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Research Article

On Fractional Diffusion Equation with Caputo-Fabrizio Derivative and Memory Term

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In this paper, we examine a nonlinear fractional diffusion equation containing viscosity terms with derivative in the sense of Caputo-Fabrizio. First, we establish the local existence and uniqueness of lightweight solutions under some assumptions about the input data. Then, we get the global solution using some new techniques. Our main idea is to combine theories of Banach's fixed point theorem, Hilbert scale theory of space, and some Sobolev embedding.

1. Introduction

The fractional calculation has a long history and plays an important role in the simulation of physical phenomena or real life, for example, mechanics, electricity, chemistry, biology, economics, notably control theory, and images. It should be noted that the standard mathematical models of integer derivatives, including nonlinear models, do not work fully in many cases. Therefore, the advent of fractional calculus was significant in modeling physical and engineering processes, and it can be said that it is one of the best descriptors using fractional differential equations. In a series of research directions on fractional differential equations (FDE), the most prominent is the appearance of two derivatives: Caputo derivative and Riemann-Liouville derivative. Some works are attracting the attention of the community, like Debbouche and his group [1–3], Karapinar et al. [4–11], Inc and his group [12–16], Tuan and his group [17–20], and the references as follows: [21–23].

In this paper, we consider the fractional Sobolev equation:

$$\begin{cases} {}_{CF}D_t^\alpha u = \Delta u + G(u) + \displaystyle\int_0^t \psi(t-z)K(u(z))dz, (x,t) \in \mathscr{M} \times (0,T), \\ u = 0, (x,t) \in \partial\mathscr{M} \times (0,T), \\ u(x,0) = u0(x), \end{cases} \tag{1}$$

where ${}_{CF}D_t^\alpha$ is the Caputo-Fabrizio operator for fractional derivatives of order α which is defined as (see [24])

$${}_{CF}D_t^\alpha v(t) = \frac{H(\alpha)}{1-\alpha}\int_0^t \mathscr{D}_\alpha(t-\nu)\frac{\partial v(\nu)}{\partial \nu} d\nu, \quad \text{for } t \geq 0, \tag{2}$$

where we denote by the kernel $\mathscr{D}_\alpha(z) = \exp(-(\alpha/(1-\alpha))z)$ and $H(\alpha)$ satisfies $H(0) = H(1) = 1$ (see, e.g., [25, 26]).

The Caputo-Fabrizio fractional derivative was presented in 2015 [25] with the aim of avoiding singular kernels. It is also the convolution of the exponential function and the first-order derivative. The Caputo-Fabrizio fraction derivative is an operator that has been widely applied to several derivative modes in many fields, such as biology, physics, control systems, materials science, dynamics, and liquid learning [27–32].

Our main aim in this paper is to provide the local and global existence for problem (1) under some various assumptions on the input data. The difficulty in studying this problem is from the memory viscoelastic model appearing in the main equation. This term makes some of the assessments more complicated. Another difficulty is the study of the existence of global solutions. The topic of the existence of global solutions is still challenging for many mathematicians today. In the paper, we have to use a new norm in weighted space, thanks to the work of [33], to establish the global solution. The two main results in the paper are shown as follows:

(i) The first result is related to the existence of local solutions. The main technique is to apply Banach's fixed point theorem

(ii) The second result is very interesting, proving the existence of a global solution. To do this, we have to thank a lemma in [33], where we have chosen suitable assumptions for functions G and K, to obtain our purpose

The paper is organized as follows. In Section 2, we give preliminaries which are useful for the next results. Section 3 shows the local existence results. In Section 4, we provide global existence results.

2. Preliminaries

We recall the Hilbert scale space, which is given as follows:

$$H^r(\mathscr{M}) = \left\{ f \in L^2(\mathscr{M}),\ \sum_{n=1}^{\infty} \lambda_n^r \langle f, e_n \rangle_{L^2(\mathscr{M})}^2 < \infty \right\}, \tag{3}$$

for any $r \geq 0$. Here, the symbol $\langle \cdot, \cdot \rangle_{L^2(\mathscr{M})}$ denotes the inner product in $L^2(\mathscr{M})$. It is well known that $\mathscr{H}^r(\mathscr{M})$ is a Hilbert space corresponding to the following norm:

$$\|f\|_{H^r(\mathscr{M})} = \sqrt{\sum_{n=1}^{\infty} \lambda_n^r \langle f, e_n \rangle_{L^2(\mathscr{M})}^2}, \quad f \in \mathscr{H}^r(\mathscr{M}). \tag{4}$$

$H^\nu(\Omega) \equiv D((-\mathbb{L})^\nu)$ is a Hilbert space. Then, $D((-\mathbb{L})^{-\nu})$ is a Hilbert space with the norm:

$$\|v\|_{D((-\mathbb{L})^{-\nu})} = \left(\sum_{j=1}^{\infty} \left| \langle v, e_j \rangle \right|^2 \lambda_j^{-2\nu} \right)^{1/2}, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ in the latter equality denotes the duality between $D((-\mathbb{L})^{-\nu})$ and $D((-\mathbb{L})^{\nu})$.

Definition 1. The function v is called a mild solution of problem (1) if it satisfies that

$$\begin{aligned} \theta(t) = \mathbf{P}_\alpha(t) u_0 &+ \int_0^t \mathbf{P}_\alpha(t-\tau) G(\theta(\tau)) d\tau \\ &+ \int_0^t \mathbf{P}_\alpha(t-\tau) \int_0^\tau \psi(\tau-\xi) K(\theta(\xi)) d\xi d\tau, \end{aligned} \tag{6}$$

where $\mathbf{P}_\alpha(t)$ is defined by

$$\begin{aligned} \mathbf{P}_\alpha(t) w = (1+\bar{\alpha}\lambda_n)^{-1} \exp\left(\frac{-\alpha\lambda_n}{1+\bar{\alpha}\lambda_n} t \right) \langle w, e_n \rangle_{L^2(\mathscr{M})} e_n(x), \bar{\alpha} \\ = 1-\alpha, \end{aligned} \tag{7}$$

for any $w \in L^2(\mathscr{M})$.

Lemma 1. *Let* $\theta \in H^{r-2}(\mathscr{M}) \cap H^{r-2-2\beta}(\mathscr{M})$. *Then,*

$$\|\mathbf{P}_\alpha(t)\theta\|_{H^r(\mathscr{M})} \leq \bar{C}_{1,\alpha,\beta} t^{-\beta} \|\theta\|_{H^{r-2}(\mathscr{M})} + \bar{C}_{2,\alpha,\beta} t^{-\beta} \|\theta\|_{H^{r-2-2\beta}(\mathscr{M})}, \tag{8}$$

for any $0 < \beta < 1$.

Proof. By the definitions of the norm in $H^r(\mathscr{M})$ and using the inequality $e^{-y} \leq C - \beta y^{-\beta}$ for $\beta > 0$, we get the following confirmation:

$$\begin{aligned} \|\mathbf{P}_\alpha(t)\theta\|_{H^r(\mathscr{M})} &= \sqrt{\sum_{n=1}^{\infty} \lambda_n^r (1+\bar{\alpha}\lambda_n)^{-2} \exp\left(\frac{-2\alpha\lambda_n}{1+\bar{\alpha}\lambda_n} t \right) \langle \theta, e_n \rangle_{L^2(\mathscr{M})}^2} \\ &\leq C_\beta (1-\alpha)^{-1} t^{-\beta} \sqrt{\sum_{n=1}^{\infty} \lambda_n^{r-2} \left(\frac{1+(1-\alpha)\lambda_n}{\alpha\lambda_n} \right)^{2\beta} \langle \theta, e_n \rangle_{L^2(\mathscr{M})}^2}. \end{aligned} \tag{9}$$

Since $0 < \beta < 1$, we know that

$$\left(\frac{1+(1-\alpha)\lambda_n}{\alpha\lambda_n} \right)^{2\beta} \leq \left(\frac{(1-\alpha)^2}{\alpha^2} + \frac{1}{\alpha^2\lambda_n^2} \right)^\beta \leq \left(\frac{(1-\alpha)^{2\beta}}{\alpha^{2\beta}} + \alpha^{-2\beta}\lambda_n^{-2\beta} \right). \tag{10}$$

This follows from (9) that

$$\begin{aligned} \|\mathbf{P}_\alpha(t)\theta\|_{H^r(\mathscr{M})} &\leq C_\beta (1-\alpha)^{-1} t^{-\beta} \frac{(1-\alpha)^\beta}{\alpha^\beta} \sqrt{\sum_{n=1}^{\infty} \lambda_n^{r-2} \langle \theta, e_n \rangle_{L^2(\mathscr{M})}^2} \\ &\quad + C_\beta (1-\alpha)^{-1} \alpha^{-\beta} t^{-\beta} \sqrt{\sum_{n=1}^{\infty} \lambda_n^{r-2-2\beta} \langle \theta, e_n \rangle_{L^2(\mathscr{M})}^2} \\ &\leq \bar{C}_{1,\alpha,\beta} t^{-\beta} \|\theta\|_{H^{r-2}(\mathscr{M})} + \bar{C}_{2,\alpha,\beta} t^{-\beta} \|\theta\|_{H^{r-2-2\beta}(\mathscr{M})}. \end{aligned} \tag{11}$$

□

3. Local Existence Results

In this section, we give the following theorem which shows the local existence result.

Theorem 1. *Let the two functions G and K be*

$$\begin{aligned} \|G(\theta_1) - G(\theta_2)\|_{L^2(M)} &\le B_G \|\theta_1 - \theta_2\|_{L^2(\mathscr{M})}, \\ \|K(\theta_1) - K(\theta_2)\|_{L^2(M)} &\le B_K \|\theta_1 - \theta_2\|_{L^2(\mathscr{M})}, \end{aligned} \tag{12}$$

for constants $B_G, B_K \ge 0$. Let us assume that there exists δ such that

$$|\psi(z)| \le \mathscr{D} z^{-\delta}, \delta < 1. \tag{13}$$

Let $u_0 \in H^{p-2}(\mathscr{M}) \cap H^{p-2-2\beta}(\mathscr{M})$. Then, problem (1) has a local mild solution:

$$u \in L^\infty_\vartheta(0, T; H^p(\mathscr{M})), \tag{14}$$

where

$$0 < \beta \le \vartheta < 1, 0 \le p \le 2. \tag{15}$$

Proof. Let the function $\mathfrak{F}$ be as follows:

$$\begin{aligned} \mathfrak{F}\theta(t) &= \mathbf{P}_\alpha(t) u_0 + \int_0^t \mathbf{P}_\alpha(t-\tau) G(\theta(\tau)) d\tau \\ &\quad + \int_0^t \mathbf{P}_\alpha(t-\tau) \int_0^\tau \psi(\tau - \xi) K(\theta(\xi)) d\xi d\tau \\ &= \mathbf{P}_\alpha(t) u_0 + \mathfrak{F}_1\theta(t) + \mathfrak{F}_2\theta(t). \end{aligned} \tag{16}$$

Step 1. Estimate $\|\mathfrak{F}_1\theta_1 - \mathfrak{F}_1\theta_2\|_{H^p(\mathscr{M})}$ for any θ_1, θ_2 that belongs to the space $H^p(\mathscr{M})$.

From the definition of $\mathfrak{F}_1$ as in (16), we find that

$$\begin{aligned} \|\mathfrak{F}_1\theta_1 - \mathfrak{F}_1\theta_2\|_{H^p(\mathscr{M})} &= \left\| \int_0^t \mathbf{P}_\alpha(t-\tau) G(\theta_1(\tau)) d\tau - \int_0^t \mathbf{P}_\alpha(t-\tau) G(\theta_2(\tau)) d\tau \right\|_{H^p(\mathscr{M})} \\ &\le \bar{C}_{1,\alpha,\beta} \int_0^t (t-\tau)^{-\beta} \|G(\theta_1(\tau)) - G(\theta_2(\tau))\|_{H^{p-2}(\mathscr{M})} d\tau \\ &\quad + \bar{C}_{2,\alpha,\beta} \int_0^t (t-\tau)^{-\beta} \|G(\theta_1(\tau)) - G(\theta_2(\tau))\|_{H^{p-2-2\beta}(\mathscr{M})} d\tau. \end{aligned} \tag{17}$$

Since $p \le 2$, we know the Sobolev embedding $L^2(\mathscr{M})^\circ H^{p-2}(\mathscr{M})$, and so we get

$$\begin{aligned} \|G(\theta_1(\tau)) - G(\theta_2(\tau))\|_{H^{p-2}(\mathscr{M})} &\le C_1 \|G(\theta_1(\tau)) - G(\theta_2(\tau))\|_{L^2(\mathscr{M})} \\ &\le C_1 B_G \|\theta_1(\tau) - \theta_2(\tau)\|_{L^2(\mathscr{M})}, \\ \|G(\theta_1(\tau)) - G(\theta_2(\tau))\|_{H^{p-2-2\beta}(\mathscr{M})} &\le C_{1,\beta} \|G(\theta_1(\tau)) - G(\theta_2(\tau))\|_{L^2(\mathscr{M})} \\ &\le C_{1,\beta} B_K \|\theta_1(\tau) - \theta_2(\tau)\|_{L^2(\mathscr{M})}. \end{aligned} \tag{18}$$

From two above observations, we deduce that

$$\begin{aligned} t^\vartheta \|\mathfrak{F}_1\theta_1 - \mathfrak{F}_1\theta_2\|_{H^p(\mathscr{M})} &\le \left(\bar{C}_{1,\alpha,\beta} C_1 B_G + \bar{C}_{2,\alpha,\beta} C_1 B_K\right) t^\vartheta \int_0^t (t-\tau)^{-\beta} \|\theta_1(\tau) \\ &\quad - \theta_2(\tau)\|_{L^2(\mathscr{M})} d\tau \\ &\le \bar{C}_{3,p,\alpha,\beta} t^\vartheta \int_0^t (t-\tau)^{-\beta} \tau^{-\vartheta} \tau^\vartheta \|\theta_1(\tau) - \theta_2(\tau)\|_{H^p(\mathscr{M})} d\tau \\ &\le \bar{C}_{3,p,\alpha,\beta} t^\vartheta \left(\int_0^t (t-\tau)^{-\beta} \tau^{-\vartheta} d\tau \right) \|\theta_1 - \theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &= \bar{C}_{3,p,\alpha,\beta} t^{1-\beta} \mathbf{B}(1-\beta, 1-\vartheta) \|\theta_1 - \theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &\le \bar{C}_{3,p,\alpha,\beta} T^{1-\beta} \mathbf{B}(1-\beta, 1-\vartheta) \|\theta_1 - \theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}. \end{aligned} \tag{19}$$

Because the right side of the above expression does not depend on t, we have the following assertion:

$$\begin{aligned} &\|\mathfrak{F}_1\theta_1 - \mathfrak{F}_1\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &\quad \le \bar{C}_{3,p,\alpha,\beta} T^{1-\beta} \mathbf{B}(1-\beta, 1-\vartheta) \|\theta_1 - \theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}. \end{aligned} \tag{20}$$

Step 2. Estimate $\|\mathfrak{F}_2\theta_1 - \mathfrak{F}_2\theta_2\|_{H^p(\mathscr{M})}$ for any θ_1, θ_2 that belongs to the space $H^p(\mathscr{M})$.

From the definition of $\mathscr{F}_1$ as in (16), we find that

$$\begin{aligned} \|\mathfrak{F}_2\theta_1 - \mathfrak{F}_2\theta_2\|_{H^p(\mathscr{M})} &= \Bigg\| \int_0^t \mathbf{P}_\alpha(t-\tau) \int_0^\tau \psi(\tau-\xi) K(\theta_1(\xi)) d\xi d\tau \\ &\quad - \int_0^t \mathbf{P}_\alpha(t-\tau) \int_0^\tau \psi(\tau-\xi) K(\theta_2(\xi)) d\xi d\tau \Bigg\|_{H^p(\mathscr{M})} \\ &\le \bar{C}_{1,\alpha,\beta} \int_0^t (t-\tau)^{-\beta} \Bigg\| \int_0^\tau \psi(\tau-\xi) K(\theta_1(\xi)) d\xi \\ &\quad - \int_0^\tau \psi(\tau-\xi) K(\theta_2(\xi)) d\xi \Bigg\|_{H^{p-2}(\mathscr{M})} d\tau \\ &\quad + \bar{C}_{2,\alpha,\beta} \int_0^t (t-\tau)^{-\beta} \Bigg\| \int_0^\tau \psi(\tau-\xi) K(\theta_1(\xi)) d\xi \\ &\quad - \int_0^\tau \psi(\tau-\xi) K(\theta_2(\xi)) d\xi \Bigg\|_{H^{p-2-2\beta}(\mathscr{M})} d\tau. \end{aligned} \tag{21}$$

It is easy to see that

$$\begin{aligned} &\bar{C}_{1,\alpha,\beta} \left\| \int_0^\tau \psi(\tau-\xi) K(\theta_1(\xi)) d\xi - \int_0^\tau \psi(\tau-\xi) K(\theta_2(\xi)) d\xi \right\|_{H^{p-2}(\mathscr{M})} \\ &\le \bar{C}_{1,\alpha,\beta} C_1 B_G \mathscr{D} \int_0^\tau (\tau-\xi)^{-\delta} \|\theta_1(\xi) - \theta_2(\xi)\|_{L^2(\mathscr{M})} d\xi \\ &\le \bar{C}_{1,\alpha,\beta} C_1 B_G \mathscr{D} \int_0^\tau (\tau-\xi)^{-\delta} \xi^{-\vartheta} \xi^\vartheta \|\theta_1(\xi) - \theta_2(\xi)\|_{L^2(\mathscr{M})} d\xi \\ &\le \bar{C}_{1,\alpha,\beta} C_1 B_G \mathscr{D} \left(\int_0^\tau (\tau-\xi)^{-\delta} \xi^{-\vartheta} d\xi \right) \|\theta_1 - \theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &= \bar{C}_{1,\alpha,\beta} C_1 B_G \mathscr{D} \tau^{1-\delta-\vartheta} B(1-\delta, 1-\vartheta) \|\theta_1 - \theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}. \end{aligned} \tag{22}$$

By a similar way as above, we also obtain that

$$\bar{C}_{2,\alpha,\beta}\int_0^t (t-\tau)^{-\beta}\left\|\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi - \int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi\right\|_{H^{p-2-2\beta}(\mathscr{M})} d\tau \\ \le \bar{C}_{2,\alpha,\beta}C_{1,\beta}B_K\mathscr{D}\tau^{1-\delta-\vartheta}B(1-\delta,1-\vartheta)\|\theta_1-\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}. \tag{23}$$

From two recent observations and noting that $\beta+\delta<2$, we find that

$$t^\vartheta\|\mathfrak{F}_2\theta_1-\mathfrak{F}_2\theta_2\|_{H^p(\mathscr{M})} = \bar{C}_3B(1-\delta,1-\vartheta)B(1-\beta,2-\delta-\vartheta)t^{2-\beta-\delta}\|\theta_1-\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ \le \bar{C}_3B(1-\delta,1-\vartheta)B(1-\beta,2-\delta-\vartheta)T^{2-\beta-\delta}\|\theta_1-\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}, \tag{24}$$

where

$$\bar{C}_3 = \bar{C}_{1,\alpha,\beta}C_1B_G\mathscr{D} + \bar{C}_{2,\alpha,\beta}C_{1,\beta}B_K\mathscr{D}. \tag{25}$$

Due to the right hand side of (24) being independent of t, we can deduce that

$$\|\mathfrak{F}_2\theta_1-\mathfrak{F}_2\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \le \bar{C}_3\mathbf{B}(1-\delta,1-\vartheta)\mathbf{B}(1-\beta,2-\delta-\vartheta)T^{2-\beta-\delta}\|\theta_1-\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}. \tag{26}$$

Combining (20) and (26), we derive that

$$\begin{aligned}\|\mathfrak{F}\theta_1-\mathfrak{F}\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} &\le \|\mathfrak{F}_1\theta_1-\mathfrak{F}_1\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &\quad + \|\mathfrak{F}_2\theta_1-\mathfrak{F}_2\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &\le \bar{C}_{3,p,\alpha,\beta}T^{1-\beta}\mathbf{B}(1-\beta,1-\vartheta)\|\theta_1 \\ &\quad -\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))} \\ &\quad + \bar{C}_3\mathbf{B}(1-\delta,1-\vartheta)\mathbf{B}(1-\beta,2-\delta \\ &\quad -\vartheta)T^{2-\beta-\delta}\|\theta_1-\theta_2\|_{L^\infty_\vartheta(0,T;H^p(\mathscr{M}))}.\end{aligned} \tag{27}$$

Moreover, by applying Lemma 1 and noting that $\vartheta\ge\beta$, we can confirm the following results:

$$\begin{aligned}t^\vartheta\|\mathbf{P}_\alpha(t)u_0\|_{H^p(\mathscr{M})} &\le t^\vartheta\left(\bar{C}_{1,\alpha,\beta}t^{-\beta}\|u_0\|_{H^{p-2}(\mathscr{M})} + \bar{C}_{2,\alpha,\beta}t^{-\beta}\|u_0\|_{H^{p-2-2\beta}(\mathscr{M})}\right) \\ &\le \bar{C}_{1,\alpha,\beta}t^{\vartheta-\beta}\|u_0\|_{H^{p-2}(\mathscr{M})} + \bar{C}_{2,\alpha,\beta}t^{\vartheta-\beta}\|u_0\|_{H^{p-2-2\beta}(\mathscr{M})} \\ &\le \bar{C}_{1,\alpha,\beta}T^{\vartheta-\beta}\|u_0\|_{H^{p-2}(\mathscr{M})} + \bar{C}_{2,\alpha,\beta}T^{\vartheta-\beta}\|u_0\|_{H^{p-2-2\beta}(\mathscr{M})}.\end{aligned} \tag{28}$$

□

4. Global Existence Results under a Global Lipschitz Case

In this section, we derive the global results under the assumption of the nonlinear source function F, a global Lipschitz.

Let F and g satisfy that

$$\|G(u)-F(v)\|_{H^s(\mathscr{M})} \le L_g\|u-v\|_{H^q(\mathscr{M})}, 1\le s\le q, \tag{29}$$

$$\|K(u)-K(v)\|_{H^s(\mathscr{M})} \le L_k\|u-v\|_{H^q(\mathscr{M})}, 1\le s\le q, \tag{30}$$

where K_f, K_g are postive constants. Our results in this section are to present the well-posedness of the problem. Let $\nu>0$ and $q\ge1$. In order to establish the existence of the mild solution, we need to define the following space:

$$\mathbf{X}_{d,m}((0,T];H^q(\mathscr{M})) = \left\{f:\mathscr{M}\times[0,T]\longrightarrow\mathbb{R}: t^de^{-mt}\|\|w(\cdot,,t)\|_{H^q(\mathscr{M})} \\ <\infty, t\in[0,T]\right\}, \tag{31}$$

associated with the following norm:

$$\|f\|_{X_{d,m}((0,T];H^q(\mathscr{M}))} = \sup_{0\le t\le T} t^de^{-mt}\|w(\cdot,t)\|_{H^q(\mathscr{M})}. \tag{32}$$

Let us provide the following results that will be valuable in justifying our key results. We can find and view it in Lemma 8 of [33] (page 9).

Lemma 2. *Let $c>-1$, $d>-1$ such that $c+d\ge-1$, $h>0$ and $t\in[0,T]$. For $\varepsilon>0$, the following limit holds:*

$$\lim_{\varepsilon\longrightarrow\infty}\left(\sup_{t\in[0,T]} t^h\int_0^1 \tau^c(1-\tau)^d e^{-\varepsilon t(1-\tau)}d\tau\right) = 0. \tag{33}$$

Now, we are in the position to introduce the main contributions of this work. Our main results address the global existence of the mild solution.

Theorem 2. *Let $0<\alpha<1$. Let us assume that*

$$|\theta(t)| \le C_\theta t^{-\delta}. \tag{34}$$

Let $u_0 \in H^{q-2-2\beta}(\mathscr{M}) \cap H^{q-2}(\mathscr{M})$. Then, there exists a positive number m_0 such that problem (1) has a unique solution in $\mathbf{X}_{d,m_0}((0,T];H^q(\mathscr{M}))$. Here, β, d, δ satisfy that

$$0<\beta\le d<1, \beta+d<1, \beta+\delta<2, \delta<1. \tag{35}$$

Proof. Let the function $\mathfrak{F}:\mathbf{X}_{d,m}((0,T];H^q(\mathscr{M})) \longrightarrow \mathbf{X}_{d,m}((0,T];H^q(\mathscr{M}))$ be as follows:

$$\begin{aligned}\mathfrak{F}\theta(t) &= \mathbf{P}_\alpha(t)u_0 + \int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta(\tau))d\tau \\ &\quad + \int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta(\xi))d\xi d\tau \\ &= \mathbf{P}_\alpha(t)u_0 + \mathfrak{F}_1\theta(t) + \mathfrak{F}_2\theta(t).\end{aligned} \tag{36}$$

First, we have the following observation:

$$\|\mathbf{P}_\alpha(t)u_0\|_{H^q(\mathscr{M})} \le \bar{C}_{1,\alpha,\beta}t^{-\beta}\|u_0\|_{H^{q-2}(\mathscr{M})} + \bar{C}_{2,\alpha,\beta}t^{-\beta}\|u_0\|_{H^{q-2-2\beta}(\mathscr{M})}. \tag{37}$$

By multiplying the two sides of the above inequality by $t^d e^{-mt}$ and noting that $e^{-mt}\le 1$, one has

$$\begin{aligned}t^d e^{-mt}\|\mathbf{P}_\alpha(t)u_0\|_{H^q(\mathscr{M})} &\le \bar{C}_{1,\alpha,\beta}t^{d-\beta}\|u_0\|_{H^{q-2}(\mathscr{M})} \\ &\quad + \bar{C}_{2,\alpha,\beta}t^{d-\beta}\|u_0\|_{H^{q-2-2\beta}(\mathscr{M})}.\end{aligned} \tag{38}$$

Noting that $d\ge\beta$, we deduce that if $u_0\in H^{q-2-2\beta}(\mathscr{M})\cap H^{q-2}(\mathscr{M})$, then the following holds:

$$\mathbf{P}_\alpha(t)u_0 \in X_{d,m}(\langle 0,T];H^q(\mathscr{M})). \tag{39}$$

Take two functions $\theta_1,\theta_2\in\mathbf{X}_{d,m}((0,T];H^q(\mathscr{M}))$. First, we need to derive the estimation for the term:

$$(I) = \left\|\int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta_1(\tau))d\tau - \int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta_2(\tau))d\tau\right\|_{H^q(\mathscr{M})}. \tag{40}$$

Using Lemma 1 and Sobolev embedding $H^s(\mathscr{M})^\circ H^{q-2}(\mathscr{M})$ and $H^s(\mathscr{M})^\circ H^{q-2-2\beta}(\mathscr{M})$ (since $s\ge q-2$), we arrive at

$$\begin{aligned}(I) &= \left\|\int_0^t P_\alpha(t-\tau)G(\theta_1(\tau))d\tau - \int_0^t P_\alpha(t-\tau)G(\theta_2(\tau))d\tau\right\|_{H^q(\mathscr{M})} \\ &\le \bar{C}_{1,\alpha,\beta}\int_0^t (t-\tau)^{-\beta}\|G(\theta_1(\tau))-G(\theta_2(\tau))\|_{H^{q-2}(\mathscr{M})}d\tau \\ &\quad + \bar{C}_{2,\alpha,\beta}\int_0^t (t-\tau)^{-\beta}\|G(\theta_1(\tau))-G(\theta_2(\tau))\|_{H^{q-2-2\beta}(\mathscr{M})}d\tau \\ &\le \left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)\int_0^t (t-\tau)^{-\beta}\|G(\theta_1(\tau))-G(\theta_2(\tau))\|_{H^s(\mathscr{M})}d\tau.\end{aligned} \tag{41}$$

Since the assumption (29), we know that

$$\begin{aligned}&\int_0^t (t-\tau)^{-\beta}\|G(\theta_1(\tau))-G(\theta_2(\tau))\|_{H^s(\mathscr{M})}d\tau \\ &\quad\le L_g\int_0^t (t-\tau)^{-\beta}\|\theta_1(\tau)-\theta_2(\tau)\|_{H^q(\mathscr{M})}d\tau \\ &\quad= L_g\left(\int_0^t (t-\tau)^{-\beta}\tau^{-d}e^{m\tau}\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))}.\end{aligned} \tag{42}$$

Combining (40) and (41), we find that

$$\begin{aligned}&t^d e^{-mt}\left\|\int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta_1(\tau))d\tau - \int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta_2(\tau))d\tau\right\|_{H^q(\mathscr{M})} \\ &\le \left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)L_g t^d \\ &\quad\times\left(\int_0^t (t-\tau)^{-\beta}\tau^{-d}e^{-m(t-\tau)}d\tau\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))} \\ &= \left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)L_g t^{1-\beta} \\ &\quad\times\left(\int_0^1 (1-\xi)^{-\beta}\xi^{-d}e^{-mt(1-\xi)}d\xi\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))},\end{aligned} \tag{43}$$

where we have used the fact that

$$t^d\int_0^t (t-\tau)^{-\beta}\tau^{-d}e^{-m(t-\tau)}d\tau = t^{1-\beta}\int_0^1 (1-\xi)^{-\beta}\xi^{-d}e^{-mt(1-\xi)}d\xi. \tag{44}$$

Let us continue to treat the term. First, we need to derive the estimation for the term:

$$\begin{aligned}(II) &= \left\|\int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi d\tau\right. \\ &\quad\left. - \int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi d\tau\right\|_{H^q(\mathscr{M})}.\end{aligned} \tag{45}$$

Using Lemma 1 and Sobolev embedding $H^s(\mathscr{M})^\circ H^{q-2}(\mathscr{M})$ and $H^s(\mathscr{M})^\circ H^{q-2-2\beta}(\mathscr{M})$ (since $s\ge q-2$), we get the following bound:

$$\begin{aligned}(II) &= \Bigg\| \int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi d\tau \\ &\quad - \int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi d\tau \Bigg\|_{H^q(\mathscr{M})} \\ &\le \bar{C}_{1,\alpha,\beta}\int_0^t (t-\tau)^{-\beta}\Bigg\|\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi \\ &\quad - \int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi\Bigg\|_{H^{q-2}(\mathscr{M})} d\tau \\ &\quad + \bar{C}_{2,\alpha,\beta}\int_0^t (t-\tau)^{-\beta}\Bigg\|\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi \\ &\quad - \int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi\Bigg\|_{H^{q-2-2\beta}(\mathscr{M})} d\tau \\ &\le \left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)\int_0^t (t-\tau)^{-\beta}\Bigg\|\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi \\ &\quad - \int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi\Bigg\|_{H^s(\mathscr{M})} d\tau.\end{aligned} \tag{46}$$

Using the Lipschitz property of K, we know that

$$\begin{aligned}&\Bigg\|\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi - \int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi\Bigg\|_{H^q(\mathscr{M})} \\ &\le C_\theta L_k \int_0^\tau (\tau-\xi)^{-\delta}\|\theta_1(\xi)-\theta_2(\xi)\|_{H^q(\mathscr{M})}d\xi \\ &= C_\theta L_k \int_0^\tau (\tau-\xi)^{-\delta}\xi^{-d}e^{m\xi}\xi^d e^{-m\xi}\|\theta_1(\xi)-\theta_2(\xi)\|_{H^q(\mathscr{M})}d\xi \\ &\le C_\theta L_k \left(\int_0^\tau (\tau-\xi)^{-\delta}\xi^{-d}e^{m\xi}d\xi\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))}.\end{aligned} \tag{47}$$

It is obvious to see that

$$\begin{aligned}\int_0^\tau (\tau-\xi)^{-\delta}\xi^{-d}e^{m\xi}d\xi &\le e^{m\tau}\int_0^\tau (\tau-\xi)^{-\delta}\xi^{-d}d\xi \\ &= e^{m\tau}\tau^{1-\delta-d}\mathbf{B}(1-\delta,1-d).\end{aligned} \tag{48}$$

This implies that the following estimation is for the integral term on the right hand side of (45):

$$\begin{aligned}&\int_0^t (t-\tau)^{-\beta}\Bigg\|\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi - \int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi\Bigg\|_{H^q(\mathscr{M})} d\tau \\ &\le C_\theta L_k B(1-\delta,1-d) \\ &\quad \times \left(\int_0^t (t-\tau)^{-\beta}\tau^{1-\delta-d}e^{m\tau}d\tau\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))}.\end{aligned} \tag{49}$$

This implies that

$$\begin{aligned}&t^d e^{-mt}\Bigg\|\int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi d\tau \\ &\quad - \int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi d\tau\Bigg\|_{H^q(\mathscr{M})} \\ &\le C_3 t^d \left(\int_0^t (t-\tau)^{-\beta}\tau^{1-\delta-d}e^{-m(t-\tau)}d\tau\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))} \\ &= C_3 t^{2-\beta-\delta}\int_0^1 (1-\xi)^{-\beta}\xi^{1-d-\delta}e^{-mt(1-\xi)}d\xi\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))},\end{aligned} \tag{50}$$

where we note that

$$\begin{aligned}&C_3 = \left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)C_\theta L_k \mathbf{B}(1-\delta,1-d), \\ &t^d\int_0^t (t-\tau)^{-\beta}\tau^{1-\delta-d}e^{-m(t-\tau)}d\tau = t^{2-\beta-\delta}\int_0^1 (1-\xi)^{-\beta}\xi^{1-d-\delta}e^{-mt(1-\xi)}d\xi.\end{aligned} \tag{51}$$

Combining (42) and (49), we obtain the following bound:

$$\begin{aligned}&t^d e^{-mt}\|\mathfrak{F}\theta_1(t)-\mathfrak{F}\theta_2(t)\|_{H^q(\mathscr{M})} \le t^d e^{-mt}\Bigg\|\int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta_1(\tau))d\tau \\ &\quad - \int_0^t \mathbf{P}_\alpha(t-\tau)G(\theta_2(\tau))d\tau\Bigg\|_{H^q(\mathscr{M})} \\ &\quad + t^d e^{-mt}\Bigg\|\int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_1(\xi))d\xi d\tau \\ &\quad - \int_0^t \mathbf{P}_\alpha(t-\tau)\int_0^\tau \psi(\tau-\xi)K(\theta_2(\xi))d\xi d\tau\Bigg\|_{H^q(\mathscr{M})} \\ &\le \left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)L_g t^{1-\beta} \\ &\quad \times \left(\int_0^1 (1-\xi)^{-\beta}\xi^{-d}e^{-mt(1-\xi)}d\xi\right)\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))} \\ &\quad + C_3 t^{2-\beta-\delta}\int_0^1 (1-\xi)^{-\beta}\xi^{1-d-\delta}e^{-mt(1-\xi)}d\xi\|\theta_1-\theta_2\|_{X_{d,m}((0,T];H^q(\mathscr{M}))}.\end{aligned} \tag{52}$$

From the condition (34), we can verify the following condition:

$$\begin{cases} 1-\beta > 0, \\ -\beta > -1, -d > -1, -\beta - d > -1, \\ \beta + \delta < 2, \\ 1-d-\delta > -1, -\beta + 1 - d - \delta > -1. \end{cases} \tag{53}$$

By using Lemma 2, we have two statements immediately:

$$\lim_{m\longrightarrow\infty}\left(\sup_{t\in[0,T]} t^{1-\beta}\left(\int_0^1 (1-\xi)^{-\beta}\xi^{-d}e^{-mt(1-\xi)}d\xi\right)\right)=0,$$
$$\lim_{m\longrightarrow\infty}\left(\sup_{t\in[0,T]} t^{2-\beta-\delta}\int_0^1 (1-\xi)^{-\beta}\xi^{1-d-\delta}e^{-mt(1-\xi)}d\xi\right)=0. \tag{54}$$

From the last two observations, we can find that the positive number m_0 such that

$$\left(\tilde{C}_{1,\alpha,\beta}+\tilde{C}_{2,\alpha,\beta}\right)L_g t^{1-\beta}\left(\int_0^1 (1-\xi)^{-\beta}\xi^{-d}e^{-mt(1-\xi)}d\xi\right) + C_3 t^{2-\beta-\delta}\int_0^1 (1-\xi)^{-\beta}\xi^{1-d-\delta}e^{-mt(1-\xi)}d\xi \tag{55}$$

is less than 1. By applying the Banach fixed point theorem, we know that problem (1) has a unique solution in $\mathbf{X}_{d,m_0}((0,T];H^q(\mathscr{M}))$. □

5. Conclusion

The result of the paper is one of the first works on the topic of memory for equations with Caputo-Fabrizio derivatives. We obtain the following results: first, prove the existence of local solutions. The second is a survey of the global solution. The main technique is to use the Banach fixed point theorem in combination with Sobolev embeddings.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. Four authors read and approved the final manuscript.

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