Asian Journal of Probability and Statistics

19(3): 13-20, 2022; Article no.AJPAS.91178

ISSN: 2582-0230

A Generalized α-Laplace Lévy Process

S. Satheesh [a*]

[a] *School of Data Analytics, M. G. University, Kottayam - 686 560, India.*

Authors' contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/AJPAS/2022/v19i330469

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/91178

Received: 20 June 2022

Accepted: 22 August 2022

Short Research Article

Published: 26 August 2022

Abstract

Random time changed Lévy Processes are getting increased attention of late as they can account for a variety of features in data. In this article we discuss α-Laplace Lévy Process and a generalization of it. Both are random time changed α-stable Lévy Processes. We obtained a characterization of α-Laplace Lévy Process and discuss the first passage time distribution of a generalized α-Laplace Lévy Process. Interestingly, this first passage time follows a discrete distribution.

Keywords: α-Laplace, characterization, first passage time, Laplace transform, Lévy Processes, moment generating function.

2010 Mathematics Subject Classification: 60E07, 60E10, 60G17, 60G40, 60G51, 60G52, 60H05, 60J75.

1 Introduction

Brownian motion (BM) are Lévy Processes (see, theorems 1.1 and 1.2 below) where $X(1)$ has a normal distribution. There are situations where a Laplace model is preferred to a Gaussian one. While [1] used it to model the pooled position errors in a large navigation system, [2] used a stationary autoregressive model with Laplace marginals in communication engineering. Such possibilities motivated the introduction of Laplace process in [3] as a possible alternative to BM. [4] proposed the variance gamma (VG) processes (same as the Laplace process) to model long taildness inherent in data. Typically, Laplace process accounts for distributions of increments that are more

**Corresponding author: E-mail: ssatheesh1963@yahoo.co.in;*

peaked at the mode with thick tails. See, [5] for a review of data driven models starting with BM and resulting in a variety of Lévy Processes including fractional BM, fractional Laplace process and fractional α-stable process and also [6] and [7].

A larger class of Lévy processes can be derived from an α-stable Lévy Process $\{X(t)\}$ by randomizing its time parameter t using a positive continuous random variable T. The advantage is, while certain features of $\{X(t)\}$ are retained, those of T can be augmented to alter some others to get new useful processes. See, [8] and [9], for more on this. In proposition 3.1 a generalized α-Laplace law is derived as an exponential mixture of α-stable laws. In terms of Lévy processes, this is equivalent to randomising the time parameter of α-stable Lévy process by the unit exponential.

A characterization using stochastic integrals and the first passage time distribution of Laplace process were obtained in [10] and [11], stated as corollaries 2.2 and 3.3 here. In this article we generalise these results to α-Laplace process and a generalized α-Laplace processes. We now brief the background needed.

Theorem 1.1. *([12], p.154)* $\{X(t), t \geq 0\}$ *is a Lévy process if (i)* $X(0) = 0$ *almost surely (ii)* $X(t)$ *has stationary and independent increments and (iii)* $X(t)$ *is continuous in probability, that is,* $X_s \xrightarrow[s\to t]{P} X_t$.

Theorem 1.2. *([13], p.303, [12], p.159)* $\{X(t), t \geq 0\}$ *is a Lévy process iff the distribution of* $X(1)$ *is infinitely divisible.*

Theorem 1.3. *([12], p.160) Any Lévy process can be decomposed as* $X(t) = \sigma B(t) + S(t);\ \sigma > 0$, *where* $B(t)$ *is a Brownian motion (BM) with drift and* $S(t)$ *is a pure jump process.*

Theorem 1.4. *([13], p.588) A random variable X or its distribution is in class-L (or self-decomposable) if its characteristic function (CF)* $\omega(s)$ *has the property that* $\omega(s)/\omega(cs)$ *is a CF* $\omega_c(s)$ *for each* $c \in (0, 1)$. *Similar definition in terms of moment generating functions (MGF) and Laplace Transforms (LT) holds.*

Theorem 1.5. *([14]) A random variable X or its distribution is geometrically infinitely divisible (geometrically* α*-stable) iff its CF* $\omega(s)$ *has the property that* $\omega(s) = \frac{1}{1+\psi(s)}$ *such that* $e^{-\psi(s)}$ *is infinitely divisible (*α*-stable). Similar definition in terms of MGFs and LTs holds.*

α-Laplace laws are defined by their CF $\frac{1}{1+c|s|^\alpha}$, $0 < \alpha \leq 2,\ c > 0$. They are mixtures of symmetric α-stable laws, where the mixing distribution is exponential. They are self-decomposable, geometrically infinitely divisible ([15]) and hence infinitely divisible ([16]). Hence one can define the corresponding α-Laplace Lévy processes (αLLP). For $\alpha = 2$ the α-Laplace law is Laplace and the corresponding Lévy process is Laplace process. Laplace Process was introduced and discussed in [3] and [5] as a possible alternative to BM and was compared and contrasted with BM. It is known that $\frac{1}{2}$-stable law is the first passage time distribution (FPT/FPTD) of BM with zero drift ([13], p.174).

[17] introduced the MGF of α-stable laws. They call it extreme stable since the parameter β in the stable model is set as $\beta = 1$. They have taken the location parameter also as zero. Here we refer to them as α-stable laws. [18] used this to define and discuss α-stable Lévy processes.

Theorem 1.6. *[17] The function* $\exp\{-b(1-\alpha)s^\alpha\};\ 0 \leq Re(s) < \infty;\ 0 < \alpha \leq 2,\ \alpha \neq 1, b > 0$ *are MGFs of* α*-stable laws.*

Using this we define a generalised α-Laplace law and the corresponding Lévy process, *viz.* generalized αLLP (GαLLP) and derive its FPTD. FPTD of processes are important as they give the distribution of the time taken for the process to reach/ cross a barrier/ threshold. If $\lambda > 0$ is the barrier, then the random variable $T(\lambda) = T = \inf\{t > 0 : X(t) \geq \lambda\}$ denote the FPT of $X(t)$. Here $t > 0$, since $X(0) = 0$ for a Lévy process.

[19] conceived a stochastic integral $\int_A^B g(t)\ dX(t)$ corresponding to a Lévy process $\{X(t), t \in T\}$ in the sense of convergence in probability, where $g(t)$ is continuous in $[A, B] \subset T$ and proved the following theorem.

Theorem 1.7. *[19] Let $\{X(t), t \in T\}$ be a Lévy process and $g(t)$ a continuous function in $[A, B] \subset T$. Let $f(u)$ be the CF of $X(1)$ and $h(u)$ that of the corresponding stochastic integral. Then $\ln[h(u)] = \int_A^B \ln[f(ug(t))]\ dt$.*

With this background we obtained a characterization of αLLP using the above theorem in the next section. In section 3 we derive and discuss a generalization of α-Laplace law and its divisibility properties such as self-decomposability, infinite divisibility *etc.*. Then we derive the FPTD of the GαLLP. These processes are obtained from α-stable Lévy processes by randomising the time parameter by the unit exponential law, see remark 3.1. Interestingly, the first passage time has a discrete distribution.

2 A Characterization of αLLP

Theorem 2.1. *A Lévy process $\{X(t), t \geq 0\}$ for which the distribution of $X(1)$ is symmetric, is αLLP if and only if, the CF $h(u)$ of the stochastic integral $\int_0^1 t^{1/\alpha}\ dX(t)$ is given by $\ln[h(u)] = 1 - (1 + |u|^{-\alpha})\ \ln[1 + |u|^{\alpha}]$.*

Proof. Let $h(u)$ be the CF of the stochastic integral $\int_0^1 t^{1/\alpha}\ dX(t)$ where $X(t)$ is αLLP with CF $f(u) = \frac{1}{1+|u|^{\alpha}}$. Then by theorem 1.7, $\ln[h(u)] = \int_0^1 \ln[f(ut^{1/\alpha})]\ dt$. Denoting $|u|^{\alpha}$ by k in the following integration, we have;

$$\begin{aligned}
\ln[h(u)] &= -\int_0^1 \ln(1 + |ut^{1/\alpha}|^{\alpha})\ dt = -\int_0^1 \ln(1 + kt)\ dt \\
&= -\left[t\ln(1 + kt)\right]_0^1 + \int_0^1 \frac{kt}{1 + kt}\ dt \\
&= -\ln(1 + k) + 1 - \int_0^1 \frac{1}{1 + kt}dt, \quad \left(\textit{since}\ \frac{kt}{1 + kt} = 1 - \frac{1}{1 + kt}\right) \\
&= -\ln(1 + k) + 1 - \frac{1}{k}\ln(1 + k) = 1 - (1 + k^{-1})\ln(1 + k) \\
&= 1 - (1 + |u|^{-\alpha})\ \ln[1 + |u|^{\alpha}].
\end{aligned}$$

Conversely, let $f(u)$ be the CF of $X(1)$, $\ln[h(u)] = 1 - (1 + |u|^{-\alpha})\ \ln[1 + |u|^{\alpha}]$. We need to find $f(u)$. Since $X(1)$ is symmetric, $f(u)$ is real and even and so we need to evaluate it for $u > 0$ only. Putting $\psi(u) = \ln[f(u)]$,

$$\begin{aligned}
1 - \left(1 + u^{-\alpha}\right)\ln(1 + u^{\alpha}) &= \int_0^1 \ln[f(ut^{1/\alpha})]\ dt = \int_0^1 \psi(ut^{1/\alpha})\ dt \\
&= \frac{\alpha}{u^{\alpha}} \int_0^u \psi(z) z^{\alpha-1}\ dz\ (z = ut^{1/\alpha}\ \&\ dz = \frac{z\ u^{\alpha}}{\alpha\ z^{\alpha}}dt).
\end{aligned}$$

That is, $\int_0^u \psi(z)z^{\alpha-1}\,dz = \frac{u^\alpha}{\alpha}\left\{1-(1+u^{-\alpha})\ln(1+u^\alpha)\right\}$. Hence,

$$\begin{aligned}
\psi(u)u^{\alpha-1} &= \frac{d}{du}\left[\frac{u^\alpha}{\alpha}\left\{1-(1+\frac{1}{u^\alpha})\ln(1+u^\alpha)\right\}\right]\\
&= \frac{\alpha u^{\alpha-1}}{\alpha} - \frac{d}{du}\left[\frac{u^\alpha}{\alpha}\left\{\left(\frac{u^\alpha+1}{u^\alpha}\right)\ln(1+u^\alpha)\right\}\right]\\
&= u^{\alpha-1} - \frac{d}{du}\left[\left(\frac{1+u^\alpha}{\alpha}\right)\ln(1+u^\alpha)\right]\\
&= u^{\alpha-1} - u^{\alpha-1}\ln(1+u^\alpha) - \frac{1+u^\alpha}{\alpha}\frac{1}{1+u^\alpha}\alpha u^{\alpha-1}\\
&= -u^{\alpha-1}\ln(1+u^\alpha)\\
\textit{That is, }\ \psi(u) &= \ln[f(u)] = -\ln(1+u^\alpha) \implies f(u) = \frac{1}{1+|u|^\alpha}.
\end{aligned}$$

That completes the proof. □

Corollary 2.2. *With $\alpha = 2$, theorem 2.1 characterizes Laplace Process.*

3 FPTD of GαLLP

Theorem 3.1. *The function $M(s) = \frac{1}{1+b(1-\alpha)s^\alpha}$; $0 \le Re(s) < 1$; $0 < \alpha \le 2, \alpha \ne 1, b > 0$ are MGFs of probability laws.*

Proof. By [20], p.213, if $\Psi(s)$ is analytic in the strip $0 < Re(s) < a$, continuous in $0 \le Re(s) < a$ and $\Psi(is)$ is the characteristic function (CF) of a probability law, then $\Psi(s)$ is the MGF of that probability law. Now, $\frac{1}{1-s}$ is analytic in the strip $0 < Re(s) < 1$ and continuous in $0 \le Re(s) < 1$. Again, $-b(1-\alpha)s^\alpha$ is analytic for $Re(s) > 0$ and continuous for $Re(s) \ge 0$. Hence $M(s) = \frac{1}{1+b(1-\alpha)s^\alpha}$ is analytic in the strip $0 < Re(s) < 1$ and continuous in $0 \le Re(s) < 1$. Since $\exp\{-b(1-\alpha)(is)^\alpha\}$ is the CF of α-stable laws ([17]), $M(is)$ is the CF of geometrically α-stable laws ([14]), and hence $M(s)$ is the MGF of a probability law. □

Note. For $\alpha = 2$ we get $M(s) = \frac{1}{1-bs^2}$, the MGF of Laplace law. α-Laplace laws are exponential mixtures of symmetric α-stable laws. By [17], the α-stable laws in theorem 1.6 are not symmetric. Hence we call the MGF $M(s)$ in the above theorem as that of a generalized α-Laplace (GαL) law. For $1 < \alpha \le 2$ it has finite mean. One may prove theorem 3.1 with a more probabilistic flavour, as follows.

Proposition 3.1. *The function $M(s) = \frac{1}{1+b(1-\alpha)s^\alpha}$; $0 \le Re(s) < 1$; $0 < \alpha \le 2, \alpha \ne 1, b > 0$ are MGFs of GαL laws.*

Proof. Let the random variable X be α-stable with MGF $\exp\{-b(1-\alpha)s^\alpha\}$. Then for $c > 0$, the MGF of $c^{1/\alpha}X$ is $\exp\{-c\, b(1-\alpha)s^\alpha\}$. Let c be a random variable having the unit exponential law. Then the MGF of $c^{1/\alpha}X$ is $E_c\left[e^{-c\ b(1-\alpha)s^\alpha}\right] = \frac{1}{1+b(1-\alpha)s^\alpha}$. □

We are finding the MGF of the scale mixture of α-stable laws where the mixing distribution is unit exponential. If the MGF of the random variable Y is $M(s)$ and $E \sim Exp(1)$, then $Y = E^{1/\alpha}X$ is the stochastic representation of Y.

Proposition 3.2. *GαL laws are geometric(p)-sum of its own type for every $p \in (0,1)$. Hence they are geometrically infinitely divisible, infinitely divisible and also self-decomposable.*

Proof. The probability generating function (PGF) of a geometric(p) law on $\{1, 2, 3, ...\}$ is $P(s) = \frac{ps}{1-(1-p)s}$. Hence the MGF of the geometric(p)-sum is; $P(M(s)) = \frac{pM(s)}{1-(1-p)M(s)}$. Taking $M(s)$ as the MGF of GαL we have,

$$\begin{aligned} P(M(p^{1/\alpha}s)) &= \frac{p/[1 + b(1-\alpha)(p^{1/\alpha}s)^\alpha]}{1-(1-p)/[1 + b(1-\alpha)(p^{1/\alpha}s)^\alpha]} \\ &= \frac{p}{p + b(1-\alpha)ps^\alpha} \\ &= \frac{1}{1 + b(1-\alpha)s^\alpha}. \end{aligned}$$

Since $0 < p^{1/\alpha} < 1$, and this is true for any $p \in (0, 1)$, GαL laws are geometric(p)-sum of its own type for every $p \in (0, 1)$. Hence they are geometrically infinitely divisible and infinitely divisible, [16]. Now, rewriting the third and first lines we have,

$$\begin{aligned} &\frac{1}{1 + b(1-\alpha)s^\alpha} = \frac{1}{[1 + b(1-\alpha)(p^{1/\alpha}s)^\alpha]} \times \frac{p}{1-(1-p)/[1 + b(1-\alpha)(p^{1/\alpha}s)^\alpha]} \\ &\textit{That is, } M(s) = M(p^{1/\alpha}s) \times P_1(M(p^{1/\alpha}s)), \end{aligned}$$

where P_1 is the PGF of the geometric law on $\{0, 1, 2, ...\}$. Since $P_1(M(p^{1/\alpha}s))$ is also an MGF, $0 < p^{1/\alpha} < 1$ and the above equation is true for any $p \in (0, 1)$, GαL laws are self-decomposable. □

Definition 3.1. Lévy processes $\{X(t); t \geq 0\}$ are generalized αLLP (GαLLP), if the distribution of $X(1)$ has MGF $M(s) = \frac{1}{1+b(1-\alpha)s^\alpha}; 0 \leq Re(s) < 1;\ 0 < \alpha \leq 2, \alpha \neq 1, b > 0$.

Remark 3.1. Now, in terms of Lévy processes, proposition 3.1 means that the GαLLP are obtained by randomising the time parameter of α-stable Lévy process in [18] by the unit exponential law. Similarly, by randomising the time parameter of symmetric α-stable Lévy process by the unit exponential, αLLP are obtained.

Since the location parameter is zero for the generalized α-Laplace laws considered here, the GαLLP has zero drift. We now derive the FPTD of GαLLP using standard arguments based on optional sampling theorem applied to the following martingale of $\{X(t)\}$.

Proposition 3.3. *For the GαLLP $\{X(v), v \geq 0\}$, $W(v) = \exp\{sX(v) - \theta v\}$, $s > 0$ a constant, is a martingale, where $\theta = -\ln[1 + b(1-\alpha)s^\alpha]$.*

Proof. Since, $E\left(e^{sX(v)}\right) = e^{\theta v}$, $E(|W(v)|) = E(W(v)) = e^{-\theta v}E\left(e^{sX(v)}\right) = 1 < \infty$. Since Lévy processes have stationary and independent increments, for $u \leq v$, $X(v) - X(u)$ is independent of $\mathcal{F}_u$, the filtration up to time u. Now,

$$\begin{aligned} E\left(W(v)/\mathcal{F}_u\right) &= E\left(\exp\{sX(v) - \theta v/\mathcal{F}_u\}\right) \\ &= e^{-\theta v}E\left(e^{s[X(v)-X(u)]}/\mathcal{F}_u\right)E\left(e^{sX(u)}/\mathcal{F}_u\right) \\ &= e^{-\theta v}E\left(e^{sX(v-u)}\right)e^{sX(u)} \\ &= e^{-\theta v}e^{\theta(v-u)}e^{sX(u)} \\ &= e^{sX(u)-\theta u} = W(u). \end{aligned}$$

That completes the proof. □

Theorem 3.2. *The FPTD of GαLLP for $1 < \alpha \leq 2$, is discrete $\frac{1}{\alpha}$-stable.*

Proof. Let the random variable $T(\lambda) = T$ denote the FPT for the GαLLP $\{X(t), t \geq 0\}$ to reach or cross $\lambda > 0$. We saw that for $\{X(t)\}$, $W(t) = \exp\{sX(t) - \theta t\}$ is a martingale, where $\theta = -\ln[1 + b(1-\alpha)s^{\alpha}]$. For a martingale $\{W(t)\}$ and for the FPT T (which is a stopping time), $E\{W(0)\} = E\{W(T \wedge t)\}$. As $X(0) = 0$, $W(0) = 1$ and hence $E\{W(T \wedge t)\} = 1$. That is,

$$E\left[\exp\{sX(T \wedge t) - \theta(T \wedge t)\}\right] = 1, \tag{3.1}$$

Note that for $\alpha > 1$; $\theta = -\ln[1 + b(1-\alpha)s^{\alpha}] > 0$, and so $0 \leq W(T \wedge t) \leq e^{s\lambda}$.

Now assuming $P\{T < \infty\} = 1$ (we will justify this at the end of the proof) we may pass to the limit as $t \to \infty$ under the expectation in (3.1) by the optional sampling theorem, yielding;

$$1 = \lim_{t\to\infty} E\left[\exp\{sX(T \wedge t) - \theta(T \wedge t)\}\right] = e^{s\lambda} E\left[e^{-\theta T}\right] \implies E\left[e^{-\theta T}\right] = e^{-s\lambda}.$$

Now $\theta = -\ln[1 + b(1-\alpha)s^{\alpha}] \implies s = \left\{\frac{e^{-\theta}-1}{b(1-\alpha)}\right\}^{1/\alpha} = \left\{\frac{1-e^{-\theta}}{b(\alpha-1)}\right\}^{1/\alpha}$, and we get the LT of the FPT as,

$$E\left[e^{-\theta T}\right] = \exp\left[\frac{-\lambda(1-e^{-\theta})^{1/\alpha}}{[b(\alpha-1)]^{1/\alpha}}\right] = \exp\left[-\beta(1-e^{-\theta})^{1/\alpha}\right],$$

which is that of discrete $\frac{1}{\alpha}$-stable law, see [21].
Finally, since $P\{T < \infty\} = \lim_{\theta\downarrow 0} E\left[e^{-\theta T}\right] = 1$, T has a proper distribution, justifying our assumption $P\{T < \infty\} = 1$. □

Remark 3.2. $E\left[e^{-\theta T}\right] = e^{-\beta(1-e^{-\theta})^{1/\alpha}}$ is the LT of a probability distribution only when $0 < 1/\alpha < 1 \implies \alpha > 1$ ([22], [13], p.448) and by the one-to-one correspondence $P(e^{-\theta}) = L(\theta)$; $\theta \geq 0$, between the probability generating function P and the LT L of a discrete distribution. Also, in the proof here we need $\theta > 0 \implies \alpha > 1$. These are the reasons for restricting the range of α to $1 < \alpha \leq 2$ in the above theorem.

Corollary 3.3. *When $\alpha = 2$, we have the Laplace process and the LT of T is $E\left[e^{-\theta T}\right] = e^{\left[\frac{-\lambda}{\sqrt{b}}\right](1-e^{\theta})^{1/2}}$ which is that of discrete $\frac{1}{2}$-stable. Recall that the FPTD of BM is $\frac{1}{2}$-stable.*

Remark 3.3. That the FPTD of Laplace process is **discrete** $\frac{1}{2}$-stable has intrigued the author for long, because it is the distribution of time, that is continuous for the process. If one defines an exponential Lévy process on the same lines and find its FPTD as in theorem 3.2, it is Poisson. This is not entirely surprising, knowing the close relation between exponential and Poisson laws in the context of renewal processes. But here, we need an interpretation for this conclusion. Note that the increase in an exponential Lévy process is in jumps and hence T represents the number of jumps, which is discrete, to reach or cross the barrier λ. Thus one possible reason is that the change (increase/ decrease as Laplace law is the difference of identical exponential laws) in Laplace process is in jumps. This intuition is substantiated by theorem 1.3 quoted in the introduction, which implies that among Lévy processes only BM has almost sure continuity of paths. Thus the changes in the αLLP are also in jumps and so what T, the FPT, represents here is the number of jumps required to reach or cross the barrier. One may also note that the structure of the martingale $W(t)$ here is comparable with that of the corresponding Wald's martingale, see [23], p.243.

Remark 3.4. We saw that the FPTD of exponential Lévy process is Poisson. Along with the discussion in [23], p.321, this is a martingale proof of the inter-arrival time characterization of Poisson process.

4 Summary

In this paper a characterization of αLLP using a method based on stochastic integrals, is obtained. GαL law is introduced and some of its divisibility properties are proved. Consequently the GαLLP is defined and its FPTD is derived.

Acknowledgement

Author thanks the referees for their suggestions for a much better presentation and proper orientation of the results obtained. The research reported here started with Laplace process when the author was in the Department of Statistics, University of Kerala, with a UGC Fellowship No.105/330016/86. He is thankful to the UGC, India for the support and Professor R N Pillai for the advice and encouragement.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Hsu DA. Long-tailed distributions for position errors in navigation. Journal of the Royal Statistical Society Series C. 1979;28:62-72.

[2] Sethia ML, Anderson JB. Interpolative DPCM. IEEE Transactions in Communications. 1984;32:729-736.

[3] Satheesh S. Laplace process. Current Science. 1990a;59:44.

[4] Madan DB, Seneta E. The variance gamma (V.G.) model for share market returns. Journal of Business. 1990;63:511524.

[5] Satheesh S. Some properties of Laplace process. J. Kerala Statist. Assoc. 2021;32:40-49.

[6] Samorodnitsky G, Taqqu MS. Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance. Chapman and Hall, New York/ CRC Press, Florida; 1994/2000.

[7] Jondeau E, Poon Ser-H, Rockinger M. Financial Modeling under Non-Gaussian Distributions, Springer-Verlag, London; 2007.

[8] Schoutens W. Lévy Processes in Finance: Pricing Financial Derivatives. Wiley, New York; 2003.

[9] Gajda J, Wylomańska A, Kumar A. Fractional Lévy stable motion time-changed by gamma subordinator. Communications in Statistics - Theory and Methods. 2018;48:1-18.

[10] Satheesh S, Pillai RN. Laplace process, Unpublished manuscript, Department of Statistics, University of Kerala, India; 1988.

[11] Satheesh S. Laplace process-II, Unpublished manuscript, Department of Statistics, University of Kerala, India; 1990b.

[12] Capasso V, Bakstein D. An Introduction to Continuous-Time Stochastic Processes. 2^{nd} Edition, Birkhäuser, Boston; 2012.

[13] Feller W. An Introduction to Probability Theory and Its Applications. 2^{nd} Edition, John Wiley & Sons, New York; 1971.

[14] Klebanov LB, Maniya GM, Melamed IA. A problem of Zolatarev and analogs of infinitely divisible and stable distributions in the scheme of summing a random number of random variables. Theor. Probab.Appl. 1984;29:791-794.

[15] George S. Studies on Certain Distributions Normally Attracted to Stable, Unpublished Ph.D. Thesis, Department of Statistics, University of Kerala, India; 1990.

[16] Sandhya E, Pillai RN. On geometric infinite divisibility. J. Kerala Statist. Assoc. 1999;10:1-7.

[17] Eaton ML, Morris C, Rubin H. On extrem stable laws and some applications. J. Appl. Probab. 1971;8:794-801.

[18] Satheesh S. A note on α-stable and α-inverse Gaussian laws. Asian J. Probab. Statist. 2022;19(2):29-34.

[19] Lukacs E. A characterisation of stable process. J. Appl. Probab. 1969; 6:409-418.

[20] Loève, M. Probability Theory. D. Van Nostrand Co., New York; 1960.

[21] Steutel FW, Van Harn K. Discrete analogs of self-decomposability and stability. Ann. Probab. 1979;7:893-899.

[22] Satheesh S, Nair NU. Some classes of distributions on the non-negative lattice. J. Ind. Statist. Assoc. 2002;40:41-58.

[23] Karlin S, Taylor HM. A First Course in Stochastic Processes. 2^{nd} Edition, Academic Press, New York; 1975.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/91178