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Research Article

A General Inequality for CR-Warped Products in Generalized Sasakian Space Form and Its Applications

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In the present paper, by considering the Gauss equation in place of the Codazzi equation, we derive new optimal inequality for the second fundamental form of CR-warped product submanifolds into a generalized Sasakian space form. Moreover, the inequality generalizes some inequalities for various ambient space forms.

1. Introduction

The fundamental idea of warped product manifolds was first initiated in [1] with manifolds of negative curvature. Let N_1 and N_2 be two Riemannian manifolds endowed with Riemannian matrices g_1 and g_2, respectively, such that $f : N_1 \longrightarrow (0,\infty)$ is a positive smooth function on N_1. Then, the warped product $M = N_1 \times_f N_2$ is characterized as the product manifold $N_1 \times N_2$ with the equipped metric $g = g_1 + f^2 g_2$. In particular, if f = constant, then M turned to be a Riemannian product manifold; otherwise, M is called a nontrivial warped product manifold. Let $M = N_1 \times_f N_2$ be a nontrivial warped product manifold. Then,

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{1}$$

for any vector fields $X, Y \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$. If we consider a local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ such that $\{e_i\}_{1\le i\le n_1} \in N_1$ and $\{e_j\}_{n_1+1\le j\le n} \in N_2$, we have

$$\sum_{1\le i\le n_1} \sum_{n_1+1\le j\le n} K\left(e_i \wedge e_j\right) = \frac{n_2 \Delta f}{f}. \tag{2}$$

In [2], Chen established the inequality for the squared norm of the mean curvature and the warping function f of a CR-warped product $N_T \times_f N_\perp$, where $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold, isometrically immersed in a complex space form as follows.

Theorem 1 (see [2]). *$N_T^{n_1} \times_f N_\perp^{n_2}$ be a CR-warped product into a complex space form $\tilde{M}(4c)$ with constant sectional curvature c. Then,*

$$\|h\|^2 \ge 2n_2\left\{\|\nabla \ln f\|^2 + \Delta(\ln f) + 2n_1 c\right\}, \tag{3}$$

where Δ is the Laplacian operator of N_T. Moreover, the equality holds if and only if N_T is totally geodesic and $N_\perp$ is totally umbilical in $\tilde{M}(4c)$.

Moreover, Theorem 1 is extended to CR-warped product submanifolds in a generalized Sasakian space form by using the same technique.

Theorem 2 (see [3]). *Let $N_T^{n_1} \times_f N_\perp^{n_2}$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ such that the structure vector field ξ is tangent to base manifold. Then, the following inequality is satisfied:*

$$\|h\|^2 \geq 2n_2\left(\|\nabla(\ln f)\|^2 - \Delta(\ln f) + 1\right) + 4n_1n_2(\lambda_1 + 1), \quad (4)$$

where Δ denotes the Laplace operator on $N_T^{n_1}$. The equality holds if and only if $N_T^{n_1}$ is a totally geodesic submanifold of $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$; in this case, $N_T^{n_1}$ is a generalized Sasakian space form of $(\lambda_1 + 3\lambda_3)$.

Furthermore, Mustafa et al. [4] recalled some fundamental problems of CR-warped products in Kenmotsu space forms as to simple relationships between the second fundamental form and the main intrinsic invariants by using the Gauss equation. In [5–7], some sharp inequalities are established for the sectional curvature of warped product pointwise semislant submanifolds in various space forms such as a Sasakian space form, a cosymplectic space form, a Kenmotsu space form, and a complex space form in terms of the Laplacian and the squared norm of a warping function with pointwise slant immersions. Afterward, several geometers [1, 2, 4, 8–18] obtained similar inequalities for different types of warped products in different kinds of structures.

Al-Ghefari et al. [3] proved the existence of CR-warped product submanifolds of type $N_T \times_f N_\perp$ in trans-Sasakian manifolds. They obtained an inequality for the second fundamental form with constant sectional curvature in terms of a warping function. Moreover, the nonexistence of CR-warped products of the form $N_\perp \times_f N_T$ in a generalized Sasakian space form was proved in [19].

In this paper, we shall establish a Chen-type inequality for CR-warped product submanifolds in a generalized Sasakian space form by considering the nontrivial case $N_T \times_f N_\perp$. We also find some applications of the inequality in the compact Riemannian manifold by using integration theory on manifolds. Our future work then is combining the work done in this paper with the techniques of singularity theory presented in [20–23] to explore new results on manifolds.

2. Preliminaries

An almost contact metric manifold $(\tilde{M}, g, \varphi, \eta, \xi)$ is an odd-dimensional manifold $\tilde{M}$, endowed with a field φ of an endomorphism on the tangent space, the Reeb vector field ξ, a 1-form η and admits Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (5)$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(X)\eta(Y), \quad (6)$$

$$\eta(U) = g(U, \xi), \quad (7)$$

for any $U, V \in \Gamma(T\tilde{M})$. An almost contact metric manifold $(\tilde{M}, g, \varphi, \eta, \xi)$ is said to be trans-Sasakian manifold (cf. [12, 13]) if

$$(\tilde{\nabla}_U \varphi)V = \alpha(g(U, V)\xi - \eta(U)V) - \beta(g(\varphi U, V) - \eta(V)\varphi U), \quad (8)$$

for any $U, V \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Riemannian connection on $(\tilde{M}, g)$. If we replace $U = \xi$ and $V = \xi$ in (8), we find that $(\tilde{\nabla}_\xi \varphi)\xi = 0$, which implies that $\tilde{\nabla}_\xi \xi = 0$. For a trans-Sasakian manifold, (8) implies

$$\tilde{\nabla}_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi). \quad (9)$$

Remark 3. We classify a trans-Sasakian manifold in the following way:

(a) If $\alpha = 0$ and $\beta = 0$ in (8), a trans-Sasakian manifold becomes a cosymplectic manifold [7]

(b) If $\alpha = 1$ and $\beta = 0$ in (8), it is a Sasakian manifold [5]

(c) If $\alpha = 0$ and $\beta = 1$ in (8), it is a Kenmotsu manifold [6]

(d) α-Sasakian manifold and β-Kenmotsu manifold can be derived from the tans-Sasakian manifold when $\beta = 0$ and $\alpha = 0$ in (8), respectively

Given an almost contact metric manifold $(\tilde{M}, \varphi, \eta, \xi)$, it is said to be a generalized Sasakian space form $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ if there exist three functions λ_1, λ_2, and λ_3 on $\tilde{M}$ such that the curvature tensor $\tilde{R}$ is

$$\begin{aligned}\tilde{R}(X, Y, Z, W) = \lambda_1(&g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ &+ \lambda_2(g(X, \varphi Z)g(\varphi Y, W) - g(Y, \varphi Z)g(\varphi X, W) \\ &+ 2g(X, \varphi Y)g(\varphi Z, W)) + \lambda_3(\eta(X)\eta(Z)g(Y, W) \\ &- \eta(Y)\eta(Z)g(X, W) + g(X, Z)\eta(Y)\eta(W) \\ &- g(Y, Z)\eta(X)\eta(W)),\end{aligned} \quad (10)$$

for any $X, Y, Z, W \in \Gamma(T\tilde{M})$ [24].

Remark 4. The characteristics are as follows:

(a) If $\lambda_1 = c + 3/4$ and $\lambda_2 = \lambda_3 = (c - 1)/4$, then $\tilde{M}$ is a Sasakian space form [25]

(b) If $\lambda_1 = (c - 3)/4$ and $\lambda_2 = \lambda_3 = (c + 1)/4$, then $\tilde{M}$ is a Kenmotsu space form [6]

(c) If $\lambda_1 = \lambda_2 = \lambda_3 = c/4$, then $\tilde{M}$ is a cosymplectic space form [26]

Let ∇ and $\nabla^\perp$ be the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^\perp M$ of a submanifold M of an almost contact metric manifold $(\tilde{M}, \varphi, \eta, \xi)$ with the induced metric g. Then, the Gauss and Weingarten formulas are given by

$$\text{(i) } \tilde{\nabla}_U V = \nabla_U V + h(U, V), \text{(ii) } \tilde{\nabla}_U N = -A_N U + \nabla_U^\perp N, \quad (11)$$

for $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator on

M. We have the relation:

$$g(h(U,V),N)=g(A_N U,V), \tag{12}$$

for $U,V\in\Gamma(TM)$ and $N\in\Gamma(T^\perp M)$. For any tangent vector $U\in\Gamma(TM)$ and normal vector $N\in\Gamma(T^\perp M)$, we have

$$\begin{aligned}&\text{(i) } \varphi U = TU + FU,\\ &\text{(ii) } \varphi N = tN + fN,\end{aligned} \tag{13}$$

where $TU(tN)$ and $FU(fN)$ are tangential and normal components of $\varphi U(\varphi N)$, respectively. If T is identically zero, then a submanifold M is called *a totally real submanifold*. The Gauss equation with curvature tensors $\tilde{R}$ and R on $\tilde{M}$ and M, respectively, is defined by

$$\begin{aligned}\tilde{R}(U,V,Z,W) = R(U,V,Z,W) + g(h(U,Z),h(V,W))\\ - g(h(U,W),h(V,Z)),\end{aligned} \tag{14}$$

for any $U,V,Z,W\in\Gamma(TM)$. The mean curvature vector H for a local frame $\{e_1,e_2,\cdots,e_n\}$ of the tangent space T M on M is defined by

$$\begin{aligned}&\text{(i) } H=\frac{1}{n}\text{trace}(h)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i),\\ &\text{(ii) } \|T\|^2=\sum_{i,j=1}^{n}g^2(\varphi e_i,e_j).\end{aligned} \tag{15}$$

The scalar curvature τ for a Riemannian submanifold M is given by

$$\tau=\sum_{1\le i=j\le n}K(e_i\wedge e_j), \tag{16}$$

where $K(e_i\wedge e_j)$ is the sectional curvature of section plane and spanned by e_i and e_j. Let G_r be an r-plane section on TM and let $\{e_1,e_2,\cdots,e_r\}$ be a orthonormal basis of G_r. Then, the scalar curvature $\tau(G_r)$ of G_r is given by

$$\tau(G_r)=\sum_{1\le i=j\le r}K(e_i\wedge e_j). \tag{17}$$

Similarly, we classify a Riemannian submanifold M said to be *totally umbilical* and *totally geodesic* if $h(U,V)=g(U,V)H$ and $h(U,V)=0$, respectively, for any $U,V\in\Gamma(TM)$.

Furthermore, if $H=0$, then M is *minimal* in $(\tilde{M},\varphi,\eta,\xi)$. If φ preserves any tangent space of M tangent to the structure vector field ξ, i.e., $\varphi(T_pM)\subseteq T_pM$, for each $p\in M$; then, M is called *an invariant submanifold*. Similarly, M is called *an anti-invariant submanifold* tangent to the Reeb vector field ξ if $\varphi(T_pM)\subseteq T^\perp M$, for each $p\in M$. To generalize these definitions, we give the following definition.

Definition 5. A submanifold M including the structure vector field ξ of an almost contact metric manifold $(\tilde{M},\varphi,\eta,\xi)$ is characterized to be a contact CR-submanifold if the pair of orthogonal distributions $\mathscr{D}$ and $\mathscr{D}^\perp$ exists such that

(i) $TM=\mathscr{D}\oplus\mathscr{D}^\perp\oplus\langle\xi\rangle$, where $\langle\xi\rangle$ is 1-dimensional distribution spanned by ξ

(ii) the distribution $\mathscr{D}$is invariant, i.e., $\varphi(\mathscr{D})\subseteq\mathscr{D}$

(iii) the distribution $\mathscr{D}^\perp$ is anti-invariant, i.e., $\varphi\mathscr{D}^\perp\subseteq(T^\perp M)$

If the dimensions of *invariant* distribution $\mathscr{D}$ and anti-invariant distribution $\mathscr{D}^\perp$ of a contact CR-submanifold of $(\tilde{M},\varphi,\eta,\xi)$ are d_1 and d_2, respectively, such that $d_2=0$, then M is invariant and anti-invariant if $d_1=0$. It is called *a proper contact* CR-*submanifold* if neither $d_1=0$ nor $d_2=0$. The normal bundle $T^\perp M$ of a contact CR-submanifold with an invariant subspace μ under φ can be decomposed as

$$T^\perp M=\varphi\mathscr{D}^\perp\oplus\mu. \tag{18}$$

M is a compact orientable Riemannian submanifold without boundary. Thus, we have

$$\int_M \Delta f\,dV=0, \tag{19}$$

where dV is the volume element of M [27].

3. Main Inequalities of CR-Warped Products

We are mentioning that in the following study, we shall consider the structure field ξ tangent to the base manifold of warped product manifold. In this main section, we classify the contact CR-warped product submanifolds in a trans-Sasakian manifold.

Lemma 6. *Let $M=N_T\times_f N_\perp$ be a CR-warped product submanifold in a trans-Sasakian manifold. Then,*

$$\begin{aligned}g(h(\varphi X,Y),\varphi Z)=g(h(X,Y),\varphi Z)=0,\\ g(h(X,X),\beta)=-g(h(\varphi X,\varphi X),\beta),\end{aligned} \tag{20}$$

for $X,Y\in\Gamma(TN_T)$, $Z,W\in\Gamma(TN_\perp)$, and $\beta\in\Gamma(\mu)$.

Proof. From (11)(i), (8), and (5), we obtain

$$g(h(\varphi X,Y),\varphi Z)=g(\bar{\nabla}_Y\varphi X,\varphi Z)=g(\bar{\nabla}_Y X,Z)-\eta(\bar{\nabla}_Y X)\eta(Z). \tag{21}$$

Since N_T is totally geodesic in M with $\xi\in\Gamma(TN_T)$, (9) implies the results. ☐

Lemma 7. *Let $\ell: M=N_T\times_f N_\perp\longrightarrow(\tilde{M},\varphi,\eta,\xi)$ be an isometric immersion from an n-dimensional contact CR-*

warped product submanifold into a trans-Sasakian manifold $(\tilde{M}, \varphi, \eta, \xi)$ *such that* N_T *is invariant submanifold of dimension* $n_1 = 2d_1 + 1$ *tangent to* ξ. *Then,* N_T *is always* ℓ*-minimal submanifold of* $\tilde{M}$.

Proof. We skip the proof of the above lemma due to the similar proof of Theorem 4.2 in [4]. □

By helping the above lemma, the following result can be obtained as follows.

Proposition 8. *Assume that* $\ell : M = N_T \times_f N_\perp \longrightarrow \tilde{M}$ *is an isometric immersion of an* n *-dimensional contact CR-warped product submanifold* M *into a trans-Sasakian manifold* $\tilde{M}$. *Thus,*

(i) *the squared norm of the second fundamental form of* M *is satisfied:*

$$\|h\|^2 \geq 2\left(n_2\|\nabla \ln f\|^2 + \tilde{\tau}(TM) - \tilde{\tau}(TN_T) - \tilde{\tau}(TN_\perp) - n_2\Delta(\ln f)\right), \tag{22}$$

where n_2 *is the dimension of anti-invariant submanifold* $N_\perp$ *and* Δ *is the Laplacian operator of* N_T

(ii) *the equality holds in (22) if and only if* N_T *is totally geodesic and* $N_\perp$ *is totally umbilical in* $\tilde{M}$. *Moreover,* M *is minimal submanifold of* $\tilde{M}$

Proof. It can be easily proven as the proof of Theorem 4.4 in [4] if we consider a Riemannian submanifold as a CR-warped product submanifold, and the base manifold is a trans-Sasakian manifold instead of a Kenmotsu manifold.

Now, we prove our main theorem using Proposition 8 for a generalized Sasakian space form. □

Theorem 9. *Let* $\ell : M = N_T \times_f N_\perp \longrightarrow \tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ *be an isometric immersion from an* n *-dimensional contact CR-warped product submanifold of a generalized Sasakian space form* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$. *Then, the second fundamental form is given by*

$$\|h\|^2 \geq 2n_2\left\{\|\nabla \ln f\|^2 + \lambda_1 n_1 + \frac{3}{2}\lambda_2 - \lambda_3 - \Delta(\ln f)\right\}, \tag{23}$$

where $n_1 = \dim N_T$, $n_2 = \dim N_\perp$, *and* Δ *is the Laplacian operator on* N_T. *The equality holds in (23) if and only if* N_T *and* $N_\perp$ *are totally geodesic and totally umbilical submanifolds in* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$, *respectively, and hence,* M *is a minimal submanifold of* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$.

Proof. Substituting $X = W = e_i$ and $Y = Z = e_j$ in (10), we get

$$\begin{aligned}\tilde{R}(e_i, e_j, e_j, e_i) = {}& \lambda_1\left\{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_i, e_j)\right\} \\ & + \lambda_2\left\{g(e_i, \varphi e_j)g(\varphi e_j, e_i) - g(e_i, \varphi e_i)g(e_j, \varphi e_j) \right.\\ & \left. + 2g^2(\varphi e_j, e_i)\right\} + \lambda_3\left\{g(e_i, e_i)\eta(e_j)\eta(e_j) \right.\\ & - g(e_i, e_j)\eta(e_i)\eta(e_i) + g(e_j, e_j)\eta(e_i)\eta(e_i) \\ & \left. \cdot\, g(e_j, e_i)(e_i)\eta(e_j)\right\}.\end{aligned} \tag{24}$$

Summing up along the orthonormal vector fields of M, it can be derived from the above as

$$2\tilde{\tau}(TM) = \lambda_1 n(n-1) + 3\lambda_2 \sum_{1\leq i=j\leq n} g^2(\varphi e_i, e_j) - 2\lambda_3(n-1). \tag{25}$$

As for an n-dimensional CR-warped product submanifold tangent ξ, one can derive $\|T\|^2 = n - 1$ from (15)(ii); we obtain

$$2\tilde{\tau}(TM) = \lambda_1 n(n-1) + 3(n-1)\lambda_2 - 2\lambda_3(n-1). \tag{26}$$

On the other hand, by helping the frame field of $TN_\perp$, we have

$$2\tilde{\tau}(TN_\perp) = \lambda_1 n_2(n_2 - 1). \tag{27}$$

Similarly, we considered that ξ is tangent to invariant submanifold N_T. Then, using the frame vector fields of $T N_T$, we get from (24)

$$2\tilde{\tau}(TN_T) = \lambda_1 n_1(n_1 - 1) + 3\lambda_2(n_1 - 1) - 2\lambda_3(n_1 - 1). \tag{28}$$

Therefore, using (26), (27), and (28) in Proposition 8, we get the required result. The equality case follows from Proposition 8. Thus, the proof is completed. □

4. Geometric Applications

Remark 10. Consider $\lambda_1 = (c - 3)/4$ and $\lambda_2 = \lambda_3 = (c + 1)/4$ in Theorem 9. It is the generalization of Theorem 4.6 in [4] for the result of contact CR-warped products in Kenmotsu space forms.

Remark 11. If we put $\lambda_1 = (c + 3)/4$ and $\lambda_2 = \lambda_3 = (c - 1)/4$ in Theorem 9, then it generalizes Corollary 4.6 in [5].

Remark 12. If $\lambda_1 = \lambda_2 = \lambda_3 = c/4$ in Theorem 9, then Theorem 9 coincides with Theorem 1.2 in [26].

Corollary 13. *Let* $\ln f$ *be a harmonic function on* N_T. *Then there does not exist any CR-warped product submanifold* $N_T \times_f N_\perp$ *into a generalized Sasakian space form* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ *with* $c \leq -\lambda_1$.

Corollary 14. *Assume that* $\ln f$ *is a nonnegative eigenfunction on* N_T *with the corresponding nonzero positive eigenvalue. Then, there does not exist any CR-warped product submanifold* $N_T \times_f N_\perp$ *into a generalized Sasakian space form* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$ *with* $c \le -\lambda_1$.

Theorem 15. *Let* $M = N_T \times_f N_\perp$ *be a compact orientated CR-warped product into a generalized Sasakian space form* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$. *Then,* M *is a simply Riemannian product if*

$$\|h\|^2 \ge 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2, \tag{29}$$

where $n_1 = \dim N_T$ *and* $n_2 = \dim N_\perp$.

Proof. From Theorem 9, we get

$$\begin{aligned} &\|h\|^2 \ge 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_3 - n_2 \Delta(\ln f) + n_2\|\nabla \ln f\|^2, \\ &n_2\|\nabla \ln f\|^2 + 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_3 - \|h\|^2 \le n_2 \Delta(\ln f). \end{aligned} \tag{30}$$

We obtain

$$\begin{aligned} &\int_{N_T \times q} \left(2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2 + n_2\|\nabla \ln f\|^2 - \|h\|^2\right) \mathrm{d}V \\ &\quad \le n_2 \int_{N_T \times q} \Delta(\ln f) \mathrm{d}V = 0. \end{aligned} \tag{31}$$

Now, if

$$\|h\|^2 \ge 2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2. \tag{32}$$

Then, from (31), we find

$$\int_{N_T \times q} \left(\|\nabla \ln f\|^2\right) dV \le 0, \tag{33}$$

which is impossible for a positive integral function, and hence, $\nabla \ln f = 0$, i.e., f is a constant function on N_T. Thus, by the definition of a warped product manifold, M is trivial. The converse part is straightforward. □

Corollary 16. *Assume that* $M = N_T \times_f N_\perp$ *is a CR-warped product submanifold in a generalized Sasakian space form* $\tilde{M}(\lambda_1, \lambda_2, \lambda_3)$. *Let* N_T *be a compact invariant submanifold and* γ *be nonzero eigenvalue of the Laplacian on* N_T. *Then,*

$$\begin{aligned} \int_{N_T \times q} \|h\|^2 dV_T &\ge (2\lambda_1 n_1 n_2 + 3\lambda_2 n_2 - 2\lambda_3 n_2) Vol(N_T) \\ &\quad + 2n_2 \gamma \int_{N_T \times q} (\ln f)^2 dV_T. \end{aligned} \tag{34}$$

Proof. From the minimum principle property, we obtain

$$\int_{N_T} \|\nabla \ln f\|^2 dV_T \ge \gamma \int_{N_T} (\ln f)^2 dV_T. \tag{35}$$

From (23) and (35), we get the required result (34). □

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no competing interest.

Authors' Contributions

All authors have equal contribution and finalized.

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