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A Comparison of Implicit and Modified Implicit Finite Difference Schemes for Solving Parabolic Equations

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Authors' contributions

This work was carried out in collaboration among all authors. Author OBJ designed the work, organized the writing and carried out the derivations and solving. Author LIO checked the derivations and read the work. Author OCN checked the solving and checked typo errors. Authors AEO and EMC checked the solving and took care of the draft. All authors read and approved the final manuscript.

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Abstract

This paper presents the comparison of implicit scheme and modified implicit scheme for solving parabolic partial differential equations, the modified implicit scheme is compared with the implicit scheme using its stability, local truncation error, derivation and numerical examples. Following this, it was discovered that the modified implicit scheme can be used as an alternative scheme to the implicit scheme for solving problems on parabolic partial differential equations.

Keywords: Implicit scheme; stability; local truncation error; modified implicit scheme; parabolic equations.

2010 Mathematics Subject Classification: $35A20$, $35A35$, $35B35$, $35K05$.

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1 Introduction

The simplest and well known heat conduction equation is the one dimensional parabolic partial differential equation called the diffusion equation. It is first order in time and second order in space. Most of the problems in Engineering, Physics, Fluid mechanics, Chemistry and other areas of application rely heavily on this equation.

The solutions of these type of parabolic partial differential equations can be found using analytical and numerical methods. Since all the partial differential differential equations can not be solve analytically, therefore, numerical method is mostly employed. The process of obtaining an approximate solutions which converge closely to the exact solution of a parabolic partial differential equation is called numerical method. There are different types of numerical methods that can be used to get the very good approximate results for parabolic equations but in this work, we shall restrict our method to the finite difference methods (FDM).

Finite difference method is an approximation method for solving partial differential equations. It is a numerical technique for solving differential equations by approximating derivatives with finite differences. The process requires the discretization of both the time interval and spatial domain. The process converts partial differential equations which may be non-linear into a system of linear equations that can be solved by matrix algebra. The prominent and commonly used types of finite difference methods are, *explicit scheme*, *implicit scheme*, *Crank-Nicolson scheme*. These schemes perform better than each other, mostly in terms of stability, accuracy and convergence.

Different numerical experts and researchers in Mathematics and related fields have used the finite difference methods a lot. [1] Compared the exact solution of parabolic equations with its numerical solution using modified Crank-Nicolson scheme. [2] considered the practical methods for numerical solution to partial differential equations of heat conduction type. [3] Investigated the stability of Modified Crank-Nicolson scheme using Fourier method (von-Newmann method). They prove that the scheme is consistent, convergent and stable. [4] compared Crank-Nicolson scheme with modified Crank-Nicolson scheme. They show that the modified Crank-Nicolson scheme is efficient and good for solving parabolic equations. [6] modified the simple explicit scheme and prove that it is much more stable than the simple explicit case which enables using of larger time steps. [7] established an improved θ method to improve the θ-iterated Crank-Nicolson scheme to second order accuracy. [8] Modified the Crank-Nicolson scheme to a 3-level implicit finite difference scheme similar to the Crank-Nicolson scheme, their method utilizes an extra grid point at the lower level and the result is shown to be more accurate than the Crank-Nicolson scheme. There are lot of interesting and comprehensive texts on finite difference method. They include [9,10,11,12,13 and 14]

In this work, focus is on the implicit scheme and it modification: in terms of local truncation error, stability and also comparing the results of the schemes using numerical examples.

2 Problem Definition and Methodology

The following initial boundary value parabolic problem of the form

$$\left.\begin{array}{c} \frac{1}{c^2}\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \\ f(x,0) = f(x),\ 0 < x < 1 \\ f(0,t) = f(l,t) = 0 \leq t \leq l \end{array}\right\} \tag{1}$$

is considered. Equation (1) is typical of the heat equation with initial and boundary conditions. For the derivation of the schemes, the finite difference approximations (2) is substituted into equation (1), this is done for both the time derivative and the spatial derivatives.

2.1 Finite difference approximation

For the purpose of this work, the following finite difference approximation [14, 15] are required. Where $\triangle$ means a forward difference operator, $O(k)$ means error of order k, δ_x means central difference operator in the $x-$direction, such that $\delta(\delta_x) = \delta_x^2$.

$$\left.\begin{aligned} \left(\frac{\partial f}{\partial t}\right)_{i,j} &= \frac{1}{k} \triangle_t\, f_{i,j} = \frac{f_{i,j}-f_{i,j-1}}{k} + O(k) \\ \left(\frac{\partial f}{\partial t}\right)_{i,j} &= \frac{1}{k} \triangle_t\, f_{i,j} = \frac{f_{i,j+1}-f_{i,j}}{k} + O(k) \\ \left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j+1} &= \frac{1}{h^2}\delta_x^2 f_{i,j+1} = \frac{f_{i+1,j+1}-2f_{i,j+1}+f_{i-1,j+1}}{h^2} + O(h^2) \\ \frac{\partial^2 f}{\partial x^2} &= \frac{1}{h^2}\delta_x^2 f_{i,j-1} = \frac{f_{i+1,j}-2f_{i,j}+f_{i-1,j}}{h^2} + O(h^2) \end{aligned}\right\} \tag{2}$$

2.2 Implicit scheme: The derivation

Substituting the time derivative $\frac{\partial f}{\partial t}$ in equation (1) with the finite approximation $\frac{1}{k} \triangle_t\, f_{i,j}$ and the second derivative $\frac{\partial^2 f}{\partial x^2}$ with the finite approximation $\frac{1}{h^2}\delta_x^2 f_{i,j+1}$ gives the following finite difference approximation

$$\frac{1}{k} \triangle_t\, f_{i,j} = \frac{c^2}{h^2}\delta_x^2 f_{i,j+1}$$

which is analogue to equation (1). Solving further gives

$$f_{i,j} = -r\left(f_{i+1,j+1} + f_{i-1,j+1}\right) + (1+2r) f_{i,j+1} \tag{3}$$

Equation (3) is the implicit scheme, where $r = \frac{kc^2}{h^2}$ and it can be written in matrix form $Af = a$, defined as follows:

$$\begin{bmatrix} 1+2r & -r & & & \\ -r & 1+2r & -r & & \\ & -r & 1+2r & \ddots & \\ & & \ddots & \ddots & -r \\ & & & -r & 1+2r \end{bmatrix} \begin{bmatrix} f_{1,j+1} \\ f_{2,j+1} \\ f_{3,j+1} \\ \vdots \\ f_{n,j+1} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

in the matrix above $a_1 = f_{1,j}, a_2 = f_{2,j}, ..., a_n = f_{n,j}$ such that $j = 0, 1, 2, ...$

2.3 Modified implicit scheme: The derivation

The spatial and time derivatives in equation (1) is replaced with the following finite difference approximations $\frac{1}{h^2}\delta_x^2 f_{i,j-1}$ and $\frac{1}{k} \triangle_t\, f_{i,j}$ respectively, then the finite difference analogue to equation (1) becomes

$$\frac{1}{k} \triangle_t\, f_{i,j} = \frac{c^2}{h^2}\delta_x^2 f_{i,j-1}$$

solving further gives

$$f_{i,j-1} = -r\left(f_{i+1,j} + f_{i-1,j}\right) + (1+2r) f_{i,j} \tag{4}$$

Equation (4) is the modified implicit scheme. where $r = \frac{c^2 k}{h^2}$ and it can be written in matrix form $Af = b$, defined as follows:

$$\begin{bmatrix} 1+2r & -r & & & \\ -r & 1+2r & -r & & \\ & -r & 1+2r & \ddots & \\ & & \ddots & \ddots & -r \\ & & & -r & 1+2r \end{bmatrix} \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \\ \vdots \\ f_{n,j} \end{bmatrix} = \begin{bmatrix} f_{1,j-1} \\ f_{2,j-1} \\ f_{3,j-1} \\ \vdots \\ f_{n,j-1} \end{bmatrix}$$

where $b = f_{1,j-1}$ ..., $f_{n,j-1}$ where $j = 1, 2, 3, ...$

2.4 Local Truncation error

The local truncation error of the implicit scheme can be found [10], [11] and [12]. We derive below the local truncation error of the modified Implicit scheme.

2.5 Modified Implicit scheme (Local truncation error)

Considering the parabolic partial differential equation (1) given as

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2},$$

and the finite difference approximation by

$$\frac{f_{i,j} - f_{i,j-1}}{k} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} \tag{5}$$

If the approximation (5) equals $F_{i,j}(f)$ such that

$$F_{i,j}(f) = \frac{f_{i,j} - f_{i,j-1}}{k} - \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2}$$

also, if $T_{i,j} = f_{i,j}(\tilde{f})$ where $f_{i,j}(\tilde{f})$ is the error, so we have

$$T_{i,j} = f_{i,j}(\tilde{f}) = \frac{\tilde{f}_{i,j} - \tilde{f}_{i,j-1}}{k} - \frac{\tilde{f}_{i+1,j} - 2\tilde{f}_{i,j} + \tilde{f}_{i-1,j}}{h^2} \tag{6}$$

using Taylor's expansion and substituting into equation (6) we have

$$\frac{1}{k}\left[\tilde{f}_{i,j} - \left(\tilde{f}_{i,j} - k\left(\frac{\partial \tilde{f}}{\partial t}\right) - \frac{1}{2}k^2\left(\frac{\partial^2 \tilde{f}}{\partial t^2}\right) - \frac{1}{6}k^3\left(\frac{\partial^3 \tilde{f}}{\partial t^3}\right) - \frac{1}{24}k^4\left(\frac{\partial^4 \tilde{f}}{\partial t^4}\right) - ...\right)\right]$$

$$-\frac{1}{h^2}\left[\tilde{f}_{i,j} + \left(\tilde{f}_{i,j} + h\left(\frac{\partial \tilde{f}}{\partial x}\right) + \frac{1}{2}h^2\left(\frac{\partial^2 \tilde{f}}{\partial x^2}\right) + \frac{1}{6}h^3\left(\frac{\partial^3 \tilde{f}}{\partial x^3}\right) + \frac{1}{24}h^4\left(\frac{\partial^4 \tilde{f}}{\partial x^4}\right) + ...\right)\right]$$

$$\frac{1}{h^2}\left[-2\tilde{f}_{i,j} + \tilde{f}_{i,j} - h\left(\frac{\partial \tilde{f}}{\partial x}\right) + \frac{1}{2}h^2\left(\frac{\partial^2 \tilde{f}}{\partial x^2}\right) - \frac{1}{6}h^3\left(\frac{\partial^3 \tilde{f}}{\partial x^3}\right) + \frac{1}{24}h^4\left(\frac{\partial^4 \tilde{f}}{\partial x^4}\right) - ...\right]$$

by cancelation of opposite signs, we obtain

$$T_{i,j} = \frac{\partial \tilde{f}}{\partial t} - \frac{\partial^2 \tilde{f}}{\partial x^2} + \frac{1}{2}k^2\frac{\partial^2 \tilde{f}}{\partial t^2} - \frac{1}{12}h^2\frac{\partial^4 \tilde{f}}{\partial x^4} + ...$$

since $\tilde{f}$ is the exact solution of $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ then $\frac{\partial \tilde{f}}{\partial t} - \frac{\partial^2 \tilde{f}}{\partial x^2} = 0$ therefore the non-zero part of the local truncation error is

$$\frac{1}{2}k\frac{\partial^2 \tilde{f}}{\partial t^2} - \frac{1}{12}h^2\frac{\partial^4 \tilde{f}}{\partial x^4} \tag{7}$$

therefore,

$$T_{i,j} = \left(\frac{1}{2}k\frac{\partial^2 \tilde{f}}{\partial t^2} - \frac{1}{12}h^2\frac{\partial^4 \tilde{f}}{\partial x^4}\right) + ... \tag{8}$$

such that $\phi_1 k - \phi_2 h^2 \in O(k + h^2)$, where $\phi_1 = \frac{1}{2}\frac{\partial^2 \tilde{f}}{\partial t^2}, and\ \phi_2 = \frac{1}{12}\frac{\partial^4 \tilde{f}}{\partial x^4}$ hence $T_{i,j} \in O(k + h^2)$. This shows that the local truncation error of the modified Implicit approximation is $O(k + h^2)$, which is also the order of the scheme.

Next, we investigate the stability of the schemes using different stability analysis methods:

2.6 Stability Analysis 1 (von Newmann Method)

The stability of the Implicit scheme using the von Newmann method can be seen in standard text such as [11] and [12] and is verify as follows:
Considering the approximation

$$f_{i,j} = -r(f_{i+1,j+1} + f_{i-1,j+1}) + (1 + 2r)f_{i,j+1}$$

Given the finite difference approximation solution in separable form as

$$\zeta_{i,j} = \xi^{\gamma ih}\xi^{z\beta jk} = \xi^{\gamma ih + z\beta jk} \tag{9}$$

In equation (9) we let $\gamma(\beta) = \gamma$, let $\xi = \varsigma^{\gamma h}$ be the amplification factor, so that equation (9) becomes

$$\zeta_{i,j} = \xi^i \varsigma^{z\beta jk} \tag{10}$$

substituting equation (10) into equation (3) gives

$$\xi^i\varsigma^{z\beta jk} = -r(\xi^{i+1}\varsigma^{z\beta(j-1)k} + \xi^{i+1}\varsigma^{z\beta(j+1)k}) + (1+2r)\xi^{i+1}\varsigma^{z\beta jk}$$
$$\xi^i\varsigma^{z\beta jk} = -r\xi^i\varsigma^{z\beta jk}(\xi\varsigma^{-z\beta k} + \xi\varsigma^{z\beta k}) + (1+2r)\xi\varsigma^{z\beta jk}$$

dividing both sides by $\xi^i\varsigma^{z\beta jk}$ we get

$$1 = -r(\xi\varsigma^{-z\beta k} + \xi\varsigma^{z\beta k}) + (1+2r)\xi$$

which can be written as

$$1 = \left[-r\left(\varsigma^{-z\beta k} + \varsigma^{z\beta k}\right) + (1+2r)\right]\xi \tag{11}$$

using the following trigonometric Identities

$$\left.\begin{aligned} \varsigma^{-z\beta k} + \varsigma^{z\beta k} &= 2\cos\beta k \\ and \qquad & \\ 2\sin^2\left(\tfrac{\beta k}{2}\right) &= 1 - \cos\beta k \end{aligned}\right\} \tag{12}$$

in equation (11) we get

$$[-r(2\cos\beta k) + (1+2r)]\xi = 1$$

which gives

$$\left[1 + 4r\sin^2\left(\frac{\beta k}{2}\right)\right]\xi = 1$$

and therefore

$$\xi = \left[\frac{1}{1 + 4r\sin^2\left(\frac{\beta k}{2}\right)}\right]$$

showing that $|\xi| \leq 1$ for all values of r and therefore the implicit approximation is unconditionally stable.

2.7 Stability of modified Implicit scheme by von Newmann

For the stability of modified Implicit scheme by von Newmann, see [5] but for the purpose of this paper we have the following after substituting the trigonometric identities (12),

$$\xi^{-1} = (1+2r) - r(2\cos\beta k) = 1 + 2r(1-\cos\beta k)$$

$$\xi^{-1} = \left[1 + 4r\sin^2\left(\frac{\beta k}{2}\right)\right]$$

$$\xi = \frac{1}{\left[1 + 4r\sin^2\left(\frac{\beta k}{2}\right)\right]} \tag{13}$$

from equation (13), it is apparent that $|\xi| \leq 1$ for all values of r, and therefore, the modified implicit scheme is unconditionally stable.

2.8 Stability Analysis II (Implicit scheme)

The stability of the implicit scheme is derived as follows:

$$f_{i,j} = -r(f_{i+1,j+1} + f_{i-1,j+1}) + (1+2r)f_{i,j+1}$$

consider the case $f_{i,j} = \xi^j(-1)^i$ substituting this into the equation above, we have

$$-r\xi^{j+1}(-1)^{i-1} - r\xi^{j+1}(-1)^{i+1} + (1+2r)\xi^{j+1}(-1)^i = \xi^j(-1)^i$$

$$\xi[-r(-1) - 1 - r(-1) + 1 + (1+2r)] = 1$$

which implies

$$\xi = \frac{1}{1+4r},\ 0 < r < 1\ \forall\ r > 0$$

This shows that the implicit scheme is unconditionally stable.

2.9 Stability Analysis II (modified Implicit scheme)

The modified implicit schemes, the stability is verified as shown below:

$$f_{i,j-1} = -r(f_{i+1,j} + f_{i-1,j}) + (1+2r)f_{i,j}$$

the case $f_{i,j} = \xi^j(-1)^i$ substituting this into the equation we have

$$-r\xi^j(-1)^{i-1} - r\xi^j(-1)^{i+1} + (1+2r)\xi^j(-1)^i = \xi^{j-1}(-1)^i$$

which gives

$$\xi^j[-r(-1) - 1 - r(-1) + 1 + (1+2r)]\xi^{j-1}$$

from where we have

$$\xi[-r(-1) - 1 + (1+2r) - r(-1) + 1] = 1$$

which implies

$$\xi = \frac{1}{1+4r}$$

$0 < r < 1\forall r > 0$. Note the magnitude of all eigenvalues of $[A]$ is < 1, such that $|\xi| < 1$, which shows that the modified implicit scheme is unconditionally stable.

3 Numerical Examples

For clarity and numerical comparison, the following parabolic partial differential equation is considered for $h = 0.1,\ r = 0.05$.

Example 1:

$$\left.\begin{array}{r l} & \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2},\ 0 \leq x \leq 1 \\ \textit{with initial condition} & \\ & f(x,0) = \sin(\pi x),\ 0 < x < 1 \\ \textit{and boundary condition} & \\ & f(0,t) = f(l,t) = 0 \leq t \leq l \end{array}\right\} \quad (14)$$

Solution
solution of problems (14) using equation both the Implicit and modified Implicit schemes gives the following tri-diagonal matrix for $1 \leq i \leq 9$ at $j = 1$, we get a tri-diagonal matrix which is represented below;

$$\begin{bmatrix} 1.1 & -0.05 & & & & & & & \\ -0.05 & 1.1 & -0.05 & & & & & & \\ & -0.05 & 1.1 & -0.05 & & & & & \\ & & -0.05 & 1.1 & -0.05 & & & & \\ & & & -0.05 & 1.1 & -0.05 & & & \\ & & & & -0.05 & 1.1 & -0.05 & & \\ & & & & & -0.05 & 1.1 & -0.05 & \\ & & & & & & -0.05 & 1.1 & -0.05 \\ & & & & & & & -0.05 & 1.1 \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{5,1} \\ f_{6,1} \\ f_{7,1} \\ f_{8,1} \\ f_{9,1} \end{bmatrix} = \begin{bmatrix} 0.3090 \\ 0.5878 \\ 0.8090 \\ 0.9511 \\ 1.0000 \\ 0.9511 \\ 0.8090 \\ 0.5878 \\ 0.3090 \end{bmatrix}$$

Table 1 present the results of example 1 above using the Implicit scheme with computation at $1 \leq i \leq 9$, and $0 \leq j \leq 8$

Table 1. Table of results at $k = 0.0005,\ r = 0.05$ and $h = 0.1$

t	x	j	$f_{1,\ j}$	$f_{2,\ j}$	$f_{3,\ j}$	$f_{4,\ j}$	$f_{5,\ j}$	$f_{6,\ j}$	$f_{7,\ j}$	$f_{8,\ j}$	$f_{9,\ j}$
0.0005	0.1	0	0.3075	0.3060	0.3045	0.3030	0.3015	0.3000	0.2986	0.2972	0.2958
0.001	0.2	1	0.5895	0.5821	0.5793	0.5765	0.5737	0.5709	0.5681	0.5653	0.5625
0.0015	0.3	2	0.8051	0.8012	0.7973	0.7934	0.7895	0.7857	0.7819	0.7781	0.7743
0.002	0.4	3	0.9465	0.9419	0.9373	0.9327	0.9282	0.9237	0.9192	0.9147	0.9102
0.0025	0.5	4	0.9951	0.9903	0.9855	0.9807	0.9759	0.9712	0.9665	0.9618	0.9571
0.003	0.6	5	0.9465	0.9419	0.9373	0.9327	0.9282	0.9237	0.9192	0.9147	0.9102
0.0035	0.7	6	0.8051	0.8012	0.7973	0.7934	0.7895	0.7857	0.7819	0.7781	0.7743
0.004	0.8	7	0.5849	0.5821	0.5793	0.5765	0.5737	0.5709	0.5681	0.5653	0.5625
0.0045	0.9	8	0.3075	0.3060	0.3045	0.3030	0.3015	0.3000	0.2986	0.2972	0.2958

Table 2 presents the result of example 1 above using the modified Implicit scheme, with computation at $1 \leq i \leq 9$, and $1 \leq j \leq 9$

Example 2.
Compare the solution of the following parabolic partial differential equation using Implicit and modified Implicit scheme for $h = 1$.

$$\left.\begin{array}{r l} & \frac{1}{4}\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}, \\ \textit{with initial condition} & \\ & f(x,0) = \left(\frac{8x - x^2}{2}\right), \\ \textit{and boundary condition} & \\ & f(0,t) = f(8,t) = 0 \end{array}\right\} \quad (15)$$

Table 2. Table of results at $k = 0.0005,\ r = 0.05$ and $h = 0.1$

t	x	j	$f_{1,\ j}$	$f_{2,\ j}$	$f_{3,\ j}$	$f_{4,\ j}$	$f_{5,\ j}$	$f_{6,\ j}$	$f_{7,\ j}$	$f_{8,\ j}$	$f_{9,\ j}$
0.0005	0.1	1	0.3075	0.3060	0.3045	0.3030	0.3015	0.3000	0.2986	0.2972	0.2958
0.001	0.2	2	0.5895	0.5821	0.5793	0.5765	0.5737	0.5709	0.5681	0.5653	0.5625
0.0015	0.3	3	0.8051	0.8012	0.7973	0.7934	0.7895	0.7857	0.7819	0.7781	0.7743
0.002	0.4	4	0.9465	0.9419	0.9373	0.9327	0.9282	0.9237	0.9192	0.9147	0.9102
0.0025	0.5	5	0.9951	0.9903	0.9855	0.9807	0.9759	0.9712	0.9665	0.9618	0.9571
0.003	0.6	6	0.9465	0.9419	0.9373	0.9327	0.9282	0.9237	0.9192	0.9147	0.9102
0.0035	0.7	7	0.8051	0.8012	0.7973	0.7934	0.7895	0.7857	0.7819	0.7781	0.7743
0.004	0.8	8	0.5849	0.5821	0.5793	0.5765	0.5737	0.5709	0.5681	0.5653	0.5625
0.0045	0.9	9	0.3075	0.3060	0.3045	0.3030	0.3015	0.3000	0.2986	0.2972	0.2958

Solution:

in example 2 above, we have that $t = \frac{1}{8},\ c^2 = 4$ such that $r = 0.5$ solving the parabolic partial differential equation above results into the following tri-diagonal matrix:

$$\begin{bmatrix} 2 & -0.5 & & & & & \\ -0.5 & 2 & -0.5 & & & & \\ & -0.5 & 2 & -0.5 & & & \\ & & -0.5 & 2 & -0.5 & & \\ & & & -0.5 & 2 & -0.5 & \\ & & & & -0.5 & 2 & -0.5 \\ & & & & & -0.5 & 2 \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{5,1} \\ f_{6,1} \\ f_{7,1} \end{bmatrix} = \begin{bmatrix} 3.5000 \\ 6.0000 \\ 7.5000 \\ 8.0000 \\ 7.5000 \\ 6.0000 \\ 3.5000 \end{bmatrix}$$

Table 3 shows the result of the Implicit scheme for example 2, which is compute at $1 \leq i \leq 7$, and $0 \leq j \leq 6$.

Table 3. Table of results at $k = 0.125,\ r = 0.5$ and $h = 1$

t	x	j	$f_{1,\ j}$	$f_{2,\ j}$	$f_{3,\ j}$	$f_{4,\ j}$	$f_{5,\ j}$	$f_{6,\ j}$	$f_{7,\ j}$
0.125	1	0	3.1340	5.5361	7.0103	7.5052	7.0103	5.5361	3.1340
0.25	2	1	2.8456	5.1143	6.5393	7.0222	6.5393	5.1143	2.8456
0.375	3	2	2.6057	4.7315	6.0918	6.5570	6.0918	4.7315	2.6057
0.5	4	3	2.3986	4.3829	5.6700	6.1135	5.6700	4.3829	2.3986
0.625	5	4	2.2153	4.0639	5.2745	5.6940	5.2745	4.0639	2.2153
0.75	6	5	2.0503	3.7707	4.9048	5.2994	4.9048	3.7707	2.0503
0.875	7	6	1.9002	3.5004	4.5599	4.9297	4.5599	3.5004	1.9002

Table 4 present the result of the modified Implicit scheme for example 2, with computation at $1 \leq i \leq 7$, and $1 \leq j \leq 7$

Table 4. Table of results at $k = 0.125,\ r = 0.5$ and $h = 1$

t	x	j	$f_{1,\ j}$	$f_{2,\ j}$	$f_{3,\ j}$	$f_{4,\ j}$	$f_{5,\ j}$	$f_{6,\ j}$	$f_{7,\ j}$
0.125	1	1	3.1340	5.5361	7.0103	7.5052	7.0103	5.5361	3.1340
0.25	2	2	2.8456	5.1143	6.5393	7.0222	6.5393	5.1143	2.8456
0.375	3	3	2.6057	4.7315	6.0918	6.5570	6.0918	4.7315	2.6057
0.5	4	4	2.3986	4.3829	5.6700	6.1135	5.6700	4.3829	2.3986
0.625	5	5	2.2153	4.0639	5.2745	5.6940	5.2745	4.0639	2.2153
0.75	6	6	2.0503	3.7707	4.9048	5.2994	4.9048	3.7707	2.0503
0.875	7	7	1.9002	3.5004	4.5599	4.9297	4.5599	3.5004	1.9002

4 Discussion

From the comparison above, it can be deduced that the modified Implicit scheme is good and efficient for solving parabolic partial differential equations. The results of the local truncation error shows that the modified implicit scheme has the same order $0(k + h^2)$ as the implicit scheme, and the stability proofs confirm that the amplification factor $|\xi| \leq 1$ for all values of r, which shows that the modified implicit scheme is unconditionally stable. Also, from the numerical computations, it is observed that the computation for the next step for the implicit scheme starts from $j = 0$ while that of the modified Implicit scheme starts from $j = 1$ but produces the same results at the different stages, also, the step $j = 1$ for the implicit scheme produces same results for the step $j = 2$ for the modified implicit scheme and same as other steps, see Tables 1 - 4.

5 Conclusion

From the above results, it is clear that the modified implicit scheme is fast and effective for solving parabolic equations (heat equations) since, it also require solving a tri-diagonal matrix at every level. We therefore conclude that the modified implicit scheme can be use as alternative scheme to the implicit scheme for solving the heat equations.

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Competing Interests

Authors have declared that no competing interests exist.

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