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Adaptive Synchronization of Fractional-Order Complex-Valued Chaotic Neural Networks with Time-Delay and Unknown Parameters

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Abstract: The purpose of this paper is to study and analyze the concept of fractional-order complex-valued chaotic networks with external bounded disturbances and uncertainties. The synchronization problem and parameter identification of fractional-order complex-valued chaotic neural networks (FOVCNNs) with time-delay and unknown parameters are investigated. Synchronization between a driving FOVCNN and a response FOVCNN, as well as the identification of unknown parameters are implemented. Based on fractional complex-valued inequalities and stability theory of fractional-order chaotic complex-valued systems, the paper designs suitable adaptive controllers and complex update laws. Moreover, it scientifically estimates the uncertainties and external disturbances to establish the stability of controlled systems. The computer simulation results verify the correctness of the proposed method. Not only a new method for analyzing FOVCNNs with time-delay and unknown complex parameters is provided, but also a sensitive decrease of the computational and analytical complexity.



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Keywords: adaptive synchronization; fractional-order; complex-valued chaotic neural networks; time-delay; unknown complex parameter

1. Introduction

Compared with real-valued neural networks, complex-valued neural networks have the advantages of simpler network structure, simpler training process, and stronger ability to handle complex signals. This is mainly due to the fact that the state vectors, connection weights, and activation functions in complex-valued neural networks are all represented by complex values. In addition, complex-valued neural networks can solve problems that cannot be solved by real-valued neural networks. For example, two-layer real-valued neural networks cannot solve the problems of exclusive “OR” (XOR) and symmetry detection, while two-layer complex-valued neural networks can easily do so, which shows that the computational ability of complex-valued neurons is remarkable. In recent years, practical applications of complex-valued neural networks in physical systems such as electromagnetic, optical, ultrasonic, and quantum waves, as well as in the fields of filtering, speech synthesis, and remote sensing, have attracted widespread attention [1–19].

With the development of fractional calculus, more and more researchers recognize that fractional-order models can better describe various substances and processes with memory and genetic properties in neural networks than integer-order models, and can also effectively facilitate the information process. Meanwhile, fractional-order calculus in neural networks can improve the accuracy and flexibility of computation, and have a great application value in computational optimization and control performance improvement. Therefore, combining fractional-order calculus with neural network models to

form fractional-order neural network models expands the basic theory and application capabilities of neural networks.

The analysis of fractional-order neural networks (FNNs) has become a research area attracting increasing interest (see [20–40] and references therein). Moreover, the simultaneous analysis of stability of fractional-order real-valued and complex-valued neural networks has received extensive attention. In [20,21], projection synchronization and adaptive synchronization of fractional-order memristor neural networks are discussed. In [10], synchronization of fractional-order complex-valued neural networks (FOCVNNs) was studied using the linear delay feedback control. The authors investigated the global Mittag-Leffler synchronization problem of fractional-order neural networks in Refs. [22–24]. In Refs. [25–32], the authors analyzed the stability, finite-time stability, and global Mittag-Leffler stability of fractional-order time-delayed complex-valued neural networks, respectively. In [31], several sufficient conditions for achieving finite-time projection synchronization of fractional-order complex-valued neural networks are derived by applying set-valued mappings, differential inclusion theory, and Gronwalls' inequality. In [32], Li et al. implemented the adaptive synchronization of fractional-order complex-valued neural networks with discrete and distributed delays. The modified function projective synchronization (MFPS) for complex dynamical networks with mixed time-varying and hybrid asymmetric coupling delays was investigated in [33]. In [34], the authors studied a novel delay-dependent asymptotic stability of a differential and Riemann-Liouville fractional differential neutral system with constant delays and nonlinear perturbation. In [35], Dai et al. reported that in populations with cooperative and competitive oscillators, the transition between continuous and explosive can be tuned simply by adjusting the balance between the two oscillator types. Furthermore, Dai et al. [36] proposed a unified framework for the analysis of system synchronization and conducted an in-depth study of network synchronization laws in different dimensions.

It should be noted that the aforementioned papers on neural network synchronization all assume that the network is predetermined. In fact, in many practical engineering situations, most system parameters cannot be accurately determined in advance, and chaotic synchronization will be disrupted by these uncertainties. In addition, there are usually delays in neural networks due to the limited speed of signal transmission between neurons. Time-delay can have an impact on the dynamic properties of a neural network and can even destroy it. Although, authors in [37] investigated the controller design problem for finite-time and fixed-time stabilization of fractional-order memristive complex-valued bidirectional associative memory (BAM) neural networks with uncertain parameters and time-varying delays, but the nonlinear complex-valued activation functions are split into two (real and imaginary) components. Therefore, to the best of our knowledge, there are few studies on the synchronization of fractional-order complex-valued chaotic neural networks (FOVCNNs) with time-delays and unknown parameters, especially without dividing the real and imaginary components into two real-valued systems. Therefore, it is very important and useful to efficiently synchronize fractional-order complex-valued chaotic neural networks with time-delays and unknown parameters in practical applications.

Inspired by the above discussion, this paper investigates the synchronization problem of FOVCNNs with time-delay and unknown complex parameters. Using inequalities containing fractional-order derivatives of complex variables and the stability theory of fractional-order complex-valued chaotic systems, synchronization and parameter identification of FOVCNNs are achieved.

The main contributions of this paper can be summarized as follows.

- (i) Most of the existing studies on the synchronization methods of fractional-order neural networks are about fractional-order real-valued neural networks. On the other hand, existing studies on fractional-order complex-valued neural networks are on the known parameters or with no time-delay or without identifying the parameters.

- (ii) A new adaptive controller and update laws are designed to synchronize the driving and response systems. This is the first study of synchronization of fractional-order complex-valued neural networks with time-delay and unknown complex parameters.
- (iii) Compared with previous synchronization models of fractional-order complex neural networks, the model proposed in this paper is more tractable and easier to be implemented in practical systems.
- (iv) For fractional-order complex neural networks with known parameters and time-delay or known parameters without time-delay, the synchronization model proposed in this paper is also applicable, and only the control strategies need to be adjusted accordingly.
- (v) This paper proposes the novel perspective that chaos occurs in fractional-order complex-valued neural networks as long as the parameters are suitable, and two new FOCVCNNs are given to broaden the application of fractional-order complex-valued neural networks.

2. Preliminaries

Fractional calculus plays an important role in modern science. In this paper, Riemann-Liouville and Caputo’s fractional operators are used as the main tools.

Notation: \mathbb{C}^n denotes a complex n -dimensional space. For $z \in \mathbb{C}$, $\text{Re}(z)$, $\text{Im}(z)$ and \bar{z} are the real part, imaginary part, and conjugate of z , respectively.

Definition 1 ([41]). *The fractional integral form of order α for function f is defined as follows:*

$$I^\alpha f(t) = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \tag{1}$$

where t denotes the time and t_0 is the initial time, $t \geq t_0$, and $\alpha > 0$.

Definition 2 ([41]). *Caputo’s fractional derivative form of order α for function $f \in \mathbb{R}^n$ is defined by:*

$${}_{t_0}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha-n+1}} d\tau, \tag{2}$$

where $t \geq t_0$ and n are a positive integer, then $n - 1 < \alpha < 1$.

Lemma 1. *Let z be a differentiable complex-valued function. Then, $\forall t \geq t_0$ and $\alpha \in (0, 1]$, and the following inequality holds [42]:*

$${}_{t_0}^C D_t^\alpha \overline{z(t)} z(t) \leq \overline{z(t)} {}_{t_0}^C D_t^\alpha z(t) + ({}_{t_0}^C D_t^\alpha \overline{z(t)}) z(t). \tag{3}$$

Lemma 2. *(Stability theory for fractional-order system [41]). Let $V_1(t)$ be a uniformly continuous and derivable Lyapunov function, and let $V_2(t)$ be a derivable and nonnegative function.*

If

$$V(t) = V_1(t) + V_2(t), \tag{4}$$

and

$${}_{t_0}^C D_t^\alpha V(t) \leq -\theta V_1(t), \tag{5}$$

where θ is a positive constant. Then, $\lim_{t \rightarrow \infty} V_1(t) = 0$.

Lemma 3 ([43]). *Let $V(t)$ and $U(t)$ be two nonnegative continuous functions, and satisfy*

$${}_{t_0}^C D_t^\alpha (V(t) + U(t)) \leq -\sigma V(t) + \rho, \tag{6}$$

where $0 < \alpha < 1, \sigma < 0$, and $\rho > 0$, then

$$V(t) \leq (V(0) + U(0) - \frac{\rho}{\sigma})E_{\alpha,1}(-\sigma t^\alpha) + \frac{\rho}{\sigma}. \tag{7}$$

Remark 1. Since $E_{\alpha,1}(-\sigma t^\alpha) \rightarrow 0 (\sigma > 0)$, as $t \rightarrow \infty$, if $\rho = 0$, one can deduce Lemma 2, by this Lemma.

Lemma 4 ([44]). For any $\alpha, \beta \in \mathbb{C}$, and any $\delta > 0$, the following inequality holds:

$$\alpha\bar{\beta} + \bar{\alpha}\beta \leq \delta\alpha\bar{\alpha} + \frac{1}{\delta}\beta\bar{\beta}. \tag{8}$$

3. Main Results

Let us consider a kind of FOCVCNNs described by the following equations:

$$\begin{cases} {}^C_0D_t^\alpha z_i(t) = -c_i z_i(t) + \sum_{j=1}^n a_{ij} f_j(z_j(t)) + \sum_{j=1}^n b_{ij} g_j(z_j(t - \tau)) + I_i(t), \quad t \geq 0, \quad i = 1, 2, \dots, n, \\ z_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \end{cases} \tag{9}$$

where $0 < \alpha < 1$, and $z_i(t) \in \mathbb{C}$ is the complex state variable of the i th neuron; $f_j(\cdot), g_j(\cdot) \in \mathbb{C}$ represent the activation functions without and with delay; $a_{ij}, b_{ij} \in \mathbb{C}$ denote the connection weight and delayed connection weight, respectively; $c_i \in \mathbb{R}, \tau > 0$ are constant delays; and I_i represents the corresponding external inputs. The complex-valued functions $f_j(\cdot), g_j(\cdot)$ and I_i satisfy the following assumptions.

Assumption 1. For any $\mu, \nu \in \mathbb{C}$, there exist real numbers $l_j, h_j > 0$, then

$$|f_j(\mu) - f_j(\nu)| \leq l_j |\mu - \nu|, \quad |g_j(\mu) - g_j(\nu)| \leq h_j |\mu - \nu|. \tag{10}$$

Assumption 2. For any $v \in \mathbb{C}$, there exist real numbers $\gamma_i > 0$ and $\varepsilon_i > 0$, then

$$|f_j(v)| \leq \gamma_i, \quad |I_i(v)| \leq \varepsilon_i. \tag{11}$$

Choose system (9) as the master system, and a_{ij}, b_{ij} are unknown constants which need to be identified, then the controlled response system is given by:

$$\begin{cases} {}^C_0D_t^\alpha w_i(t) = -c_i w_i(t) + \sum_{j=1}^n \alpha_{ij} f_j(w_j(t)) + \sum_{j=1}^n \beta_{ij} g_j(w_j(t - \tau)) + I_i(t) + u_i(t), \quad t \geq 0, \quad i = 1, 2, \dots, n, \\ w_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \end{cases} \tag{12}$$

where $w_i(t) \in \mathbb{C}$ is the complex state variable of the i th neuron of the response system; $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ represent the estimated connection weights and delayed connection weights, respectively; and $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ are controllers to be determined.

Let $e_i(t) = w_i(t) - z_i(t)$ be the synchronization errors between master system (9) and slave system (12), then one can get the following error dynamical system:

$$\begin{aligned} {}^C_0D_t^\alpha e_i(t) &= -c_i e_i(t) + \sum_{j=1}^n [\alpha_{ij} f_j(w_j(t)) - a_{ij} f_j(z_j(t))] + \sum_{j=1}^n [\beta_{ij} g_j(w_j(t - \tau)) - b_{ij} g_j(z_j(t - \tau))] + u_i(t) \\ &= -c_i e_i(t) + \sum_{j=1}^n \{a_{ij} [f_j(w_j(t)) - f_j(z_j(t))] + (\alpha_{ij} - a_{ij}) f_j(w_j(t))\} \\ &\quad + \sum_{j=1}^n \{b_{ij} [g_j(w_j(t - \tau)) - g_j(z_j(t - \tau))] + (\beta_{ij} - b_{ij}) g_j(w_j(t - \tau))\} + u_i(t). \end{aligned} \tag{13}$$

Theorem 1. *If Assumptions 1 and 2 hold, the asymptotic synchronization and parameter identification of systems (9) and (12) can be achieved under adaptive controllers, described as Equation (14) and adaptive update laws (15)–(18):*

$$u_i(t) = -k_i(t)e_i(t) - m_i \frac{e_i(t)}{e_i(t)} \overline{e_i(t - \tau)} e_i(t - \tau), \quad i = 1, 2, \dots, n, \tag{14}$$

$${}_0^C D_t^\alpha k_i = \sigma_i \overline{e_i(t)} e_i(t), \tag{15}$$

$${}_0^C D_t^\alpha m_i = \varepsilon_i \overline{e_i(t - \tau)} e_i(t - \tau), \tag{16}$$

$${}_0^C D_t^\alpha \alpha_{ij} = -\eta_{ij} \overline{f_j(w_j(t))} e_i(t), \tag{17}$$

$${}_0^C D_t^\alpha \beta_{ij} = -\zeta_{ij} \overline{g_j(w_j(t - \tau))} e_i(t), \tag{18}$$

where $\sigma_i, \varepsilon_i, \eta_{ij}, \zeta_{ij}$ are positive constants.

Proof. Let us present the following Lyapunov functional candidate:

$$\begin{aligned} V_1(t) &= \sum_{i=1}^n \bar{e}_i(t) e_i(t) = e^H(t) e(t), \\ V_2(t) &= \sum_{i=1}^n \left[\frac{1}{\sigma_i} (k_i - k_i^*)^2 + \frac{1}{\varepsilon_i} (m_i - m_i^*)^2 \right. \\ &\quad \left. + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\alpha}_{ij} - \bar{a}_{ij})(\alpha_{ij} - a_{ij}) + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\beta}_{ij} - \bar{b}_{ij})(\beta_{ij} - b_{ij}) \right], \end{aligned} \tag{19}$$

where k_i^*, m_i^* are two positive constants to be determined. \square

Using Lemma 1:

$$\begin{aligned} {}_0^C D_t^\alpha (V_1(t) + V_2(t)) &= {}_0^C D_t^\alpha \sum_{i=1}^n \bar{e}_i(t) e_i(t) + {}_0^C D_t^\alpha \sum_{i=1}^n \left[\frac{1}{\sigma_i} (k_i - k_i^*)^2 + \frac{1}{\varepsilon_i} (m_i - m_i^*)^2 \right. \\ &\quad \left. + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\alpha}_{ij} - \bar{a}_{ij})(\alpha_{ij} - a_{ij}) + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\beta}_{ij} - \bar{b}_{ij})(\beta_{ij} - b_{ij}) \right] \\ &\leq \sum_{i=1}^n \bar{e}_i(t) {}_0^C D_t^\alpha e_i(t) + \sum_{i=1}^n e_i(t) \overline{{}_0^C D_t^\alpha e_i(t)} \\ &\quad + \sum_{i=1}^n \left[\frac{2}{\sigma_i} (k_i - k_i^*) {}_0^C D_t^\alpha k_i + \frac{2}{\varepsilon_i} (m_i - m_i^*) {}_0^C D_t^\alpha m_i \right. \\ &\quad \left. + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\alpha}_{ij} - \bar{a}_{ij}) {}_0^C D_t^\alpha (\alpha_{ij} - a_{ij}) + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\alpha}_{ij} - \bar{a}_{ij}) \overline{{}_0^C D_t^\alpha (\alpha_{ij} - a_{ij})} \right. \\ &\quad \left. + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\beta}_{ij} - \bar{b}_{ij}) {}_0^C D_t^\alpha (\beta_{ij} - b_{ij}) + \sum_{j=1}^n \frac{1}{\eta_{ij}} (\bar{\beta}_{ij} - \bar{b}_{ij}) \overline{{}_0^C D_t^\alpha (\beta_{ij} - b_{ij})} \right]. \end{aligned} \tag{20}$$

Along with Equation (13) and qualities (14)–(18), one gets:

$$\begin{aligned} {}_0^C D_t^\alpha (V_1(t) + V_2(t)) &\leq - \sum_{i=1}^n 2\bar{e}_i(t) c_i e_i(t) + \sum_{i=1}^n \sum_{j=1}^n \{ \bar{e}_i(t) a_{ij} [f_j(w_j(t)) - f_j(z_j(t))] \\ &\quad + e_i(t) \bar{a}_{ij} [\bar{f}_j(w_j(t)) - \bar{f}_j(z_j(t))] \\ &\quad + \bar{e}_i(t) b_{ij} [g_j(w_j(t - \tau)) - g_j(z_j(t - \tau))] \\ &\quad + e_i(t) \bar{b}_{ij} [\bar{g}_j(w_j(t - \tau)) - \bar{g}_j(z_j(t - \tau))] \} \\ &\quad - \sum_{i=1}^n 2[k_i^* \bar{e}_i(t) e_i(t) + m_i^* \overline{e_i(t - \tau)} e_i(t - \tau)]. \end{aligned} \tag{21}$$

According to Lemma 4 and Assumption 1:

$$\begin{aligned}
 {}_0^C D_t^\alpha (V_1(t) + V_2(t)) &\leq -\sum_{i=1}^n 2c_i \bar{e}_i(t) e_i(t) + \sum_{i=1}^n \sum_{j=1}^n [(a_{ij} \bar{a}_{ij} + l_j^2) \bar{e}_i(t) e_i(t) + b_{ij} \bar{b}_{ij} \bar{e}_i(t) e_i(t) \\
 &\quad + h_j^2 \overline{e_i(t-\tau)} e_i(t-\tau)] - \sum_{i=1}^n 2[k_i^* \bar{e}_i(t) e_i(t) - m_i^* \overline{e_i(t-\tau)} e_i(t-\tau)] \\
 &= \sum_{i=1}^n [-2c_i - 2k_i^* + \sum_{j=1}^n (a_{ij} \bar{a}_{ij} + l_j^2 + b_{ij} \bar{b}_{ij})] \bar{e}_i(t) e_i(t) \\
 &\quad + \sum_{i=1}^n (\sum_{j=1}^n h_j^2 - 2m_i^*) \overline{e_i(t-\tau)} e_i(t-\tau).
 \end{aligned} \tag{22}$$

Letting $k_i^* = \frac{1}{2} \sum_{j=1}^n (a_{ij} \bar{a}_{ij} + l_j^2 + b_{ij} \bar{b}_{ij}) - c_i + 1$, $m_i^* = \frac{1}{2} \sum_{j=1}^n h_j^2$:

$${}_0^C D_t^\alpha (V_1(t) + V_2(t)) \leq -V_1(t). \tag{23}$$

From **Lemma 2** or **Lemma 3**, one can obtain: $\lim_{t \rightarrow \infty} V_1(t) = \lim_{t \rightarrow \infty} e^H(t) e(t) = 0$, indicating $\lim_{t \rightarrow \infty} e(t) = 0$, which shows that systems (9) and (12) can obtain asymptotic synchronization. Meanwhile, according to **Remark 1** of **Theorem 1** of Ref. [45], the parameter identification is achieved.

Remark 2. *Theorem 1 provides a stability criterion for fractional-order nonlinear uncertain systems with time-delay by choosing a Lyapunov function that includes $V_1(t)$ and $V_2(t)$.*

Remark 3. *Theorem 1 provides a Lyapunov-based adaptive control method for stability analysis and synchronization of FOCVCNNs.*

Remark 4. *Lemma 3 is applied to verify the stability of fractional-order with unknown parameters and external disturbances, as well as to the design of synchronous controllers for these systems.*

Remark 5. *For FOCVCNNs with known parameters, the update laws will be reduced to (14) and (15).*

Remark 6. *For FOCVCNNs with known parameters and without time-delay, the synchronization between systems (9) and (12) can be achieved under the following control strategy (24).*

Remark 7. *It is worth mentioning that the synchronization problem discussed in this paper is about fractional-order complex-valued neural networks with time-varying delays and unknown parameters, while most of the existing work on parameter identification methods for synchronization is about fractional-order real-valued models. On the other hand, previous studies have mainly focused on fractional-order complex-valued models with known parameters [10,31,43] or fractional-order complex-valued models without time-delay [46]:*

$$\begin{cases} u_i(t) = -k_i e_i(t), \\ {}_0^C D_t^\alpha k_i = \overline{\sigma_i e_i(t)} e(t), \end{cases} \tag{24}$$

where σ_i is a positive constant.

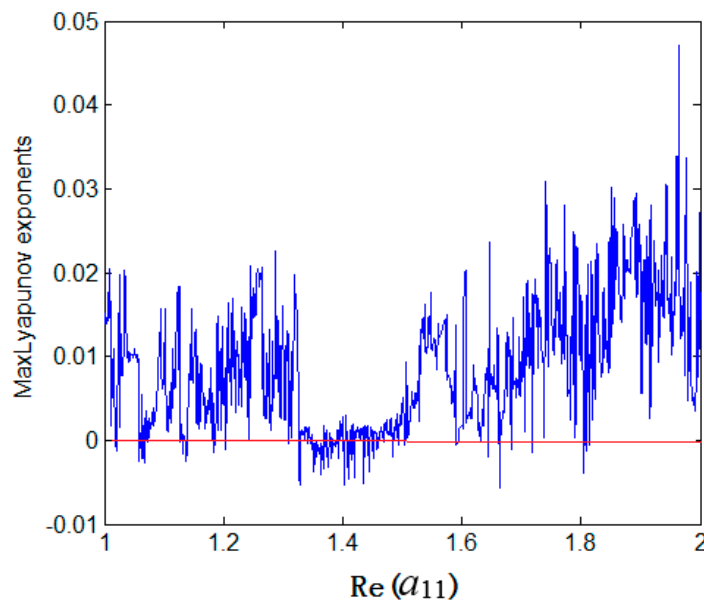
4. Numerical Simulations

In this Section, several numerical examples of fractional-order complex-valued neural networks are given to show the effectiveness of the scheme proposed in previous Sections. For the numerical solution of these systems, the predictor–corrector method [45] of the MATLAB platform is adopted. The Lyapunov exponents of systems are calculated by the algorithm of Wolf et al. [47], with some adaptations.

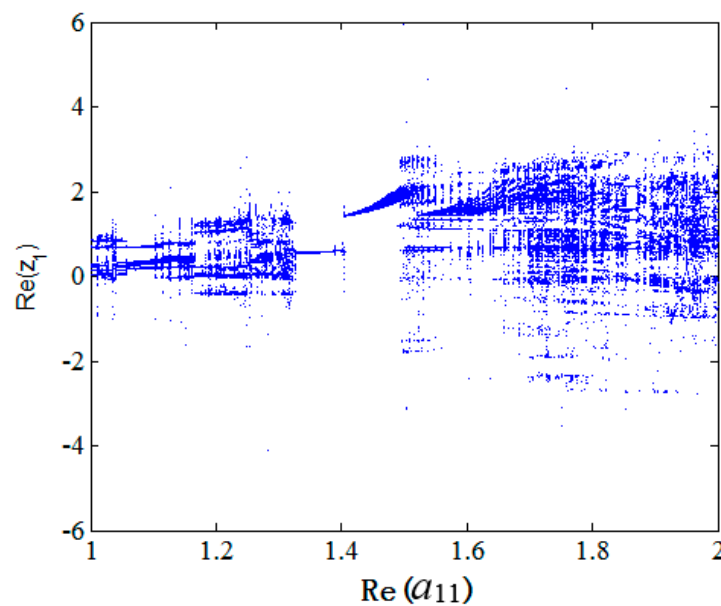
Example 1. Consider a class of FOCVCNNs, which is described as follows:

$$\begin{cases} {}_0^C D_t^\alpha z_i(t) = -c_i z_i(t) + \sum_{j=1}^n a_{ij} f_j(z_j(t)) + \sum_{j=1}^n b_{ij} g_j(z_j(t - \tau)) + I_i(t), \\ z_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad t \geq 0, \quad i = 1, 2. \end{cases} \quad (25)$$

If one selects $\tau = 1$, $a_{11} = 2 + 0.1i$, $a_{12} = 0.4 - 0.1i$, $a_{21} = 5 - 0.5i$, $a_{22} = 3 - 0.2i$, $b_{11} = -2 + 0.1i$, $b_{12} = 0.2 + 0.1i$, $b_{21} = 0.3 + 0.1i$, $b_{22} = -2.5 - 0.3i$, $f(z) = g(z) = \tanh(z)$, $I_1(t) = I_2(t) = 0$, $[z_1(s), z_2(s)]^T = [0.01 + 0.01i, 0.1 - 0.1i]^T$, $\forall s \in [-1, 0]$, $\alpha = 0.96$, then let $\text{Re}(a_{11}) = 1 \sim 2$. Figure 1a depicts the maximum Lyapunov exponent (MLE) spectrum of system (25), and Figure 1b shows its bifurcation diagram. Figure 1 shows that system (25) is chaotic at fractional-order, $\text{Re}(a_{11}) \in [1.85, 2]$.



(a)



(b)

Figure 1. Dynamic behaviors of fractional-order complex-valued chaotic neural networks (FOCVCNNs) (25) with $\text{Re}(a_{11})$: (a) maximal Lyapunov exponent, (b) bifurcation diagram.

System (25) can exhibit chaotic behaviors, which can be called fractional-order complex-valued chaotic neural networks. While $a_{11} = 2 + 0.1i$ and the other parameters are the same as above, the attractor trajectory with the initial condition $[z_1(s), z_2(s)]^T = [0.1 - 0.1i, 0.1]^T$ is shown in Figure 2. The state trajectory is shown in Figure 3.

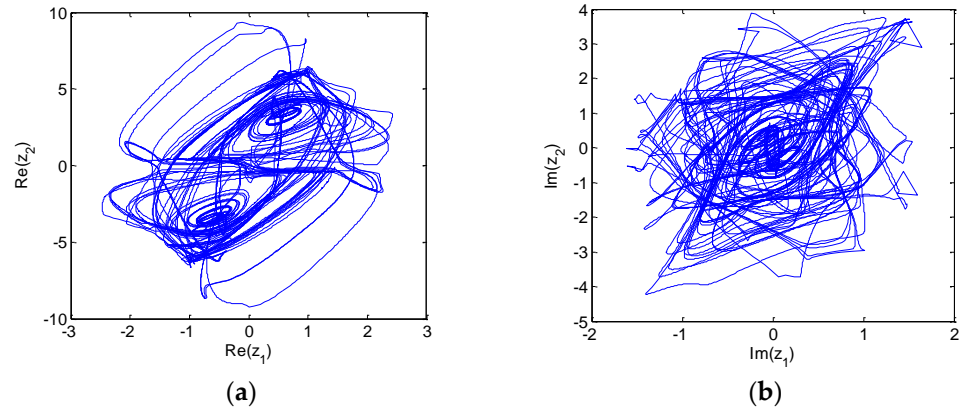


Figure 2. Chaotic attractors of FOCVCNNs (25) with $\tau = 1$, $a_{11} = 2 + 0.1i$, $a_{12} = 0.4 - 0.1i$, $a_{21} = 5 - 0.5i$, $a_{22} = 3 - 0.2i$, $b_{11} = -2 + 0.1i$, $b_{12} = 0.2 + 0.1i$, $b_{21} = 0.3 + 0.1i$, $b_{22} = -2.5 - 0.3i$ and fractional-order $\alpha = 0.96$: (a) $\text{Re}(z_1)$ vs. $\text{Re}(z_2)$, (b) $\text{Im}(z_1)$ vs. $\text{Im}(z_2)$.

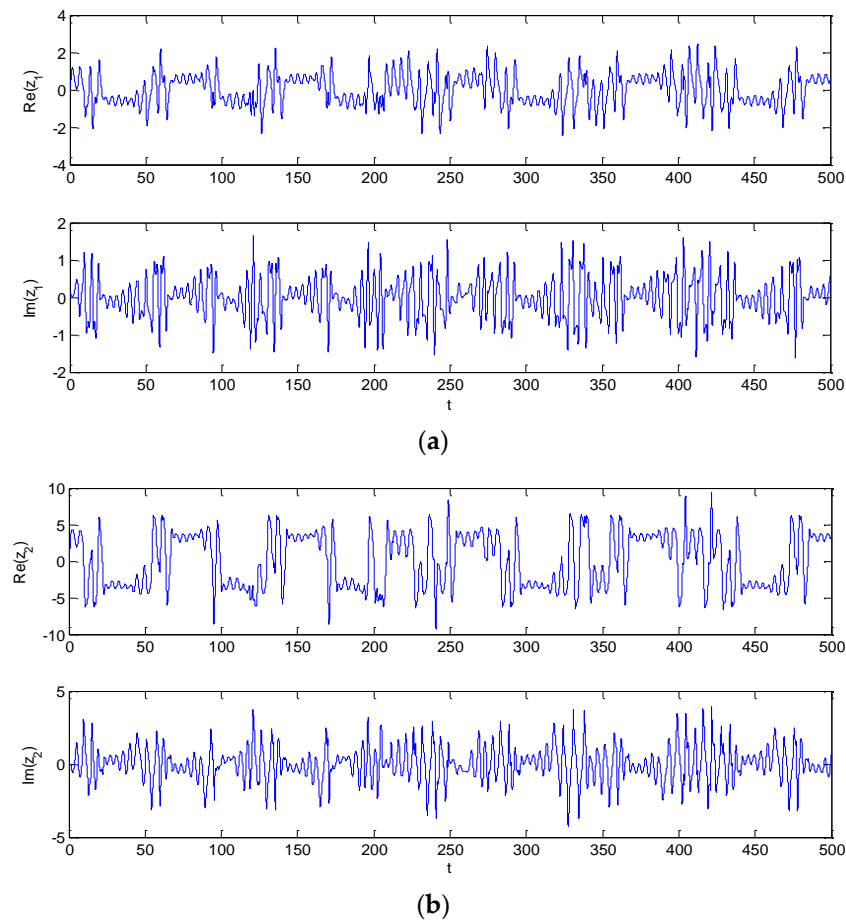


Figure 3. The state trajectories of FOCVCNNs with $\tau = 1$, $a_{11} = 2 + 0.1i$, $a_{12} = 0.4 - 0.1i$, $a_{21} = 5 - 0.5i$, $a_{22} = 3 - 0.2i$, $b_{11} = -2 + 0.1i$, $b_{12} = 0.2 + 0.1i$, $b_{21} = 0.3 + 0.1i$, $b_{22} = -2.5 - 0.3i$ and fractional-order $\alpha = 0.96$, t is the time: (a) $\text{Re}(z_1)$ and $\text{Im}(z_1)$, (b) $\text{Re}(z_2)$ and $\text{Im}(z_2)$.

Let system (25) be the driving system and assume that the parameters $a_{ij}, b_{ij}, (i = 1, 2, j = 1, 2)$ are unknown, then the response FOCVCNNs are given as follows:

$$\begin{cases} {}^C_0 D_t^\alpha w_i(t) = -c_i w_i(t) + \sum_{j=1}^n \alpha_{ij} f_j(w_j(t)) + \sum_{j=1}^n \beta_{ij} g_j(w_j(t - \tau)) + I_i(t) + u_i(t), \\ w_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad t \geq 0, \quad i = 1, 2, \end{cases} \quad (26)$$

where α_{ij}, β_{ij} are estimated values of a_{ij}, b_{ij} , respectively and $u_i(t)$ are controllers. The controllers and the update laws are selected as Equations (14)–(18). The following initial conditions are chosen:

$$\alpha_{11}(0) = \alpha_{12}(0) = \alpha_{21}(0) = \alpha_{22}(0) = \beta_{11}(0) = \beta_{12}(0) = \beta_{21}(0) = \beta_{22}(0) = k_1(0) = k_2(0) = 0.1, m_1(0) = m_2(0) = 0, [w_1(s), w_2(s)]^T = [-0.1, 0.1i]^T, \forall s \in [-1, 0] \text{ and } \varepsilon_1 = \varepsilon_2 = 1, \sigma_1 = \sigma_2 = 10, \eta_{11} = 8, \eta_{12} = \eta_{21} = \eta_{22} = \zeta_{11} = \zeta_{12} = \zeta_{21} = \zeta_{22} = 6, \xi_{21} = \xi_{22} = 6, I_1(t) = I_2(t) = 0.$$

Two FOCVCNNs can achieve synchronization and the parameters are identified, as shown in Figures 4 and 5. Figure 4 shows that the above two pairs of FOCVCNNs achieve asymptotic synchronization through the adaptive controller and adaptive update laws. Figure 5 indicates that all the unknown parameters of the driving system are identified.

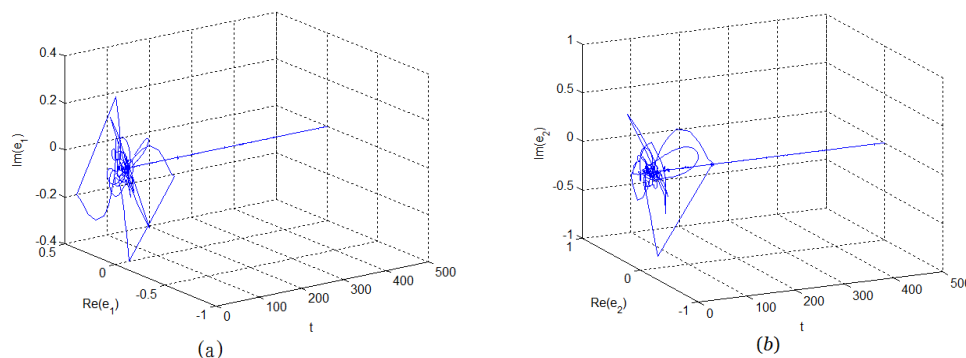


Figure 4. Synchronization errors of FOCVCNNs (25) and (26): (a) $e_1(t) = w_1(t) - z_1(t)$, (b) $e_2(t) = w_2(t) - z_2(t)$.

It is shown that with this approach one can rapidly achieve global synchronization of these networks, while dynamically identifying all the unknown parameters. Additionally, this method is quite robust against noise effects.

Example 2. To further illustrate the effectiveness and wider application of the proposed scheme, a higher dimensional FOCVNN is considered described by the following equation:

$$\begin{cases} {}^C_0 D_t^\alpha z_1(t) = -z_1(t) + a_{11}f(z_1(t)) + a_{12}f(z_2(t)) + a_{13}f(z_3(t - \tau)), \\ {}^C_0 D_t^\alpha z_2(t) = -z_2(t) + a_{21}f(z_1(t)) + a_{22}f(z_2(t)) + a_{23}f(z_3(t - \tau)), \\ {}^C_0 D_t^\alpha z_3(t) = -z_3(t) + a_{31}f(z_1(t)) + a_{32}f(z_2(t)) + a_{33}f(z_3(t - \tau)). \end{cases} \quad (27)$$

If $\tau = 2$ is selected, then:

$a_{11} = 2 + 0.1i, a_{12} = 16 - i, a_{13} = -6 + 0.5i, a_{21} = -6 + 0.5i, a_{22} = 1.6 + 0.5i, a_{23} = 2 + 0.1i, a_{31} = -3 + 0.5i, a_{32} = 4 + i, a_{33} = 0.2 + 0.1i, f(z_i) = (|z_i + 1| - |z_i - 1|)/2, i = 1, 2, 3, [z_1(s), z_2(s), z_3(s)]^T = [1 + 0.1i, 1 - 0.1i, -1 + 0.1i]^T, \forall s \in [-1, 0], \alpha = 0.96$, then system (27) can exhibit chaotic behaviors. The attractor trajectory is shown in Figure 6. The state trajectory is shown in Figure 7.

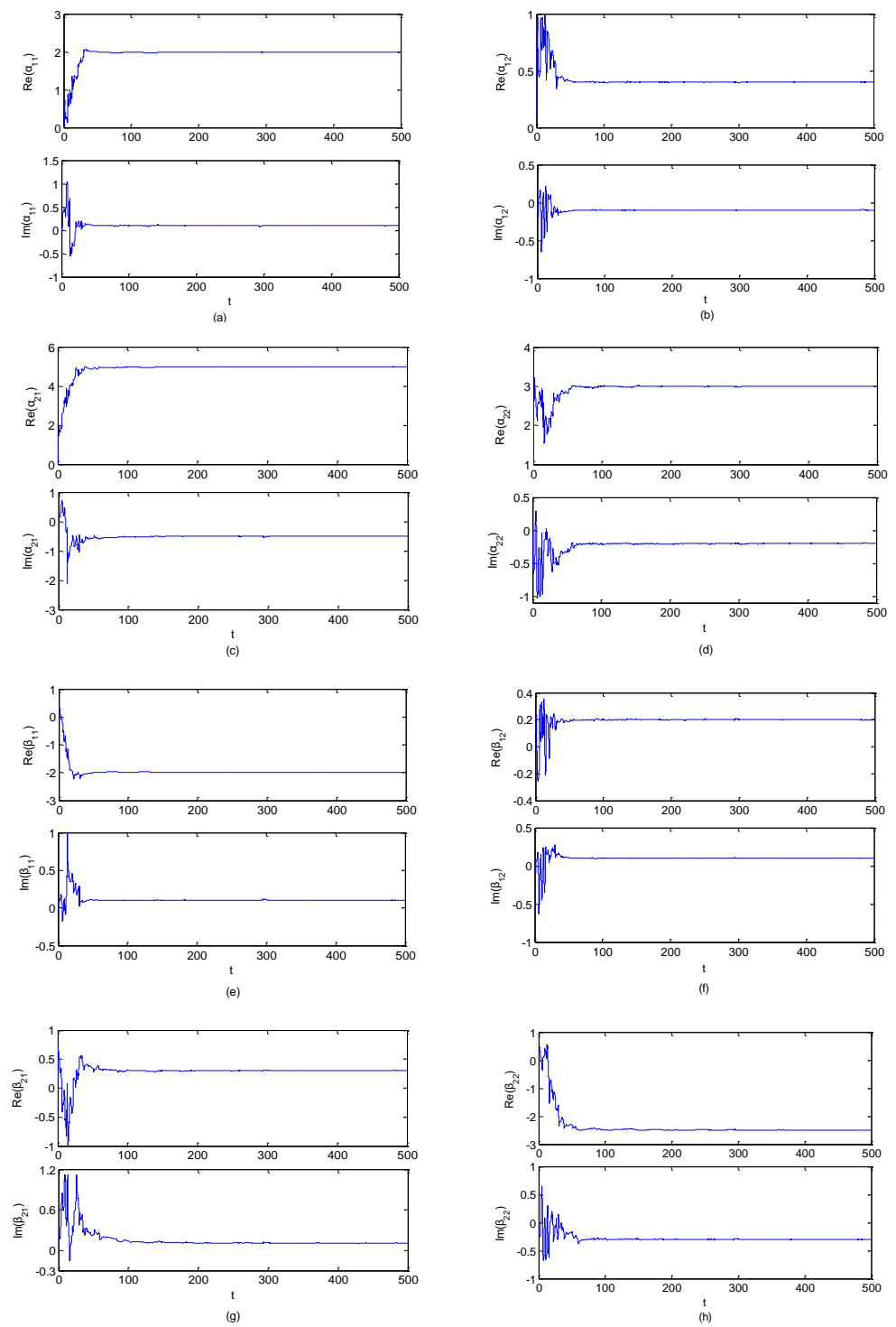


Figure 5. Estimated complex parameters of FOCVCNNs (26): (a) $\alpha_{11} \rightarrow a_{11} = 2 + 0.1i$, (b) $\alpha_{12} \rightarrow a_{12} = 0.4 - 0.1i$, (c) $\alpha_{21} \rightarrow a_{21} = 5 - 0.5i$, (d) $\alpha_{22} \rightarrow a_{22} = 3 - 0.2i$, (e) $\beta_{11} \rightarrow b_{11} = -2 + 0.1i$, (f) $\beta_{12} \rightarrow b_{12} = 0.2 + 0.1i$, (g) $\beta_{21} \rightarrow b_{21} = 0.3 + 0.1i$, (h) $\beta_{22} \rightarrow b_{22} = -2.5 - 0.3i$.

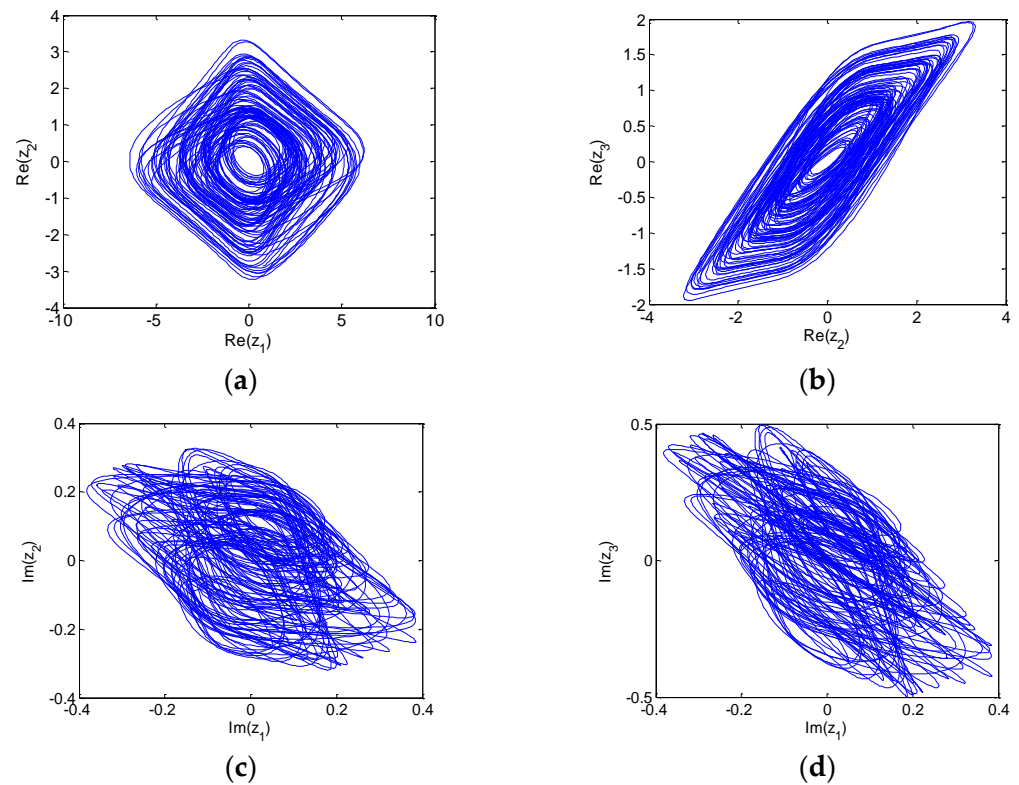


Figure 6. Chaotic attractors of FOCVCNNs (27): (a) $Re(z_1)$ vs. $Re(z_2)$, (b) $Re(z_2)$ vs. $Re(z_3)$, (c) $Im(z_1)$ vs. $Im(z_2)$, (d) $Im(z_1)$ vs. $Im(z_3)$.

Taking system (27) as the driving system and assuming that the coefficients $a_{ij}(i, j = 1, 2, 3)$ are unknown, the corresponding controlled response system is as follows:

$$\begin{cases} {}^C_0 D_t^\alpha w_1(t) = -w_1(t) + b_{11}f(w_1(t)) + b_{12}f(w_2(t)) + b_{13}f(w_3(t - \tau)) + u_1(t), \\ {}^C_0 D_t^\alpha w_2(t) = -w_2(t) + b_{21}f(w_1(t)) + b_{22}f(w_2(t)) + b_{23}f(w_3(t - \tau)) + u_2(t), \\ {}^C_0 D_t^\alpha w_3(t) = -w_3(t) + b_{31}f(w_1(t)) + b_{32}f(w_2(t)) + b_{33}f(w_3(t - \tau)) + u_3(t), \end{cases} \quad (28)$$

where $b_{ij}(i, j = 1, 2, 3)$ are estimated values of a_{ij} and $u_i(t)$ are controllers. Let the system errors be $e_i(t) = w_i(t) - z_i(t)$, ($i = 1, 2, 3$). The controllers and the update laws are selected as Equations (14)–(18), and the initial conditions are chosen as follows:

$$\begin{pmatrix} b_{11}(0) & b_{12}(0) & b_{13}(0) \\ b_{21}(0) & b_{22}(0) & b_{23}(0) \\ b_{31}(0) & b_{32}(0) & b_{33}(0) \end{pmatrix} = \begin{pmatrix} 1 & 10 - i & -8 + i \\ -8 + i & 2 + i & 2 \\ -3 + i & 3 + 0.5i & 0.3 - 0.1i \end{pmatrix}, \quad (29)$$

$$\begin{pmatrix} z_1(0) & w_1(0) \\ z_2(0) & w_2(0) \\ z_3(0) & w_3(0) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix},$$

$$k_1(0) = k_2(0) = k_3(0) = 1.$$

The simulation results are shown in Figures 8 and 9. Figure 8 shows that two pairs of high-dimensional FOCVCNNs achieve asymptotic synchronization through the adaptive controllers and adaptive update laws. Figure 9 indicates that all the unknown parameters of the driving system (27) are identified.

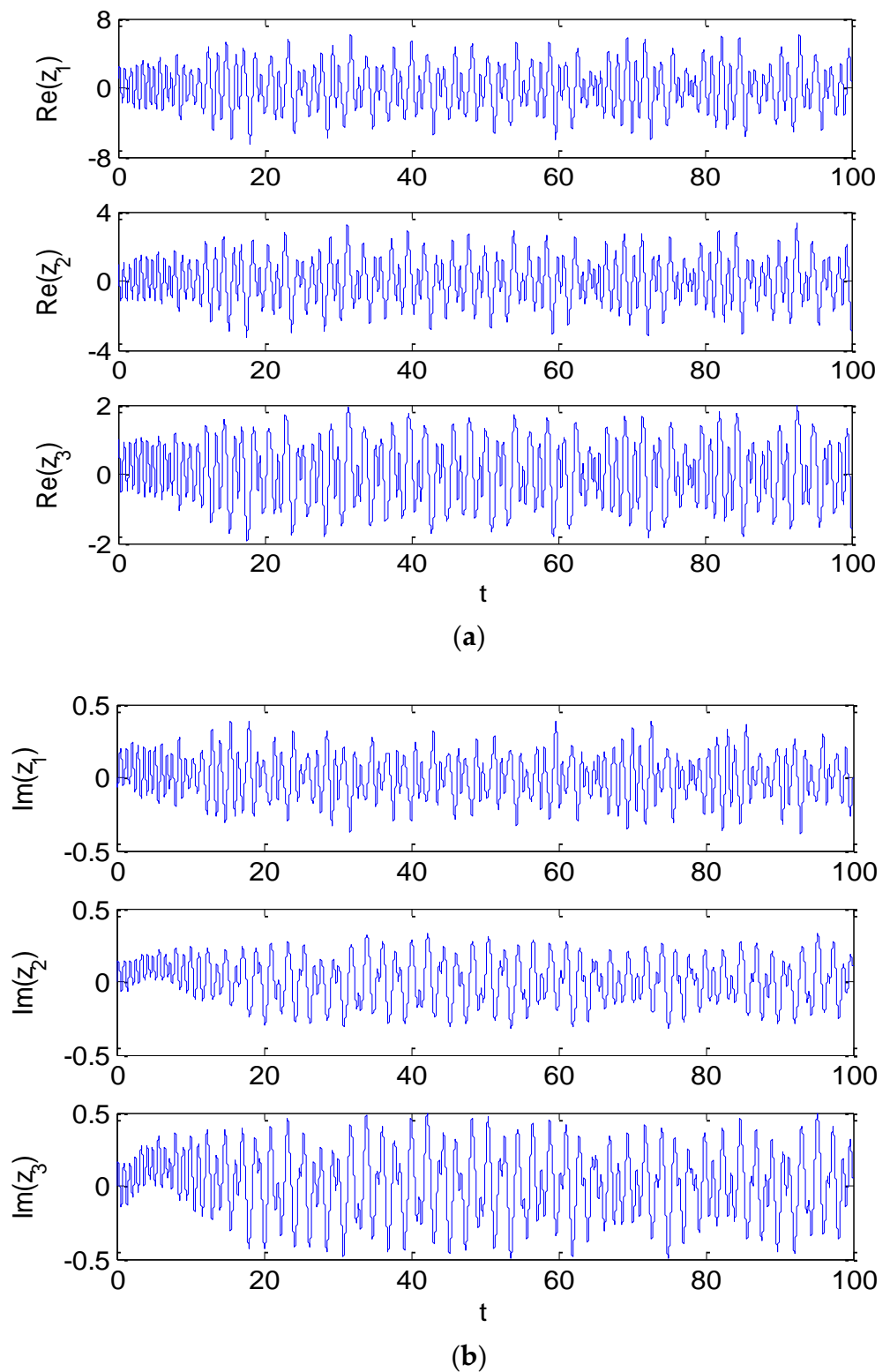


Figure 7. The state trajectories of FOCVCNNs (27): (a) $\text{Re}(z_1)$, $\text{Re}(z_2)$ and $\text{Re}(z_3)$ vs. t , (b) $\text{Im}(z_1)$, $\text{Im}(z_2)$ and $\text{Im}(z_3)$ vs. t .

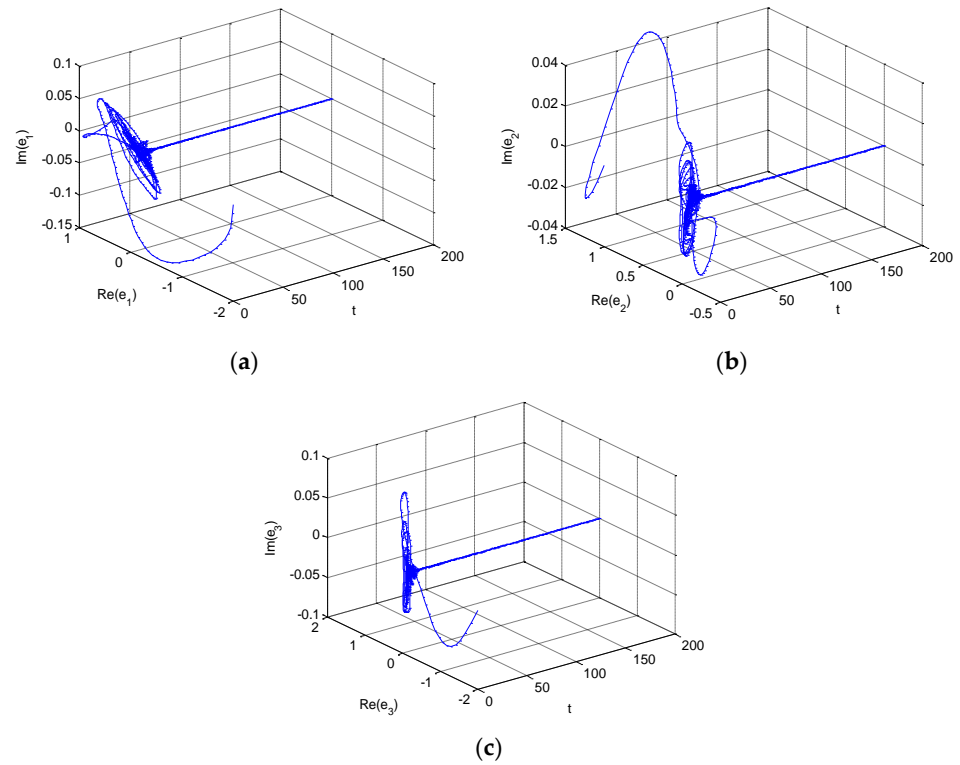


Figure 8. Synchronization errors of FOCVCNNs (27) and (28): (a) $e_1(t) = w_1(t) - z_1(t)$, (b) $e_2(t) = w_2(t) - z_2(t)$, (c) $e_3(t) = w_3(t) - z_3(t)$.

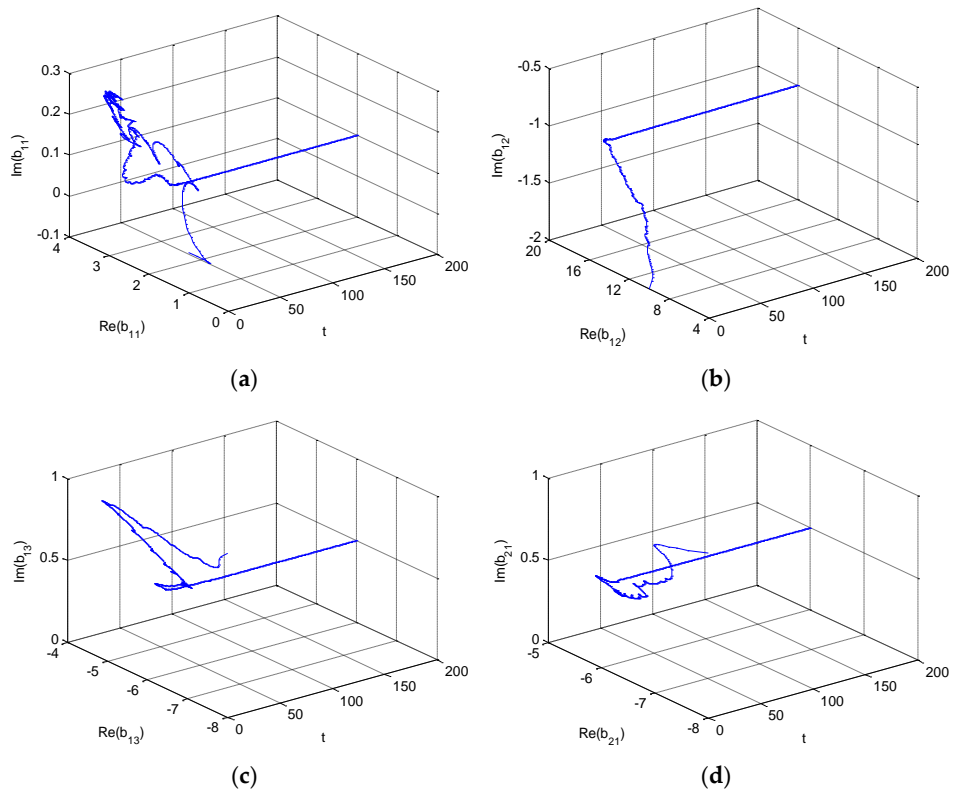


Figure 9. Estimated complex parameters of FOCVCNNs (28): (a) $b_{11} \rightarrow a_{11} = 2 + 0.1i$, (b) $b_{12} \rightarrow a_{12} = 16 - i$, (c) $b_{13} \rightarrow a_{13} = -6 + 0.5i$, (d) $b_{21} \rightarrow a_{21} = -6 + 0.5i$, (e) $b_{22} \rightarrow a_{22} = 1.6 + 0.5i$, (f) $b_{23} \rightarrow a_{23} = 2 + 0.1i$, (g) $b_{31} \rightarrow a_{31} = -3 + 0.5i$, (h) $b_{32} \rightarrow a_{32} = 4 + i$, (i) $b_{33} \rightarrow a_{33} = 0.2 + 0.1i$.

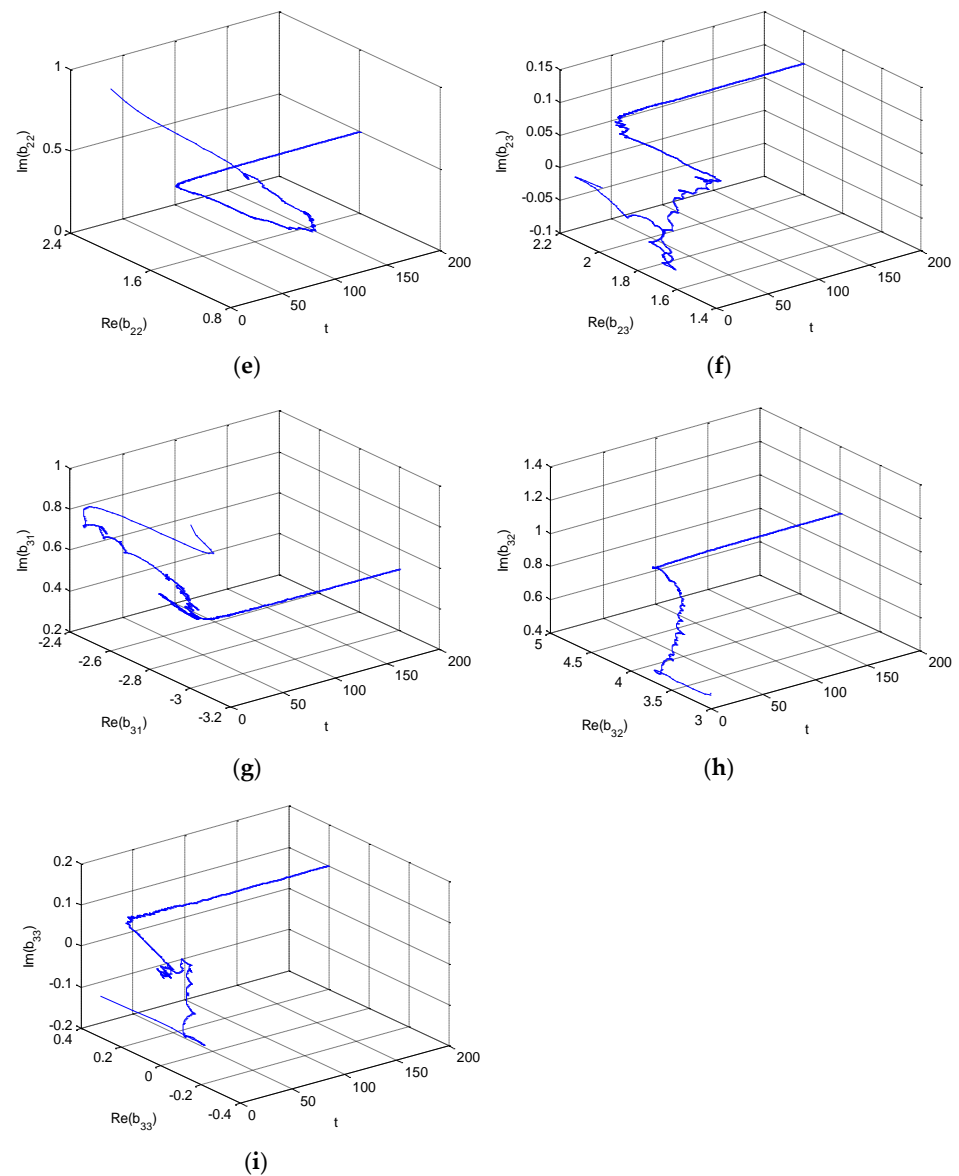


Figure 9. Cont.

It is not difficult to see from Example 2 that the research results of this paper can be easily extended to the synchronization and parameter identification for high-dimensional FOCVCNNs. Meanwhile, the synchronization control and parameter identification schemes proposed in this paper have very loose conditions, which make them easy to implement in practical applications. In addition, the synchronization control strategies are quite robust to external disturbances.

5. Conclusions

This paper focuses on the synchronization and parameter identification of fractional-order complex-valued chaotic neural networks (FOVCNNs) with time-delay and unknown complex parameters. Using the complex-valued inequalities of fractional derivatives and stability theory of fractional-order complex-valued systems, the adaptive controllers and complex update laws for synchronizing these systems are proposed. The proposed synchronization scheme preserves the complex nature of FOCVCNNs. Not only a new method for analyzing FOCVCNNs with time-delay and unknown complex parameters is provided here, but also a sensible decrease of the computational and analytical complexity.

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