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Fixed Point Approximation for Suzuki Generalized Nonexpansive Mapping Using $B_{(\delta,\mu)}$ Condition

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Abstract

In this paper, we introduce AK^* iteration scheme to approximate fixed point for Suzuki generalized nonexpansive mapping satisfying $B_{(\delta,\mu)}$ condition in the framework of Banach spaces. Also, an example is given to confirm the efficiency of AK^* iteration scheme. Our results are generalizations in the existing literature of fixed points in Banach spaces.

Keywords

AK^* Iterative Scheme, $B_{(\delta,\mu)}$ Condition, Suzuki Genertalized Nonexpansive Mappings, Banach Spaces

1. Introduction

Let $\mathcal{B}_s$ be a Banach space and $\mathcal{C}_b \subseteq \mathcal{B}_s$. For a mapping $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$, a point $q \in \mathcal{C}_b$ is said to be a fixed point if $q = \mathcal{F}q$. Also, a mapping $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ is said to be nonexpansive if

$$\|\mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b}\| \le \kappa\|\mathfrak{a} - \mathfrak{b}\| \;\; \forall\, \mathfrak{a},\mathfrak{b} \in \mathcal{C}_b.$$

We will refer to the set of natural numbers as $\mathbb{N}$ and the set of real numbers as $\mathbb{R}$ throughout the whole study and the set of all fixed points of $\mathcal{F}$ is referred by $\mathfrak{p}^{\mathcal{F}}$. If a mapping $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ then it is said to be quasi-nonexpansive mappings if $\mathfrak{p}^{\mathcal{F}} \ne \varnothing$ and $\|\mathcal{F}\mathfrak{a} - \kappa\| \le \|\mathfrak{a} - \kappa\|$ $\forall\, \mathfrak{a} \in \mathcal{C}_b$ and $q \in \mathfrak{p}^{\mathcal{F}}$. Browder [1] (also refer [2] [3]), Gohde [4], and Kirk [5] independently scrutinised the significance of fixed points for nonexpansive mappings in the framework of Banach spaces. They exemplified that if $\mathcal{C}_b$ is a nonempty, closed, bounded, and convex subset of a uniformly convex Banach space, then each nonexpansive mapping $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ seems to have at least one fixed point. Several other researchers have examined an amount of generalisations of nonexpansive mappings in recent decades. Suzuki introduced a new class of mappings (weaker than nonexpansiveness and stronger than quasi-nonexpansiveness) known as Suzuki generalised nonexpansive mappings, which is really a consequence on mappings regarded as Condition (C), and successfully obtained several other convergence and existence findings for these kinds of mappings in [6]. A mapping $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ is said to satisfy Condition (C) (oftentimes Suzuki generalised nonexpansive) if

$$\frac{1}{2}\|\mathfrak{a}-\mathcal{F}\mathfrak{a}\| \le \kappa\|\mathfrak{a}-\mathfrak{b}\| \text{ implies } \|\mathcal{F}\mathfrak{a}-\mathcal{F}\mathfrak{b}\| \le \|\mathfrak{a}-\mathfrak{b}\|,$$

for each $\mathfrak{a},\mathfrak{b} \in \mathcal{C}_b$.

Suzuki illustrated that Condition (C) is relatively weak than nonexpansion and stronger than quasi-nonexpansion. Falset *et al.* [7] introduced two new classes of generalised nonexpansive mappings that are wider than those satisfying the (C) condition whilst also retaining their fixed point attributes in 2011. We established a novel category of mappings in this paper that is relatively large than the class order to satisfy the Condition (C). Including some examples, we scrutinise the existence of fixed points for this category of mapping. First, we'll go over some key concepts. Every nonexpansive mapping evidently ensures the Condition (C).

Suzuki [6] [7] exemplified that Condition (C) is much more general than nonexpansiveness through the following example.

Example: [8] Define a mapping $\mathcal{F}:[0,3] \to \mathbb{R}$ by

$$\mathcal{F}\mathfrak{a} = \begin{cases} 0 & \mathfrak{a} \ne 3, \\ 1 & \mathfrak{a} = 3. \end{cases} \tag{1}$$

It is worth noting that $\mathcal{F}$ appeases Condition (C), however it is not nonexpansive. In 2018, Patir *et al.* [8] recently standardised the conception of Condition (C), and is as continues to follow:

[8] Consider a $\mathcal{B}_s$ and $\varnothing \ne \mathcal{C}_b \subseteq \mathcal{B}_s$, a mapping $\mathcal{F}$ such that $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ is known to achieve $B_{(\delta,\mu)}$ condition if there is an existence of $\gamma \in [0,1]$ and $\mu \in \left[0,\frac{1}{2}\right]$ satisfying the condition $2\mu \le \gamma$ in such a manner that $\forall\, \mathfrak{a},\mathfrak{b} \in \mathcal{C}_b$,

$$\gamma\|\mathfrak{a}-\mathcal{F}\mathfrak{a}\| \le \|\mathfrak{a}-\mathfrak{b}\| + \mu\|\mathfrak{b}-\mathcal{F}\mathfrak{b}\|$$

signifies

$$\|\mathcal{F}\mathfrak{a}-\mathcal{F}\mathfrak{b}\| \le (1-\gamma)\|\mathfrak{a}-\mathfrak{b}\| + \mu\left(\|\mathfrak{a}-\mathcal{F}\mathfrak{b}\| + \|\mathfrak{b}-\mathcal{F}\mathfrak{a}\|\right).$$

Remark: It is observable that a mapping with Condition (C) achieves the $B(\delta,\mu)$ condition.

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Example: [8] Define a mapping $\digamma:[0,2]\to\mathbb{R}$ by

$$\digamma\mathfrak{a}=\begin{cases}0 & \mathfrak{a}\neq 2,\\ 1 & \mathfrak{a}=2,\end{cases} \tag{2}$$

Here, $\digamma$ satisfies $B_{(\delta,\mu)}$ condition, but not Condition (C).

It is instinctual to investigate the processing of fixed points for known existence results, and that's not an easy process. The Picard iteration process is being used in the Banach contraction mapping criterion. The Picard iteration process is as follows:

$$\mathfrak{a}^{(\eta+1)}=\digamma\mathfrak{a}^{(\eta)}$$

and is used to the approximate unique fixed point. Mann [9], Ishikawa [10], S [11], Noor [12], Abbas [13], Thakur *et al.* [14], and so forth are other excellently iteration techniques. The convergence rate is absolutely essential for an iteration process to be favoured over the other iteration process. Rhoades [15] suggested that the Mann iteration process converges faster than that of the Ishikawa iterative procedure for significantly decreasing function as well as the Ishikawa iterative model is better for significantly increasing function than that of the Mann iterative procedure. The renowned Mann [9] and Ishikawa [10] iteration procedures are described as follows:

$$\begin{cases}\mathfrak{a}^1\in\mathcal{C}_b,\\ \mathfrak{a}^{(\eta+1)}=\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}\digamma\mathfrak{a}^{(\eta)},\eta\in\mathbb{N},\end{cases} \tag{3}$$

where $j_0^{(\eta)}\in(0,1)$.

$$\begin{cases}\mathfrak{a}^1\in\mathcal{C}_b,\\ \mathfrak{b}^{(\eta)}=(1-j_1^{(\eta)})\mathfrak{a}^{(\eta)}+j_1^{(\eta)}\digamma\mathfrak{a}^{(\eta)},\\ \mathfrak{a}^{(\eta+1)}=\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}\digamma\mathfrak{b}^{(\eta)},\eta\in\mathbb{N},\end{cases} \tag{4}$$

where $j_0^{(\eta)},j_1^{(\eta)}\in(0,1)$. The following iteration approach, known as S iteration, was established by Agarwal *et al.* [11] in 2007:

$$\begin{cases}\mathfrak{a}^1\in\mathcal{C}_b,\\ \mathfrak{b}^{(\eta)}=\left(1-j_1^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_1^{(\eta)}\digamma\mathfrak{a}^{(\eta)},\\ \mathfrak{a}^{(\eta+1)}=\left(1-j_0^{(\eta)}\right)\digamma\mathfrak{a}^{(\eta)}+j_0^{(\eta)}\digamma\mathfrak{b}^{(\eta)},\eta\in\mathbb{N},\end{cases} \tag{5}$$

where $0<j_0^{(\eta)},j_1^{(\eta)}<1$. They observed that for the class of contraction mappings, the speed of convergence of the (5) iteration process is much like the Picard iteration and speedier than the Mann iteration process. Thakur *et al.* [14] used a modified iterative algorithm, which was described as follows:

$$\begin{cases}\mathfrak{a}^1\in\mathcal{C}_b,\\ \mathfrak{c}^{(\eta)}=\left(1-j_1^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_1^{(\eta)}\digamma\mathfrak{a}^{(\eta)},\\ \mathfrak{b}^{(\eta)}=\digamma\left(\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}\mathfrak{c}^{(\eta)}\right),\\ \mathfrak{a}^{(\eta+1)}=\digamma\mathfrak{b}^{(\eta)},\eta\in\mathbb{N},\end{cases} \tag{6}$$

where $j_0^{(\eta)}, j_1^{(\eta)} \in (0,1)$.

They asserted that (6) is significantly faster than Picard, Mann, Ishikawa, Agarwal, Noor, and Abbas iteration algorithms for the class of Suzuki generalised nonexpansive mappings through numerical examples.

Recently in 2018, Ullah and Arshad [16] introduced K^* iteration process:

$$\begin{cases} \mathfrak{a}^1 \in \mathcal{C}_b, \\ \mathfrak{c}^{(\eta)} = \left(1 - j_1^{(\eta)}\right)\mathfrak{a}^{(\eta)} + j_1^{(\eta)} \digamma \mathfrak{a}^{(\eta)}, \\ \mathfrak{b}^{(\eta)} = \digamma\left(\left(1 - j_0^{(\eta)}\right)\mathfrak{c}^{(\eta)} + j_0^{(\eta)} \digamma \mathfrak{c}^{(\eta)}\right), \\ \mathfrak{a}^{(\eta+1)} = \digamma \mathfrak{b}^{(\eta)}, \eta \in \mathbb{N}, \end{cases} \tag{7}$$

where $j_0^{(\eta)}, j_1^{(\eta)} \in (0,1)$. They contended that iteration (8) had a faster rate of convergence than that of the other iteration methods.

Question. Is it feasible to establish an iteration process that has a faster convergence rate than that of the iteration processes (7)?

As a response, we propose the AK^* iterative approach, which is a newer version, and is as follows:

$$\begin{cases} \mathfrak{a}^1 \in \mathcal{C}_b, \\ \mathfrak{c}^{(\eta)} = \digamma\left(\left(1 - j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)} + j_0^{(\eta)} \digamma \mathfrak{a}^{(\eta)}\right), \\ \mathfrak{b}^{(\eta)} = \digamma\left(\left(1 - j_1^{(\eta)}\right)\digamma \mathfrak{c}^{(\eta)} + j_1^{(\eta)} \mathfrak{c}^{(\eta)}\right), \\ \mathfrak{a}^{(\eta+1)} = \digamma \mathfrak{b}^{(\eta)}, \eta \in \mathbb{N}, \end{cases} \tag{8}$$

where $j_0^{(\eta)}, j_1^{(\eta)} \in (0,1)$. In this way, we approximate fixed points of mapping which satisfies condition $B_{(\delta,\mu)}$. We compare the convergence rate of our novel AK^* iteration approach to current faster iteration schemes using a numerical example.

2. Numerical Example

In this section, an example is given to support the assertion that AK^* iteration scheme converges faster than the K^* and S iteration scheme.

Example Let $\mathcal{B}_s = (-\infty, \infty)$ and $\mathcal{C}_b = [1, 50]$. Let $\digamma : \mathcal{C}_b \to \mathcal{C}_b$ be mapping defined as $\digamma(\mathfrak{a}) = \frac{1 + 4\mathfrak{a}}{5}$ $\forall \mathfrak{a} \in \mathcal{C}_b$. Obviously $\mathfrak{a} = 1$ is an invariant point of $\digamma$. Let $\mathfrak{a}^{(1)} = 40$ $\forall \eta \in \mathbb{N}$ and $j_0^{(\eta)} = 0.95$, $j_1^{(\eta)} = 0.30$ and $j_2^{(\eta)} = 0.90$. The iterative values for $\mathfrak{a}^{(\eta)}$ are given in Table 1 where as the Study of AK^* for initial value $\mathfrak{a}^{(1)} = 0.7$ for function $(1 - \mathfrak{a})^8$ with $j_0^{(\eta)} = \frac{1}{\sqrt{1+\eta}}$ and $j_1^{(\eta)} = \frac{1}{\sqrt{1+\eta}}$ for AK^*, K^* and S iteration processes is studied in Table 2.

In compared to conventional iteration processes, the proposed AK^* iterative model clearly converges faster to the fixed point of $\digamma$.

Table 1. Study of AK' for initial value $\mathfrak{a}^{(1)} = 40$.

Empirical Study of AK' iteration algorithm for initial value $\mathfrak{a}^{(1)} = 40$.			
Iterative Sequence	$AK'(8)$	$K^*(7)$	$S(5)$
$\mathfrak{a}^{(1)}$	40	40	40
$\mathfrak{a}^{(2)}$	14.9097	20.0045	30.4216
$\mathfrak{a}^{(3)}$	1.30275	10.2608	23.1957
$\mathfrak{a}^{(4)}$	1.00014	3.19905	13.632
$\mathfrak{a}^{(5)}$	1.	2.07159	10.5296
$\mathfrak{a}^{(6)}$	1.	1.52218	8.1891
$\mathfrak{a}^{(7)}$	1.	1.25446	6.42346
$\mathfrak{a}^{(8)}$	1.	1.124	5.09146
$\mathfrak{a}^{(9)}$	1.	1.	3.

Table 2. Study of AK' for initial value $\mathfrak{a}^{(1)} = 0.7$ for function $(1-\mathfrak{a})^8$.

Empirical Study of AK' iteration algorithm			
Iterative Sequence	$AK'(8)$	$K^*(7)$	$S(5)$
$\mathfrak{a}^{(1)}$	0.7	0.7	0.7
$\mathfrak{a}^{(2)}$	1.	1	0.99847
$\mathfrak{a}^{(3)}$	0.244059	0.0637599	0.0627778
$\mathfrak{a}^{(4)}$	0.311435	0.143016	0.285937
$\mathfrak{a}^{(5)}$	0.209555	0.171169	0.161419
$\mathfrak{a}^{(6)}$	0.192273	0.19578	0.240008
$\mathfrak{a}^{(7)}$	0.217227	0.202928	0.170926
$\mathfrak{a}^{(8)}$	0.206056	0.203451	0.241304
$\mathfrak{a}^{(9)}$	0.204142	0.203456	0.163171
$\mathfrak{a}^{(10)}$	0.203397	0.203456	0.258741
$\mathfrak{a}^{(11)}$	0.203465	0.203456	0.141754

3. Preliminaries

In this section, we give some preliminaries. Let $\mathcal{B}_s$ be a Banach space and $\mathcal{C}_b$ be a nonempty closed convex subset of $\mathcal{B}_s$. Let $\left\{\mathfrak{a}^{(\eta)}\right\}$ be a bounded sequence in $\mathcal{C}_b$. For $\mathfrak{a} \in \mathcal{C}_b$, set

$$r\left(\mathfrak{a},\left\{\mathfrak{a}^{(\eta)}\right\}\right) = \limsup_{\eta\to\infty}\left\|\mathfrak{a}-\mathfrak{a}^{(\eta)}\right\|.$$

The asymptotic radius of $\left\{\mathfrak{a}^{(\eta)}\right\}$ relative to $\mathcal{C}_b$ is given by

$$r\left(\mathcal{C}_b,\left\{\mathfrak{a}^{(\eta)}\right\}\right) = \inf\left\{r\left(\mathfrak{a},\left\{\mathfrak{a}^{(\eta)}\right\}\right) : \mathfrak{a} \in \mathcal{C}_b\right\}.$$

The asymptotic centre of $\left\{\mathfrak{a}^{(\eta)}\right\}$ relative to $\mathcal{C}_b$ is the set

$$A\left(\mathcal{C}_b,\left\{\mathfrak{a}^{(\eta)}\right\}\right)=\left\{\mathfrak{a}\in\mathcal{C}_b:r\left(\mathfrak{a},\left\{\mathfrak{a}^{(\eta)}\right\}\right)=r\left(\mathcal{C}_b,\left\{\mathfrak{a}^{(\eta)}\right\}\right)\right\}.$$

It's also commonly acknowledged that $A\left(\mathcal{C}_b,\mathfrak{a}^{(\eta)}\right)$ encompasses essentially one point in a uniformly convex Banach space. Furthermore, when $\mathcal{C}_b$ is nonempty and convex in the case when $A\left(\mathcal{C}_b,\mathfrak{a}(\eta)\right)$ is weakly compact and convex, see [2] [3] [16] [17] [18] also refer [19]-[29] for fixed point based literature.

So, here are a few effective approaches and consequences. Let $\mathcal{B}_s$ is a Banach space, it is known as uniformly convex if for each $\varepsilon\in(0,2]$, there is an existence of $\lambda>0$ in such a manner that for every $\mathfrak{a},\mathfrak{b}\in\mathcal{B}_s$,

$$\left.\begin{array}{l}\|\mathfrak{a}\|\le 1\\ \|\mathfrak{b}\|\le 1\\ \|\mathfrak{a}-\mathfrak{b}\|>\varepsilon\end{array}\right\}\Rightarrow\frac{1}{2}\|\mathfrak{a}+\mathfrak{b}\|\le(1-\lambda).$$

Definition [17] A Banach space $\mathcal{B}_s$ is said to have Opial's property if for each sequence $\left\{\mathfrak{a}^{(\eta)}\right\}$ in $\mathcal{C}_b$ which weakly converges to $\mathfrak{a}\in\mathcal{B}_s$ and for every $\mathfrak{b}\in\mathcal{C}_b$, it satisfies the following

$$\limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\mathfrak{a}\right\|<\limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\mathfrak{b}\right\|.$$

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces ($1<p<\mathfrak{b}$).

Definition [17] Let $\varnothing\ne\mathcal{C}_b$ be subset of a Banach space $\mathcal{B}_s$. Let $\mathcal{F}:\mathcal{C}_b\to\mathcal{C}_b$. A Banach space $\mathcal{B}_s$ is said to have satisfy Condition (C) if there is a function $g:[0,\infty)\to[0,\infty)$ satisfying $g(0)=0$ and $g(r)>0\ \ \forall r\in(0,\infty)$ such that

$$\|\mathfrak{a}-\mathcal{F}\mathfrak{a}\|\ge\psi\left(d\left(\mathfrak{a},\mathfrak{p}^{\mathcal{F}}\right)\right)\tag{9}$$

$\forall\,\mathfrak{a}\in\mathcal{C}_b$, where $d\left(\mathfrak{a},\mathfrak{p}^{\mathcal{F}}\right)$ represents distance of x from $\mathfrak{p}^{\mathcal{F}}$.

Definition [7] If $\mathcal{S}$ is a closed convex and bounded subset of Banach space $\mathcal{B}$, and a self-mapping $\mathcal{F}$ on $\mathcal{S}$ is nonexpansive, then there exists a sequence $\{\mathfrak{a}_\eta\}$ in $\mathcal{S}$ such that $\|\mathfrak{a}_\eta-\mathcal{F}\mathfrak{a}_\eta\|\to 0$. Such a sequence is called almost fixed point sequence for $\mathcal{F}$.

We now list some basic facts about Suzuki generalized nonexpansive mappings, which can be found in [6]. The following useful Lemma can be found in [18].

Lemma 1 Let $\mathcal{C}_b$ be a uniformly convex Banach space and $0<p\le j_0^{(\eta)}\le q<1$ for every $\eta\in\mathbb{N}$. If $\{t^\eta\}$ and $\{s^\eta\}$ are two sequences in $\mathcal{C}_b$ such that $\limsup_{\eta\to\infty}\left\|\mathcal{F}^\eta\right\|\le\mathcal{C}_b$, $\limsup_{\eta\to\infty}\left\|s^\eta\right\|\le\mathcal{C}_b$ and $\lim_{\eta\to\infty}\left\|j_0^{(\eta)}t^\eta+\left(1-j_0^{(\eta)}\right)s^\eta\right\|=\mathcal{C}_b$ for some $c\ge 0$ then, $\lim_{\eta\to\infty}\left\|t^\eta-s^\eta\right\|=0$. also,let $\mathcal{F}$ be a Suzuki generalized nonexpansive mapping defined on a subset $\mathcal{C}_b$ of a Banach space $\mathcal{B}_s$ with the Opial property. If a sequence $\left\{\mathfrak{a}^{(\eta)}\right\}$ converges weakly to z and $\lim_{\eta\to\infty}\left\|\mathcal{F}\mathfrak{a}^{(\eta)}-\mathfrak{a}^{(\eta)}\right\|=0$, then I-$\mathcal{F}$ is democlosed at zero.

The following Lemma gives many examples of mappings with $B_{(\delta,\mu)}$ condition.

Lemma 2 [8] Let $\mathcal{C}_b \neq \varnothing$ be subset of a Banach space $\mathcal{B}_s$. Let $\mathcal{F}$ be a self mapping on $\mathcal{C}_b$. if $\mathcal{F}$ satisfies Condition (C), then $\mathcal{F}$ satisfies $B_{(\delta,\mu)}$ condition.

Lemma 3 [8] Let $\varnothing \neq \mathcal{C}_b$ be subset of a Banach space $\mathcal{B}_s$. Let $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ satisfies $B_{(\delta,\mu)}$ condition. if κ is a fixed point of $\mathcal{F}$, then for each $\mathfrak{a} \in \mathcal{C}_b$

$$\|\kappa - \mathcal{F}\mathfrak{a}\| \leq \|\kappa - \mathfrak{a}\|.$$

Theorem 4 Let $\mathcal{B}$ be a Banach space. $\mathcal{C}_b$ be a nonempty subset of $\mathcal{B}$ and $\phi_a:\mathcal{C}_b \to \mathcal{C}_b$ be a mapping satisfies condition $B_{(\delta,\mu)}$. if $\mathfrak{a}^{(\eta)} \subseteq \mathcal{C}_b$ be such that

1) $\mathfrak{a}^{(\eta)}$ converges weakly to κ;

2) $\lim_{\eta\to\infty}\left\|\mathcal{F}\mathfrak{a}^{(\eta)} - \mathfrak{a}^{(\eta)}\right\| = 0$.

Then, $\kappa = \mathcal{F}\kappa$.

Proposition 5 Let $\mathcal{B}$ be a Banach space. $\mathcal{C}_b$ be a nonempty subset of $\mathcal{B}$ and $\phi_a:\mathcal{C}_b \to \mathcal{C}_b$ be a mapping satisfies condition $B_{(\delta,\mu)}$ on $\mathcal{C}_b$, then $\forall\, \mathfrak{a},\mathfrak{b} \in \mathcal{C}_b$ and $\alpha \in [0,1]$

1) $\mathcal{F}\mathfrak{a} - \mathcal{F}^2\mathfrak{a} \leq \mathfrak{a} - \mathcal{F}\mathfrak{b}$

2) at least one of the following (a) and b)) holds:

a) $\left(\frac{\alpha}{2}\right)\mathfrak{a} - \mathcal{F}\mathfrak{a} \leq \mathcal{F}\mathfrak{a} - \mathfrak{b}$

b) $\left(\frac{\alpha}{2}\right)\mathcal{F}\mathfrak{a} - \mathcal{F}^2\mathfrak{a} \leq \mathcal{F}\mathfrak{a} - \mathfrak{b}$ condition a) implies

$\|\mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b}\| \leq \left(1 - \left(\frac{\alpha}{2}\right)\right)\|\mathfrak{a} - \mathfrak{b}\| + \mu\left(\|\mathfrak{a} - \mathcal{F}\mathfrak{b}\| + \|\mathfrak{b} - \mathcal{F}\mathfrak{a}\|\right)$ and condition b) implies

$\left\|\mathcal{F}^2\mathfrak{a} - \mathcal{F}\mathfrak{b}\right\| \leq \left(1 - \left(\frac{\alpha}{2}\right)\right)\|\mathcal{F}\mathfrak{a} - \mathfrak{b}\| + \mu\left(\|\mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b}\| + \left\|\mathfrak{b} - \mathcal{F}^2\mathfrak{a}\right\|\right)$.

3)
$$\begin{aligned}\|\mathfrak{a} - \mathfrak{b}\| \leq{}& (3-\alpha)\|\mathfrak{a} - \mathcal{F}\mathfrak{a}\| + \left(1 - \left(\frac{\alpha}{2}\right)\right)\|\mathcal{F}\mathfrak{a} - \mathfrak{b}\| \\ &+ \mu\left(2\|\mathfrak{a} - \mathcal{F}\mathfrak{a}\| + \|\mathfrak{a} - \mathcal{F}\mathfrak{b}\| + \|\mathfrak{b} - \mathcal{F}\mathfrak{a}\| + 2\left\|\mathcal{F}\mathfrak{a} - \mathcal{F}^2\mathfrak{a}\right\|\right)\end{aligned}$$

Then, $\kappa = \mathcal{F}\kappa$.

Lemma 6 Let $\mathcal{B}_s$ be a uniformly convex Banach space and $0 < q \leq \delta_\eta \leq p < 1 \quad \forall\, \eta \in \mathbb{N}$. If $\{\mathfrak{a}_\eta\}$ and $\{\mathfrak{b}_\eta\}$ are two sequences in $\mathcal{B}_s$ such that $\limsup_{\eta\to\infty}\|\mathfrak{a}_\eta\| \leq 1$, $\limsup_{\eta\to\infty}\|\mathfrak{b}_\eta\| \leq 1$, and $\lim_{\eta\to\infty}\left\|\delta^{(\eta)}\mathfrak{a}^{(\eta)} + \left(1-\delta^{n}\right)\mathfrak{b}^{(\eta)}\right\| = l$ for some $l \geq 0$ then $\lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)} - \mathfrak{b}^{(\eta)}\right\| = 0$.

4. Convergence Analysis

In this section, we study the convergence analysis of AK^* iteration scheme for which following Lemma plays a significant role.

Lemma 7 Let $\mathcal{C}_b$ be a nonempty closed convex subset of a Banach space $\mathcal{B}_s$ and $\mathcal{F}:\mathcal{C}_b \to \mathcal{C}_b$ satisfies condition ($B_{\delta,\mu}$) with $\mathfrak{p}^{\mathcal{F}} \neq \varnothing$. Let $\left\{\mathfrak{a}^{(\eta)}\right\}$ be a sequence generated by (8), then $\lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)} - \kappa\right\|$ exists for each $q \in \mathfrak{p}^{\mathcal{F}}$.

Let $q \in \mathfrak{p}^{\mathcal{F}}$. By Proposition (5) part 2), we have

$$\begin{aligned}\left\|\mathfrak{b}^{(\eta)}-\kappa\right\| &= \left\|F\left(\left(1-j_1^{(\eta)}\right)F\,\mathfrak{c}^{(\eta)}+j_1^{(\eta)}\mathfrak{c}^{(\eta)}\right)-\kappa\right\| \\ &\le \left\|\left(1-j_1^{(\eta)}\right)F\,\mathfrak{c}^{(\eta)}+j_1^{(\eta)}\mathfrak{c}^{(\eta)}-\kappa\right\| \\ &\le \left(1-j_1^{(\eta)}\right)\left\|F\,\mathfrak{c}^{(\eta)}-\kappa\right\|+j_1^{(\eta)}\left\|\mathfrak{c}^{(\eta)}-\kappa\right\| \\ &\le \left(1-j_1^{(\eta)}\right)\left\|\mathfrak{c}^{(\eta)}-\kappa\right\|+j_1^{(\eta)}\left\|\mathfrak{c}^{(\eta)}-\kappa\right\| \\ &= \left\|\mathfrak{c}^{(\eta)}-\kappa\right\|\end{aligned}$$

$$\begin{aligned}\left\|\mathfrak{c}^{(\eta)}-\kappa\right\| &= \left\|F\left(\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}F\,\mathfrak{c}^{(\eta)}\right)-\kappa\right\| \\ &\le \left\|\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}F\,\mathfrak{a}^{(\eta)}-\kappa\right\| \\ &\le \left(1-j_0^{(\eta)}\right)\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|+j_0^{(\eta)}\left\|F\,\mathfrak{a}^{(\eta)}-\kappa\right\| \\ &\le \left(1-j_0^{(\eta)}\right)\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|+j_0^{(\eta)}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\| \\ &= \left\|\mathfrak{a}^{(\eta)}-\kappa\right\|,\end{aligned}$$

which implies that

$$\left\|\mathfrak{a}^{(\eta+1)}-\kappa\right\| = \left\|F\,\mathfrak{b}^{(\eta)}-\kappa\right\| \le \left\|\mathfrak{b}^{(\eta)}-\kappa\right\|.$$

Using the values of $\left\|\mathfrak{b}^{(\eta)}-\kappa\right\|$ and $\left\|\mathfrak{c}^{(\eta)}-\kappa\right\|$, we have

$$\left\|\mathfrak{a}^{(\eta+1)}-\kappa\right\| \le \left\|\mathfrak{a}^{(\eta)}-\kappa\right\|.$$

Thus, the sequence $\left\{\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|\right\}$ is bounded. Also, $\left\{\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|\right\}$ is non increasing. Consequently, $\lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|$ exists for each $q\in\mathfrak{p}^F$. The following Theorem is useful for the next results.

Theorem 8 Let $\mathcal{B}_s$ is a uniformly convex Banach space and $\mathcal{C}_b$ is a non-empty convex subset of $\mathcal{B}_s$. Also, the mapping $F:\mathcal{C}_b\to\mathcal{C}_b$ satisfies Condition $B_{(\delta,\mu)}$. Let $\left\{\mathfrak{a}^{(\eta)}\right\}$ be a sequence which is formulated by (8). Then, the set of all fixed points *i.e.* $\mathfrak{p}^F\neq\varnothing$ iff $\left\{\mathfrak{a}^{(\eta)}\right\}$ is bounded and limiting value of $\left\{\left\|F\,\mathfrak{a}^{(\eta)}-\mathfrak{a}^{(\eta)}\right\|\right\}$ is 0 for $\eta\to\infty$.

Let, $\mathfrak{p}^F\neq\varnothing$ and $q\in\mathfrak{p}^F$. Consequently, by Lemma (7), there is an existence of $\lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|$ which proves that the sequence $\left\{\mathfrak{a}^{(\eta)}\right\}$ is bounded. Let

$$\lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\| = c. \tag{10}$$

Following the proof of Lemma (3),

$$\limsup_{\eta\to\infty}\left\|F\,\mathfrak{a}^{(\eta)}-\kappa\right\| \le \limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\| = c. \tag{11}$$

By the proof of Lemma (7). It follows that

$$\left\|\mathfrak{a}^{(\eta+1)}-\kappa\right\| \le \left\|F\,\mathfrak{b}^{(\eta)}-\kappa\right\| \le \left\|\mathfrak{b}^{(\eta)}-\kappa\right\|$$

and hence we have

$$\left\|\mathfrak{a}^{(\eta+1)}-\kappa\right\| \le \left\|\mathfrak{c}^{(\eta)}-\kappa\right\|.$$

Hence, we have

$$\lim_{\eta\to\infty}\left\|F\,\mathfrak{c}^{(\eta)}-\kappa\right\| \ge c \tag{12}$$

using the Equations (10) and (12)

$$\lim_{\eta\to\infty}\left\|\mathfrak{c}^{(\eta)}-\kappa\right\|. \tag{13}$$

From Equation (13)

$$\begin{aligned} c &= \liminf_{\eta\to\infty}\left\|\mathfrak{c}^{(\eta)}-\kappa\right\| \\ &= \liminf_{\eta\to\infty}\left\|F\left(\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}\mathfrak{a}^{(\eta)}\right)-\kappa\right\| \\ &\le \liminf_{\eta\to\infty}\left\|\left(\left(1-j_0^{(\eta)}\right)\mathfrak{a}^{(\eta)}+j_0^{(\eta)}\mathfrak{a}^{(\eta)}\right)-\kappa\right\| \\ &\le \liminf_{\eta\to\infty}\left\|\left(1-j_0^{(\eta)}\right)\left(\mathfrak{a}^{(\eta)}-\kappa\right)+j_0^{(\eta)}\left(\mathfrak{a}^{(\eta)}-\kappa\right)\right\|. \end{aligned}$$

Hence,

$$\liminf_{\eta\to\infty}\left\|\left(1-j_0^{(\eta)}\right)\left(\mathfrak{a}^{(\eta)}-\kappa\right)+j_0^{(\eta)}\left(\mathfrak{a}^{(\eta)}-\kappa\right)\right\| = c. \tag{14}$$

Now, using (10) and (12) with (14) with Lemma (6)

$$\lim_{\eta\to\infty}\left\|F\,\mathfrak{c}^{(\eta)}-\mathfrak{c}^{(\eta)}\right\| = 0. \tag{15}$$

Conversely, let $\kappa\in A\left(\mathcal{C}_b,\mathfrak{a}^{(\eta)}\right)$. By Proposition (5) 3), for $\gamma=\frac{\alpha}{2},\alpha\in[0,1]$,

$$\begin{aligned} \left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\| &\le (3-\alpha)\left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\|+\left(1-\frac{\alpha}{2}\right)\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|+\mu\Big(2\left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\| \\ &\quad+\left\|\mathfrak{a}^{(\eta)}-F\,\kappa\right\|+\left\|\kappa-F\,\mathfrak{a}^{(\eta)}\right\|+2\left\|F\,\mathfrak{a}^{(\eta)}-F^2\mathfrak{a}^{(\eta)}\right\|\Big) \\ &\le (3-\alpha)\left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\|+\left(1-\frac{\alpha}{2}\right)\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|+\mu\Big(2\left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\| \\ &\quad+\left\|\mathfrak{a}^{(\eta)}-F\,\kappa\right\|+\left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\|+2\left\|\mathfrak{a}^{(\eta)}-F\,\mathfrak{a}^{(\eta)}\right\|\Big). \end{aligned}$$

By Proposition (5)

$$(1-\mu)\limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-F\,\kappa\right\| \le \left(1-\frac{\alpha}{2}+\mu\right)\limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|$$

$$\limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-F\,\kappa\right\| \le \left(\frac{1-\frac{\alpha}{2}+\mu}{1-\mu}\right)\limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\| \le \limsup_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|.$$

Since,

$$\frac{1-\frac{\alpha}{2}+\mu}{1-\mu}\le 1,\ \text{ for } 2\mu\le\gamma=\frac{\alpha}{2}$$

we have

$$r\left(F\kappa,\left\{\mathfrak{a}^{(\eta)}\right\}\right)\le r\left(\kappa,\left\{\mathfrak{a}^{(\eta)}\right\}\right),$$

which confirms that $F\kappa\in A\left(\mathcal{C}_b,\mathfrak{a}^{(\eta)}\right)$. Since, $\mathcal{B}_s$ is uniformly convex Banach space. Hence, $F\kappa=\kappa$.

Now, we prove our weak convergence result.

Theorem 9 Let $\mathcal{B}_s\ne\varnothing$ be a uniformly Banach space with Opial's property. $\mathcal{C}_b$ is closed and convex subset of $\mathcal{B}_s$ and $F:\mathcal{C}_b\to\mathcal{C}_b$ satisfying Condition $B_{(\delta,\mu)}$ with $\mathfrak{p}^F\ne\varnothing$. Then, the sequence $\left\{\mathfrak{a}^{(\eta)}\right\}$ formulated by (8) is a convergent sequence, which converges weakly to κ, where $\kappa\in\mathfrak{p}^F$.

Using aforementioned Theorem (8), the sequence $\left\{\mathfrak{a}^{(\eta)}\right\}$ is bounded and we have null sequence as $\left\{\left\|F\mathfrak{a}^{(\eta)}-\mathfrak{a}^{(\eta)}\right\|\right\}$. It is given that, $\mathcal{B}_s$ is uniformly convex. Consequently, $\mathcal{C}_b$ is reflexive. So, there is an existence of the subsequence $\left\{\mathfrak{a}^{n^i}\right\}$ of $\left\{\mathfrak{a}^{(\eta)}\right\}$ in such a manner that $\left\{\mathfrak{a}^{n^i}\right\}$ is convergent and converges weakly to some $w^1\in\mathcal{C}_b$. By Proposition (5) part (v), we have $\kappa^{(1)}\in\mathfrak{p}^F$. It is sufficient to show that $\left\{\mathfrak{a}^{(\eta)}\right\}$ converges weakly to $\kappa^{(1)}$. In fact, if $\left\{\mathfrak{a}^{(\eta)}\right\}$ does not converges weakly to $\kappa^{(1)}$. Then, $\exists$ a subsequence $\left\{\mathfrak{a}^{n^j}\right\}$ of $\left\{\mathfrak{a}^{(\eta)}\right\}$ and $\kappa^{(2)}\in\mathcal{C}_b$ such that $\left\{\mathfrak{a}^{n^j}\right\}$ is convergent sequence and converges weakly to $\kappa^{(2)}$ and $\kappa^{(2)}\ne\kappa^{(1)}$. By Theorem (4), $w^2\in\mathfrak{p}^F$. Considering Opial's property together with Lemma (7), we have

$$\begin{aligned}\lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa^{(1)}\right\| &= \lim_{i\to\infty}\left\|\mathfrak{a}^{\left(\eta^i\right)}-\kappa^{(1)}\right\| < \lim_{i\to\infty}\left\|\mathfrak{a}^{\left(\eta^i\right)}-\kappa^{(2)}\right\| = \lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa^{(2)}\right\| \\ &= \lim_{j\to\infty}\left\|\mathfrak{a}^{\left(\eta^j\right)}-\kappa^{(2)}\right\| < \lim_{j\to\infty}\left\|\mathfrak{a}^{\left(\eta^j\right)}-\kappa^{(1)}\right\| = \lim_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)}-\kappa^{(1)}\right\|.\end{aligned}$$

It really is an ambiguity. So, $\kappa^{(1)}=\kappa^{(2)}$. Thus, $\left\{\mathfrak{a}^{(\eta)}\right\}$ is convergent and converges weakly to $\kappa^{(1)}\in\mathfrak{p}^F$.

Theorem 10 Let $\mathcal{C}_b\subset\mathcal{B}_s$, where $\mathcal{B}_s\ne\varnothing$ be a uniformly Banach space and $F:\mathcal{C}_b\to\mathcal{C}_b$ be a mapping satisfying $B_{(\delta,\mu)}$. Then, $\left\{\mathfrak{a}^{(\eta)}\right\}$ generated by (8) converges to an element of $\mathfrak{p}^F$ iff $\liminf_{\eta\to\infty}d\left(\mathfrak{a}^{(\eta)},\mathfrak{p}^F\right)=0$ or $\limsup_{\eta\to\infty}d\left(\mathfrak{a}^{(\eta)},\mathfrak{p}^F\right)=0$.

The necessity is self-evident. Assume, however, that

$$\liminf_{\eta\to\infty}d\left(\mathfrak{a}^{(\eta)},\mathfrak{p}^F\right)=0$$

and $0=\kappa\in\mathfrak{p}^F$, from Lemma (7), $\liminf_{\eta\to\infty}\left\|\mathfrak{a}^{(\eta)},-\kappa\right\|$ exists for each $\kappa\in\mathfrak{p}^F$. Hence, $\liminf_{\eta\to\infty}d\left(\mathfrak{a}^{(\eta)},\mathfrak{p}^F\right)$, by based on the assumption. We prove that $\left\{x^{(\eta)}\right\}$ is a sequence which is Cauchy in $\mathcal{C}_b$. It is given that $\liminf_{\eta\to\infty}d\left(\mathfrak{a}^{(\eta)},\mathfrak{p}^F\right)=0$, for a given $\varepsilon>0$, thereis an existence of $k_0\in\mathbb{N}$ sch that for each $n\ge k_0$,

$$d\left(\mathfrak{a}^{(\eta)},\mathfrak{p}^F\right)<\frac{\varepsilon}{2},$$

implies,

$$\inf\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|:\kappa\in\mathfrak{p}^F<\frac{\varepsilon}{2}. \tag{16}$$

In particular, $\inf\left\|\mathfrak{a}^{(\eta)}-\kappa\right\|:\kappa\in\mathfrak{p}^F<\frac{\varepsilon}{2}$. It confirms the existence of $\kappa\in\mathfrak{p}^F$

such that

$$\begin{aligned}\left\|\mathfrak{a}^{(\eta+k)}-\mathfrak{a}^{(\eta)}\right\| &\leq \left\|\mathfrak{a}^{(\eta+k)}-\kappa\right\|+\left\|\mathfrak{a}^{(\eta)}-\kappa\right\| \\ &\leq \left\|\mathfrak{a}^{(k_0)}-\kappa\right\|+\left\|\mathfrak{a}^{(k_0)}-\kappa\right\| \\ &\leq 2\left\|\mathfrak{a}^{(k_0)}-\kappa\right\|<\varepsilon.\end{aligned}$$

It proves that the sequence $\left\{x^{(\eta)}\right\}$ is Cauchy in $\mathcal{C}_b$. Also, $\mathcal{C}_b$ is a closed subset of a Banach space $\mathcal{B}_s$. Consequently, there is an existence of a point $\kappa' \in \mathcal{C}_b$ in such a manner that

$$\lim_{\eta\to\infty} \mathfrak{a}^{(\eta)} = \kappa'.$$

Now, $\lim_{\eta\to\infty} d\left(\mathfrak{a}^{(\eta)}, \mathfrak{p}^{F}\right)=0$ gives that $d\left(\kappa', \mathfrak{p}^{F}\right)=0$. As we know that $\mathfrak{p}^{F}$ is closed, hence from Lemma (3), $\kappa' \in \mathfrak{p}^{F}$.

We now prove the following Theorem using Condition (C).

Theorem 11 Let $\mathcal{C}_b \subset \mathcal{B}_s$, where $\mathcal{B}_s \neq \varnothing$ be a uniformly Banach space and $F: \mathcal{C}_b \to \mathcal{C}_b$ be a mapping satisfying $B_{(\delta,\mu)}$. Then, $\left\{\mathfrak{a}^{(\eta)}\right\}$ generated by (8) converges strongly to an element of $\mathfrak{p}^{F}$ provided that F satisfies Condition (C).

By using the Theorem (8), we have

$$\lim_{\eta\to\infty}\left\|F\mathfrak{a}^{(\eta)}-\mathfrak{a}^{(\eta)}\right\|=0.$$

Thus, by Condition (C), we obtain

$$\lim_{\eta\to\infty} d\left(\mathfrak{a}^{(\eta)}, \mathfrak{p}^{F}\right)=0.$$

Now that all of Theorem (11)'s have been met, $\left\{\mathfrak{a}^{(\eta)}\right\}$ converges strongly to a fixed point of F as a consequence of its conclusion.

5. Conclusion

Our work deals with AK^{*} iteration scheme to approximate fixed point for Suzuki generalized nonexpansive mapping which satisfy $B_{(\delta,\mu)}$ condition in the framework of Banach spaces. With the help of examples, it is proved that AK^{*} iteration scheme is more efficient than K^{*} and S iteration schemes. AK^{*} iteration scheme can be used to find the solution of functional Volterra-Fredholm integral equation and absolute value equations.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Browder, F.E. (1965) Nonexpansive Nonlinear Operators in a Banach Space. *Pro-*

ceedings of the National Academy of Sciences of the United States of America, **54**, 1041-1044. https://doi.org/10.1073/pnas.54.4.1041

[2] Takahashi, W. (2000) Nonlinear Functional Analysis. Yokohoma Publishers, Yokohoma.

[3] Agarwal, R.P., O'Regan, D. and Sahu, D.R. (2009) Fixed Point Theory for Lipschitzian-Type Mappings with Applications Series: Topological Fixed Point Theory and Its Applications. Springer, New York. https://doi.org/10.1155/2009/439176

[4] Gohde, D. (1965) Zum Prinzip der Kontraktiven Abbildung. *Mathematische Nachrichten*, **30**, 251-258. https://doi.org/10.1002/mana.19650300312

[5] Kirk, W.A. (1965) A Fixed Point Theorem for Mappings Which Do Not Increase Distance. *American Mathematical Monthly*, **72**, 1004-1006. https://doi.org/10.2307/2313345

[6] Suzuki, T. (2008) Fixed Point Theorems and Convergence Theorems for Some Generalized Nonexpansive Mappings. *Journal of Mathematical Analysis and Applications*, **340**, 1088-1095. https://doi.org/10.1016/j.jmaa.2007.09.023

[7] Suzuki, T., Falset, J.G. and Fuster, E.L. (2011) Fixed Point Theory for a Class of Generalised Non-Expansive Mappings. *Journal of Mathematical Analysis and Applications*, **375**, 185-195. https://doi.org/10.1016/j.jmaa.2010.08.069

[8] Patir, B., Goswami, N. and Mishra, V.N. (2018) Some Results on Fixed Point Theory for a Class of Generalized Nonexpansive Mappings. *Fixed Point Theory and Applications*, **2018**, Article No. 19. https://doi.org/10.1186/s13663-018-0644-1

[9] Mann, W.R. (1953) Some Results on Fixed Point Theory for a Class of Generalized Nonexpansive Mappings Mean Value Methods in Iterations. *Proceedings of the American Mathematical Society*, **4**, 506-510.

[10] Ishikawa, S. (1974) Fixed Points by a New Iteration Method. *Proceedings of the American Mathematical Society*, **44**, 147-150. https://doi.org/10.1090/S0002-9939-1974-0336469-5

[11] Agarwal, R.P., O'Regan, D. and Sahu, D.R. (2007) Iterative Construction of Fixed Points of Nearly Asymptotically Nonexpansive Mappings. *Journal of Nonlinear and Convex Analysis*, **8**, 61-79.

[12] Noor, M.A. (2000) New Approximation Schemes for General Variational Inequalities. *Journal of Nonlinear and Convex Analysis*, **251**, 217-229. https://doi.org/10.1006/jmaa.2000.7042

[13] Abbas, M. and Nazir, T. (2014) A New Faster Iteration Process Applied to Constrained Minimization and Feasibility Problems. *Matematički Vesnik*, **66**, 223-234.

[14] Rhoades, B.E., Thakur, B.S., Thakur, D. and Postolache, M. (2016) A New Iterative Scheme for Numerical Reckoning Fixed Points of Suzuki's Generalized Nonexpansive Mappings. *Applied Mathematics and Computation*, **275**, 147-155. https://doi.org/10.1016/j.amc.2015.11.065

[15] Rhoades, B.E. (1991) Some Fixed Point Iteration Procedures. *International Journal of Mathematics and Mathematical Sciences*, **14**, 1-16. https://doi.org/10.1155/S0161171291000017

[16] Ullah, K. and Arshad, M. (2018) New Three-Step Iteration Process and Fixed Point Approximation in Banach Spaces. *Journal of Linear and Topological Algebra*, **7**, 87-100.

[17] Opial, Z. (1967) Weak and Strong Convergence of the Sequence of Successive Approximations for Nonexpansive Mappings. *Bulletin of the American Mathematical Society*, **73**, 591-597. https://doi.org/10.1090/S0002-9904-1967-11761-0

[18] Schu, J. (1991) Weak and Strong Convergence to Fixed Points of Asymptotically Nonexpansive Mappings. *Bulletin of the Australian Mathematical Society*, **43**, 153-159. https://doi.org/10.1017/S0004972700028884

[19] Sharma, N., Mishra, V.N., Mishra, L.N. and Almusawa, H. (2021) End Point Approximation of Standard Three-Step Multivalued Iteration Algorithm for Nonexpansive Mappings. *Applied Mathematics Information Sciences*, **15**, 73-81. https://doi.org/10.18576/amis/150109

[20] Sharma, N., Mishra, L.N., Mishra, V.N. and Pandey, S. (2020) Solution of Delay Differential Equation via N_1^v Iteration Algorithm. *European Journal of Pure and Applied Mathematics*, **15**, 1110-1130. https://doi.org/10.29020/nybg.ejpam.v13i5.3756

[21] Sharma, N. and Dutta, H. (2021) Generalized Multiplicative Nonlinear Elastic Matching Distance Measure and Fixed Point Approximation. *Mathematical Methods in Applied Sciences*, **2021**, 1-17. https://doi.org/10.1002/mma.7827

[22] Sharma, N., Almusawa, H. and Hammad, H.A. (2021) Approximation of the Fixed Point for Unified Three-Step Iterative Algorithm with Convergence Analysis in Busemann Spaces. *Axioms*, **10**, 26. https://doi.org/10.3390/axioms10010026

[23] Sharma, N. and Mishra, L.N. (2021) Multi-Valued Analysis of CR Iterative Process in Banach Spaces. Springer, New York. https://doi.org/10.1007/978-981-16-1402-6_21

[24] Wu, C., Cui, J. and Zhong, L. (2019) Suzuki-Type Fixed Point Theorem in b2-Metric Spaces. *Open Access Library Journal*, **6**, e5974. https://doi.org/10.4236/oalib.1105974

[25] Kim, J., din, S. and Won Hee, W. (2017) An Iterative Algorithm for Generalized Mixed Equilibrium Problems and Fixed Points of Nonexpansive Semigroups. *Journal of Applied Mathematics and Physics*, **5**, 276-293. https://doi.org/10.4236/jamp.2017.52025

[26] Mishra, L.N., Dewangan, V., Mishra, V.N. and Karateke, S. (2021) Best Proximity Points of Admissible Almost Generalized Weakly Contractive Mappings with Rational Expressions on b-Metric Spaces. *Journal of Mathematics and Computer Science*, **22**, 97-109. https://doi.org/10.22436/jmcs.022.02.01

[27] Mishra, L.N., Dewangan, V., Mishra, V.N. and Amrulloh, H. (2021) Coupled Best Proximity Point Theorems for Mixed G-Monotone Mappings in Partially Ordered Metric Spaces. *Journal of Mathematics and Computer Science*, **11**, 6168-6192.

[28] Kumar, M., Jha, R. and Kumar, K. (2015) Common Fixed Point Theorem for Weak Compatible Mappings of Type (A) in Complex Valued Metric Space. *Journal of Analysis Number Theory*, **4**, 143-148. https://doi.org/10.18576/jant/030211

[29] Kumar, K., Sharma, N., Jha, R., Mishra, A. and Kumar, M. (2016) Weak Compatibility and Related Fixed Point Theorem for Six Maps in Multiplicative Metric Space. *Turkish Journal of Analysis and Number Theory*, **4**, 39-43.