{280},\frac{3}{350},-\frac{1}{1200}\right)^T
\end{array}\right\} \quad (9)$$

$$\left.\begin{array}{l}
(\beta_{102},\beta_{112},\beta_{122},\beta_{132},\beta_{142},\beta_{152},\beta_{162})=\\
\left(\frac{28549}{4354560},\frac{275}{20736},-\frac{5717}{483840},\frac{10621}{1088640},-\frac{7703}{1451520},\frac{403}{241920},-\frac{199}{870912}\right)^T\\
(\beta_{202},\beta_{212},\beta_{222},\beta_{232},\beta_{242},\beta_{252},\beta_{221})^T=\\
\left(\frac{1027}{68040},\frac{97}{1890},-\frac{2}{81},\frac{197}{8505},-\frac{97}{7560},\frac{23}{5670},-\frac{19}{34020}\right)^T\\
(\beta_{302},\beta_{312},\beta_{322},\beta_{332},\beta_{342},\beta_{352},\beta_{362})^T=\\
\left(\frac{253}{10752},\frac{165}{1792},-\frac{267}{17920},\frac{5}{128},-\frac{363}{17920},\frac{57}{8960},-\frac{47}{53760}\right)^T\\
(\beta_{402},\beta_{412},\beta_{422},\beta_{432},\beta_{442},\beta_{452},\beta_{462})^T=\\
\left(\frac{272}{8505},\frac{375}{2835},-\frac{2}{945},\frac{656}{8505},-\frac{2}{81},\frac{8}{945},-\frac{2}{1701}\right)^T\\
(\beta_{502},\beta_{512},\beta_{522},\beta_{532},\beta_{542},\beta_{552},\beta_{562})^T=\\
\left(\frac{35225}{870912},\frac{8375}{48384},-\frac{3125}{290304},\frac{25625}{217728},-\frac{625}{96768},\frac{275}{20736},-\frac{1375}{870912}\right)^T\\
(\beta_{602},\beta_{612},\beta_{622},\beta_{632},\beta_{642},\beta_{652},\beta_{662})^T=\\
\left(\frac{41}{840},\frac{3}{14},-\frac{3}{140},\frac{17}{105},-\frac{3}{280},\frac{3}{70}\right)^T
\end{array}\right\} \quad (10)$$

$$\left.\begin{array}{l}
(\beta_{103},\beta_{113},\beta_{123},\beta_{132},\beta_{143},\beta_{153},\beta_{163})=\\
\left(\frac{19087}{362880},\frac{2713}{15120},-\frac{15487}{120960},\frac{293}{2835},-\frac{6737}{120960},\frac{263}{15120},-\frac{863}{362880}\right)^T\\
(\beta_{203},\beta_{213},\beta_{223},\beta_{233},\beta_{243},\beta_{253},\beta_{223})^T=\\
\left(\frac{1027}{68040},\frac{97}{1890},-\frac{2}{81},\frac{197}{8505},-\frac{97}{7560},\frac{23}{5670},-\frac{19}{34020}\right)^T\\
(\beta_{303},\beta_{313},\beta_{323},\beta_{333},\beta_{343},\beta_{353},\beta_{332})^T=\\
\left(\frac{253}{10752},\frac{165}{1792},-\frac{267}{17920},\frac{5}{128},-\frac{363}{17920},\frac{57}{8960},-\frac{47}{53760}\right)^T\\
(\beta_{403},\beta_{413},\beta_{423},\beta_{433},\beta_{443},\beta_{453},\beta_{432})^T=\\
\left(\frac{272}{8505},\frac{375}{2835},-\frac{2}{945},\frac{656}{8505},-\frac{2}{81},\frac{8}{945},-\frac{2}{1701}\right)^T\\
(\beta_{503},\beta_{513},\beta_{523},\beta_{533},\beta_{543},\beta_{553},\beta_{563})^T=\\
\left(\frac{35225}{870912},\frac{8375}{48384},-\frac{3125}{290304},\frac{25625}{217728},-\frac{625}{96768},\frac{275}{20736},-\frac{1375}{870912}\right)^T\\
(\beta_{603},\beta_{613},\beta_{623},\beta_{633},\beta_{643},\beta_{653},\beta_{663})^T=\\
\left(\frac{41}{840},\frac{3}{14},-\frac{3}{140},\frac{17}{105},-\frac{3}{280},\frac{3}{70}\right)^T
\end{array}\right\} \quad (11)$$

4 Numerical Experiment of the method

To prove the accuracy and convergence of the implicit linear block method, some fourth order initial value problems are considered and textual shown.

System one: Consider the fourth order initial value problem

$$y^{iv} + y'' = 0,\ y(0) = 0,\ y'(0) = \frac{-1.1}{72 - 50pi},\ y''(0) = \frac{1}{144 - 50pi},\ y'''(0) = \frac{1.2}{144 - 100pi},$$

$$0 \le x \le \frac{pi}{2},\ h = 0.01$$

with exact solution given by

$$y(x) = \frac{1 - x - \cos x - 1.2\sin x}{144 - 100pi}$$

System two: Consider the fourth order initial value problem

$$y^{iv} = x,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 1,\ y'''(0) = 0,$$

with exact solution given by

$$y(x) = \frac{x^5}{120xpi} + x$$

System three: Consider the fourth order initial value problem

$$y^{iv} = 4y'',\ y(0) = 1,\ y'(0) = 3,\ y''(0) = 0,\ y'''(0) = 16,$$

with exact solution given by

$$y(x) = 1 - x + 2\exp(2x) - 2\exp(-2x)$$

Table 1. Comparison of new method with [11-13] for solving system one

x	Exact Solution	Computed Solution	Error in new Method	Error in [11]	Error in 12	Error in 13
0.01	0.00012889983466749742	0.00012889983466749769	2.7000E-19	4.6295E-16	0.4987E-15	2:1116E-13
0.02	0.00025720540846480060	0.00025720540846481039	9.7900E-18	6.9966E-15	0.6765E-15	5:6987E-12
0.03	0.00038490976337573504	0.00038490976337580179	6.6750E-17	2.8002E-14	0.3135E-14	6:8031E-12
0.04	0.00051200600150551386	0.00051200600150575629	2.4243E-16	7.1500E14	0.9436E-14	2:2072E-09
0.05	0.00063848728577052155	0.00063848728577116311	6.4156E-16	1.4503E-13	0.2212E-13	1:2741E-08
0.06	0.00076434684058201660	0.00076434684058341932	1.4027E-15	2.5942E-13	0.4338E-13	3:4561E-06
0.07	0.00088957795252368476	0.00088957795252638343	2.6987E-15	4.3689E-13	0.7787E-13	6:5524E-06
0.08	0.00101417397102297506	0.00101417397102771177	4.7367E-15	7.0309E-13	0.1286E-12	9:5865E-06
0.09	0.00113812830901615150	0.00113812830902391059	7.7591E-15	1.0825E-12	0.1993E-12	1:0493E-06
1.0	0.00126143444360699400	0.00126143444361903725	1.2043E-14	1.5981E-12	0.2932E-12	5:6962E-06

Source [11-13]

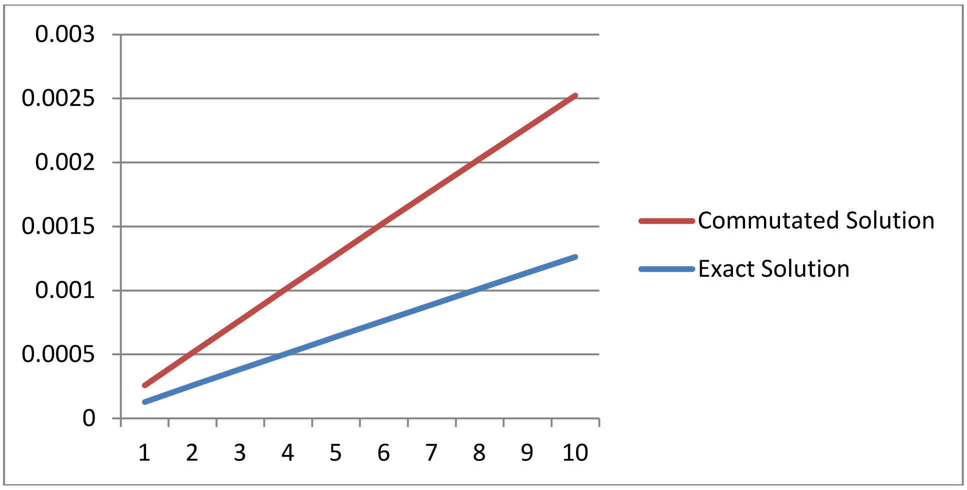

Fig. 1. Comparison of exact and numerical solution of system one

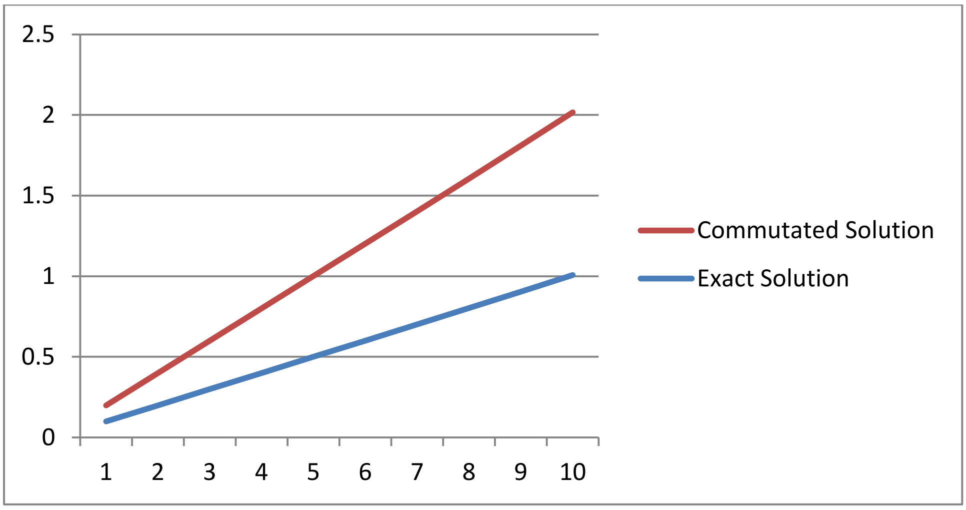

Fig. 2. Comparison of exact and numerical solution of system two

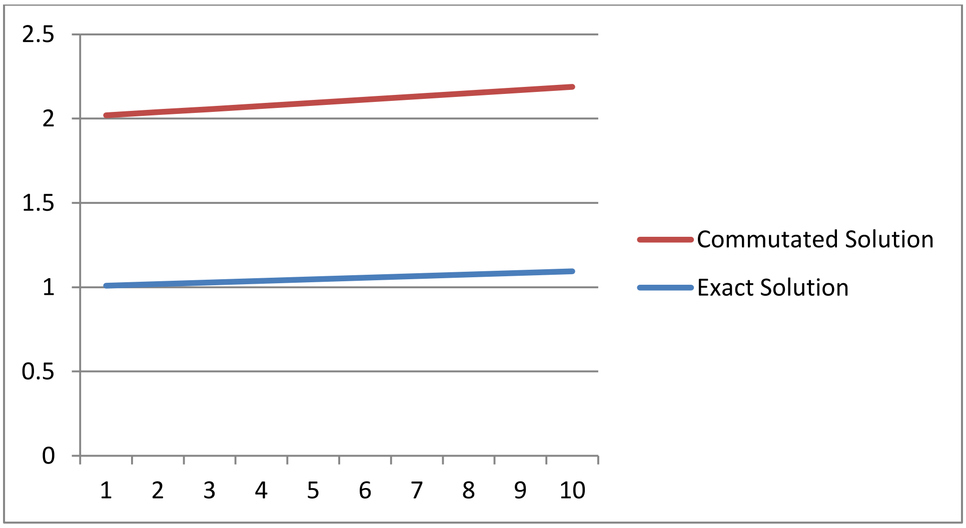

Figure 3. Comparison of exact and numerical solution of system three

Table 2. Comparison of new method with [10, 11 and 14] for solving system two

x-value	Exact Solution	Computed Solution	Error in new Method	Error in [11]	Error in [10]	Error in [14]
0.1	0.10000008333333333333	0.10000008333333333333	0.0000E-00	7.5052E-14	2.9976E-15	7.0000E-10
0.2	0.20000266666666666667	0.20000266666666666667	0.0000E-00	1.1082E-12	2.9976E-15	9.0000E-10
0.3	0.30002025000000000000	0.30002025000000000000	0.0000E-00	4.3214E-12	1.8225E-05	2.6000E-09
0.4	0.40008533333333333333	0.40008533333333333333	0.0000E-00	1.0767E-11	7.6800E-05	5.1000E-09
0.5	0.50026041666666666667	0.50026041666666666667	0.0000E-00	2.1326E-11	6.9944E-15	7.8000E-09
0.6	0.60064800000000000000	0.60064800000000000000	0.0000E-00	3.7426E-11	0.0000E-00	1.1800E-08
0.7	0.70140058333333333333	0.70140058333333333333	0.0000E-00	6.2429E-11	2.9976E-15	1.2400E-08
0.8	0.80273066666666666667	0.80273066666666666667	0.0000E-00	9.9845E-11	2.9976E-15	1.4100E-08
0.9	0.90492075000000000000	0.90492075000000000000	0.0000E-00	1.5255E-10	0.0000E-00	1.8800E-08
1.0	1.00833333333333333330	1.00833333333333333330	0.0000E-00	2.2292E-10	7.5000E-03	2.6000E-08

Source [10, 11 and 14]

Table 3. Comparison of new method with [9] for solving system three

x	Exact Solution	**Computed** Solution	Error in new Method	Error in [9] **in block method**	Error in [9] **in P-C method**
0.103125	1.00937508138036727920	1.00937508138036727920	0.0000E-00	1.5000E-17	9.0300E-10
0.206250	1.01875065104675294860	1.01875065104675294820	4.0000E-19	1.4900E-16	1.1940E-09
0.306250	1.02812719730424913310	1.02812719730424912900	4.1000E-18	1.1330E-15	2.6400E-10
0.406250	1.03750520849609617210	1.03750520849609615210	2.0000E-17	1.7200E-16	2.0900E-09
0.506250	1.04688517302275858900	1.04688517302275852200	6.7000E-17	2.0200E-16	1.5736E-08
0.603125	1.05626757936100329750	1.05626757936100311810	1.7940E-16	2.1720E-15	3.2505E-08
0.703125	1.06565291608298078600	1.06565291608298037690	4.0910E-16	3.1040E-15	5.3027E-08
0.803125	1.07504167187531003060	1.07504167187530919930	8.3130E-16	4.8900E-15	7.7515E-07
0.903125	1.08443433555816787740	1.08443433555816632970	1.5477E-15	1.1584E-14	1.1621E-07
1.003125	1.09383139610438364350	1.09383139610438095240	2.6911E-15	2.3123E-14	1.6062E-07

Source [9]

4 Summary, Conclusion and Discussion

In this paper, a new linear block method for the direct simulation of fourth order IVPs has been developed. When developing the method, we adopted the linear block approach through a single step. We have validated the accuracy of the method on some fourth order IVPs of ODEs. The direct simulation of LBA on fourth order IVPs is better than the conventional method has been presented.

The numerical experiments were given and the results obtained were compared with [9-11, 13, 14] and were found to be better in accuracy than the existing methods.

Therefore, this article has given a better convergent and highly accurate, for solving fourth order initial value problems.

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Competing Interests

Authors have declared that no competing interests exist.

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