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Research Article

Global Existence and Decay Rate of Smooth Solutions for Full System of Partial Differential Equations for Three-Dimensional Compressible Magnetohydrodynamic Flows

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We focus on the global existence and $L^p - L^q$ rates of convergence for the compressible magnetohydrodynamic equations in R^3. We prove the global existence of smooth solutions using the standard energy method under the condition that the initial data are close to a constant equilibrium state in H^3. Rates of convergence for the solution in L^q norm with $2 \leq q \leq 6$ and its first- and second-order derivatives in L^2 norm are obtained, if the initial data belong to L^p with $1 \leq p \leq \frac{6}{5}$.

1. Introduction

The study of the interaction between magnetic fields and electrically conducting fluids is of great importance for magnetohydrodynamics (MHD). From liquid metals to cosmic plasmas, involving intensely heated and ionized fluids in astrophysics, geophysics, high-speed aerodynamics, and plasma physics—the applications of MHD cover a very broad spectrum of physics. The structures of the solar system, including the outer layers, the solar wind that covers the Earth planets, and the interstellar magnetic fields are all sources of astrophysical problems. MHD is relevant to many engineering challenges, including extended plasma confinement for controlled thermonuclear fusion, liquid metal cooling of nuclear reactors, MHD power generation, electromagnetic casting of metals, and plasma accelerators for ion engines for spacecraft propulsion. Magnetic fields can induce currents to flow through conducting fluids, producing forces on the fluid and change the magnetic field. This is called MHD flows. It is necessary to consider both hydrodynamics and electrodynamics, as there is a complicated interaction between magnetic and fluid dynamic phenomena. The compressible Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism together form the set of equations describing the compressible viscous MHD. The whole system of partial differential equations for three-dimensional viscous, compressible, MHD flows in Euler coordinates is considered for $(0, \infty) \times \mathrm{R}^3$ [1, 2]:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div}\psi, \\ \varepsilon_t + \operatorname{div}\left(\mathbf{u}\left(\rho e + \frac{1}{2}\rho|\mathbf{u}|^2 + p\right)\right) = \operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H} + \nu \mathbf{H} \times (\nabla \times \mathbf{H}) + \mathbf{u}\psi + \kappa \nabla \theta), \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \operatorname{div}\mathbf{H} = 0, \end{cases} \tag{1}$$

where $\kappa > 0$ represents the fluid's heat conductivity and ρ, $\mathbf{u}$, θ, and $\mathbf{H}$ stand for the density, velocity, temperature of the fluid, and the magnetic field, respectively. The symbol Ψ denotes the viscous stress tensor and we shall assume that the Ψ is given through formula:

$$\psi := \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda \text{div}\, \mathbf{u} \mathrm{I}. \tag{2}$$

The total energy ε has contributions from the kinetic energy, internal energy, and magnetic energy given by:

$$\varepsilon := \rho e + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2. \tag{3}$$

The coefficients of viscosity μ and λ of the flow satisfy $2\mu + 3\lambda > 0$ and $\mu > 0$, I is the 3×3 identity matrix, $\nabla \mathbf{u}^T$ is the transpose of the matrix $\nabla \mathbf{u}$, and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The equations of state $p := p(\rho, \theta)$ and $e := e(\rho, \theta)$ relate the pressure p and the internal energy e to the density and temperature of the flow. MHD is studied by physicists and mathematicians likewise because of its physical significance, its complexity, its diverse phenomena, and its mathematical challenges. There are many published studies that have been carried out in this field. Viscous compressible MHD fluids in the isentropic case have been studied by Abdallah et al. [3]. Chen and Tan [4] have studied the interactions between the viscous, isentropic, compressible fluid motion and the magnetic field are modeled by the MHD system. Three-dimensional viscous compressible MHD flows in Eulerian coordinates were studied by Hu and Wang [5]. The motion of a compressible viscous heat-conductive gases, isotropic Newtonian fluid was studied by Matsumura and Nishida [6]. The flow of viscous compressible fluids, even under the influence of a magnetic field in a bounded domain, was described by Ströhmer [7]. The solvability of the Cauchy problem in a space of smooth functions is demonstrated for hyperbolic–parabolic composite systems of nonlinear equations involving a wide class of equations of mathematical physics discussed by Vol'Pert and Hudjaev [8]. The motion of a compressible viscous fluid in an external domain was studied by Kobayashi [9]. Kobayashi and Shibata [10] studied the motion of compressible viscous and heat-conductive gases in an exterior domain. Chen and Wang [11] studied a fundamental problem of MHD fluid flow in which the pressure, internal energy, and heat conductivity satisfy certain physical growth conditions of temperature. In this paper, we think of the global solution to the three-dimensional MHD problem over large time scales. Our aim is to analyze the uniqueness of the smooth solutions and global existence under the idea that the initial data are very near to the constant equilibrium position. Hence, we rewrite Equation (1) as follows:

$$\begin{cases} \rho_t + \text{div}\,(\rho \mathbf{u}) = 0, \\ \rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \text{div}\, \mathbf{u}, \\ \rho e_t + \rho \mathbf{u} \cdot \nabla e + p \text{div}\, \mathbf{u} = \nu |\nabla \mathbf{H}|^2 - \nu \nabla \mathbf{H} : \nabla \mathbf{H}^{\mathrm{T}} + \mu |\nabla \mathbf{u}|^2 + \mu \nabla \mathbf{u} : \nabla \mathbf{u}^T + \lambda |\text{div}\, \mathbf{u}|^2 + \kappa \Delta \theta, \\ \mathbf{H}_{\mathrm{t}} - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nu \nabla \times (\nabla \times \mathbf{H}), \text{div}\, \mathbf{H} = 0, \end{cases} \tag{4}$$

where $\nabla \mathbf{H} : \nabla \mathbf{H}^{\mathrm{T}} = \sum_{1 \le i, j \le 3} \partial_j H_i \partial_i H_j$. We shall assume that the fluid is ideal and barotropic, i.e., $e := c_v \theta$ and $p := \mathrm{R} \rho \theta$ with positive constants c_v, R. Moreover, without loss of generality, we also presume that the constants R, c_v, and ν to be unity, then reformulate the MHD system:

$$\begin{cases} \rho_t + \rho \text{div}\, \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \theta + \frac{1}{\rho} \theta \nabla \rho - \frac{1}{\rho} (\mu \Delta + (\mu + \lambda) \nabla \text{div}\,) \mathbf{u} = \frac{1}{\rho} (\mathbf{H} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{H}^T), \\ \theta_t + \mathbf{u} \cdot \nabla \theta + \theta \text{div}\, \mathbf{u} - \frac{1}{\rho} \Delta \theta = \frac{1}{\rho} (|\nabla \mathbf{H}|^2 - \nabla \mathbf{H} : \nabla \mathbf{H}^{\mathrm{T}} + \mu |\nabla \mathbf{u}|^2 + \mu \nabla \mathbf{u} : \nabla \mathbf{u}^{\mathrm{T}} + \lambda |\text{div}\, \mathbf{u}|^2), \\ \mathbf{H}_{\mathrm{t}} - \Delta \mathbf{H} = \mathbf{H} \cdot \nabla \mathbf{u} - \mathrm{u} \cdot \nabla \mathbf{H} - (\text{div}\, \mathbf{u}) \mathbf{H}, \text{div}\, \mathbf{H} = 0. \end{cases} \tag{5}$$

We complement Equation (5) with the Cauchy data given as follows:

$$(\rho, \mathbf{u}, \theta, \mathbf{H})(0, x) = (\rho_0(x), \mathbf{u}_0(x), \theta_0(x), \mathbf{H}_0(x)) x \in R^3. \tag{6}$$

Notation. *Throughout this paper, the norms in Lebesgue space L^p and Sobolev spaces $H^m(R^3)$, and $W^{m,p}(R^3)$ are denoted, respectively, by $\|\cdot\|_{L^p}, \|\cdot\|_{W^{m,p}}$ and $\|\cdot\|_{H^m}$. Moreover, C denotes a general constant, which may vary in different estimates. If the dependence needs to be explicitly stressed, a notation such as C_1, C_2 will be used. As usual $\partial_x = \nabla = (\partial_1, \partial_2, \partial_3), \partial_i = \partial_{x_i}, i = 1, 2, 3,$ and for any integer $k > 0$, $\nabla^{\mathrm{k}} f$ denotes all derivatives up to k-order of the function f.*

Remark 1. Note that the divergence-free magnetic field $\mathbf{H}$ can be justified by the initial assumption that $\text{div}\, \mathbf{H}_0 = 0$. Indeed, this can be easily and formally observed by taking div of the magnetic equation. Hence, the magnetic equation is purely parabolic with respect to $\mathbf{H}$. Now, we are in a position to state our main results of this paper. First, we have the following existence result of a unique small solution to the Cauchy problem (5)–(6):

Theorem 2. *Assume* $\operatorname{div}\mathbf{H}_0 = 0$ *and the initial data are close enough to the constant situation* $(\overline{\rho}, \mathbf{0}, \overline{\theta}, \overline{\mathbf{H}})$ *with* $\overline{\rho}, \overline{\theta}, \overline{\mathbf{H}} > 0$, *i.e., there exists a sufficiently small constant* δ_0 *such that for any initial data satisfy.*

$$\left\|\left(\rho_0 - \overline{\rho}, \mathbf{u}_0, \theta_0 - \overline{\theta}, \mathbf{H}_0 - \overline{\mathbf{H}}\right)\right\|_{H^3} \leq \delta_0, \tag{7}$$

then the MHD (5)–(6) possesses a unique globally smooth solution $(\rho, \mathbf{u}, \theta, \mathbf{H})$ *such that:*

$$\begin{aligned} &\left\|\left(\rho\text{--}\overline{\rho}, \mathbf{u}, \theta - \overline{\theta}, \mathbf{H} - \overline{\mathbf{H}}\right)(\cdot, t)\right\|_{H^3}^2 \\ &+ \int_0^t \|\partial_x \rho(\cdot, s)\|_{H^2}^2 + \|(\partial_x \mathbf{u}, \partial_x \theta, \partial_x \mathbf{H})(\cdot, s)\|_{H^3}^2 ds \leq C\delta_0^2, \end{aligned} \tag{8}$$

for any $t \in [0, \infty)$.

Second, we further have the following decay estimates for the solution constructed in the theorem above.

Theorem 3. *Under the Theorem 2, if in addition, there is some* $p \in [1, \frac{6}{5})$ *such that:*

$$\left\|\left(\rho_0 - \overline{\rho}, \mathbf{u}_0, \theta_0 - \overline{\theta}, \mathbf{H}_0 - \overline{\mathbf{H}}\right)\right\|_{L^p} < +\infty, \tag{9}$$

then the solution constructed in Theorem 2 satisfies the following decay estimates:

$$\left\|\left(\rho - \overline{\rho}, \mathbf{u}, \theta - \overline{\theta}, \mathbf{H} - \overline{\mathbf{H}}\right)(t)\right\|_{L^q} \leq C(1+t)^{-\sigma(p,q;0)}, \ \forall q \in [2, 6], \tag{10}$$

and

$$\left\|\left(\rho - \overline{\rho}, \mathbf{u}, \theta - \overline{\theta}, \mathbf{H} - \overline{\mathbf{H}}\right)(t)\right\|_{L^2} \leq C(1+t)^{-\sigma(p,2;1)}, \tag{11}$$

where $\sigma(p, q; k)$ *are defined by:*

$$\sigma(p, q; k) = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{k}{2}, \ k = 1, 2, 3. \tag{12}$$

If the initial data further satisfy $(\rho_0 - \overline{\rho}, \mathbf{u}_0, \theta_0 - \overline{\theta}, \mathbf{H}_0 - \overline{\mathbf{H}}) \in H^4(R^3)$ *and* $\left\|\left(\rho_0 - \overline{\rho},\ \mathbf{u}_0,\ \theta_0 - \overline{\theta},\ \mathbf{H}_0 - \overline{\mathbf{H}}\right)\right\|_{H^4}$ *is small, then the solution has the following high-order estimate:*

$$\left\|\nabla^2\left(\rho_0 - \overline{\rho}, \mathbf{u}_0, \theta_0 - \overline{\theta}, \mathbf{H}_0 - \overline{\mathbf{H}}\right)(t)\right\|_{L^2} \leq C(1+t)^{-\sigma(p,2;2)}. \tag{13}$$

The part which is left from this study is committed to demonstrate Theorems 2 and 3. In part two of this section, the priori estimates are carried out for the smooth solution. The global existence of the smooth solutions has been formulated due to the merging of the local existence and priori estimates outcomes. The issue in the third section is shaped using the model of a Laypinov-type energy inequality for all the derivatives, which are ruled by the first-order derivatives. Besides that, the decay-in-time estimates for the linearized system is also used to dominate the first-order derivatives by the higher-order derivatives. From these two types of estimates, the decay rates of the smooth solutions can be followed.

2. Global Existence

We will prove the existence part of Theorem 2. In outline, we first derive the uniform-in-time a priori estimates for smooth solutions. These estimates also hold for our H^3 local solutions, which are thoroughly authenticated by the standard method in [6] utilizing the Mollifier technique. Owing to the uniform estimates, the global existence is finally proved. To reduce complicated calculations, we recall the following useful inequalities:

$$|f|_{H^k}^2 \leq C\left|f, \partial_x^k f\right|_{L^2}^2, \ \forall f \in H^k. \tag{14}$$

This can be easily proved by combining the inequalities of Young and Gagliardo–Nirenberg:

$$|\partial_x^i f|_{L^p} \leq C(p)|f|_{L^q}^{\alpha}\left|\partial_x^k f\right|_{L^r}^{(1-\alpha)}, \ \forall f \in H^k, \tag{15}$$

where $\frac{1}{p} - \frac{i}{3} = \frac{1}{q}\alpha + \left(\frac{1}{r} - \frac{k}{3}\right)(1-\alpha)$ with $\alpha \in (0,1), r \in (1, \infty)$ and $0 \leq i \leq k$,

$$\left\|\partial_x^k(fg)\right\|_{L^2} \leq C\left[\|f\|_{L^\infty}\left\|\partial_x^k g\right\|_{L^2} + \left\|\partial_x^k f\right\|_{L^2}\|g\|_{L^\infty}\right]. \tag{16}$$

2.1. A Priori Estimates. For this objective, suppose $(\rho, \mathbf{u}, \theta, \mathbf{H})$ is a smooth solution of Equations (5) and (6) on $(0, T)$ with $\rho > 0$. We formulate the following theorem:

Theorem 4. *There exists a sufficiently small constant* δ *such that:*

$$\sup_{0 \leq t \leq T} \left\|\left(\rho - \overline{\rho}, \mathbf{u}, \theta - \overline{\theta}, \mathbf{H} - \overline{\mathbf{H}}\right)(\cdot, t)\right\|_{H^3} \leq \delta, \tag{17}$$

then for any $t \in [0, T]$, *there exists a constant* $C_1 > 1$ *such that:*

$$\begin{aligned} &\left\|\left(\rho - \overline{\rho}, \mathbf{u}, \theta - \overline{\theta}, \mathbf{H} - \overline{\mathbf{H}}\right)(\cdot, t)\right\|_{H^3}^2 \\ &+ \int_0^t \|\partial_x \rho(\cdot, s)\|_{H^2}^2 + \|(\partial_x \mathbf{u}, \partial_x \theta, \partial_x \mathbf{H})(\cdot, s)\|_{H^3}^2 ds \\ &\leq C_1\left\|\left(\rho_0 - \overline{\rho}, \mathbf{u}_0, \theta_0 - \overline{\theta}, \mathbf{H}_0 - \overline{\mathbf{H}}\right)(t)\right\|_{H^3}^2. \end{aligned} \tag{18}$$

For this purpose, we introduce new variables

$$\varrho+\overline{\rho}=\rho, \mathbf{u}=\alpha\beta^2\mathbf{v}, \theta=\alpha^2\beta^2\vartheta+\overline{\theta}, \alpha\beta\mathscr{H}+\overline{\mathbf{H}}=\mathbf{H}, \quad (19)$$

where $\alpha=\sqrt{\overline{\theta}}, \beta=\frac{1}{\sqrt{\overline{\rho}}}$. Then, Equations (5)–(6) are reformulated as:

$$\varrho_t+\alpha \mathrm{div}\,\mathbf{v}=N_1, \quad (20)$$

$$\mathbf{v}_t+\alpha\nabla\vartheta+\alpha\nabla\varrho-\mu\beta^2\Delta\mathbf{v}-\beta^2(\mu+\lambda)\nabla\mathrm{div}\,\mathbf{v}-\beta\overline{\mathbf{H}}\cdot\nabla\mathscr{H} +\beta\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T=N_2, \quad (21)$$

$$\vartheta_t+\alpha\mathrm{div}\,\mathbf{v}-\beta^2\Delta\vartheta=N_3, \quad (22)$$

$$\mathscr{H}_t+\beta\overline{\mathbf{H}}(\mathrm{div}\,\mathbf{v})-\beta\overline{\mathbf{H}}\cdot\nabla\mathbf{v}-\Delta\mathscr{H}=N_4, \quad (23)$$

$$(\varrho,\mathbf{v},\vartheta,\mathscr{H})=(\varrho_0,\mathrm{v}_0,\vartheta_0,\mathscr{H}_0)(x)\longrightarrow(0,0,0,0)\text{ as }|x|\longrightarrow\infty, \quad (24)$$

where

$$N_1=-\alpha\beta^2\,\mathrm{div}(\varrho\mathbf{v}), \quad (25)$$

$$\begin{aligned}N_2= & -\alpha\beta^2(\mathbf{v}\cdot\nabla\mathbf{v})-\alpha\frac{(\vartheta\cdot\nabla\varrho)}{\rho}-\frac{\alpha}{\beta^2}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\nabla\varrho+\mu\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\Delta\mathbf{v}+(\mu+\lambda)\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\nabla\mathrm{div}\,\mathbf{v}\\ & +\alpha\frac{(\mathscr{H}\cdot\nabla\mathscr{H})}{\rho}+\frac{1}{\beta}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)(\overline{\mathbf{H}}\cdot\nabla\mathscr{H})-\alpha\frac{(\mathscr{H}\cdot\nabla\mathscr{H}^T)}{\rho}-\frac{1}{\beta}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)(\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T),\end{aligned} \quad (26)$$

$$N_3=-\alpha\beta^2(\mathbf{v}\cdot\nabla\vartheta)-\alpha\beta^2(\vartheta\mathrm{div}\,\mathbf{v})+\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\Delta\vartheta+\frac{1}{\rho}[|\nabla\mathscr{H}|^2-\nabla\mathscr{H}:\nabla\mathscr{H}^T+\mu\beta^2|\nabla\mathbf{v}|^2+\mu\beta^2\nabla\mathbf{v}:\nabla\mathbf{v}^T+\lambda\beta^2|\mathrm{div}\,\mathbf{v}|^2], \quad (27)$$

$$N_4=\alpha\beta^2(\mathscr{H}\cdot\nabla\mathbf{v})-\alpha\beta^2(\mathbf{v}\cdot\nabla\mathscr{H})-\alpha\beta^2\,(\mathscr{H}\cdot\mathrm{div}\,\mathbf{v}). \quad (28)$$

First, we observe that a priori assumption Equation (17) with the equation of continuity Equation (19) imply:

$$\begin{aligned}&\sup_{0\le t\le T}\|(\varrho,\varrho_t,\partial_x\varrho,\mathbf{v},\partial_x\mathbf{v},\vartheta,\partial_x\vartheta,\mathscr{H},\partial_x\mathscr{H})(t)\|\\ &\le C\sup_{0\le t\le T}\left\|\left(\varrho,\mathbf{u},\theta-\overline{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3}^2\le C\delta.\end{aligned} \quad (29)$$

Because $H^2\hookrightarrow L^\infty$, we can choose δ sufficiently small that:

$$\begin{aligned}&\frac{\overline{\rho}}{2}\le\rho=\varrho+\overline{\rho}\le2\overline{\rho},\ \frac{\overline{\theta}}{2}\le\theta=\alpha^2\beta^2\vartheta\\ &+\overline{\theta}\le2\overline{\theta},\ \frac{\overline{\mathbf{H}}}{2}\le\mathbf{H}=\alpha\beta\mathscr{H}+\overline{\mathbf{H}}\le2\overline{\mathbf{H}}.\end{aligned} \quad (30)$$

In the following, we always assume δ is small and use Equations (29)–(30). The a priori estimates will be made in four steps.

A: L^2 norms of $\varrho,\mathbf{v},\vartheta,\mathscr{H}$. Multiplying the equation of continuity Equation (20) by ϱ and integrate over R^3 (by parts), we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\varrho\|_{L^2}^2+\alpha\int_{R^3}\varrho\mathrm{div}\,\mathbf{v}dx=I_1, \quad (31)$$

where $I_1=\langle N_1,\varrho\rangle$.

When momentum Equation (21) is multiplied by $\mathbf{v}$ and integrate over R^3, we get:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_{L^2}^2+\alpha\int_{R^3}(\mathbf{v}\cdot\nabla\vartheta)dx+\alpha\int_{R^3}(\mathbf{v}\cdot\nabla\varrho)dx-\mu\beta^2\\ &\int_{R^3}(\mathbf{v}\cdot\Delta\mathbf{v})dx-\beta^2(\mu+\lambda)\int_{R^3}(\mathbf{v}\cdot\nabla\mathrm{div}\,\mathbf{v})dx-\beta\\ &\int_{R^3}\mathbf{v}\cdot\left(\overline{\mathbf{H}}\cdot\nabla\mathscr{H}\right)dx+\beta\int_{R^3}\mathbf{v}\cdot\left(\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T\right)dx=I_2,\end{aligned} \quad (32)$$

where $I_2=\langle N_2,\mathbf{v}\rangle$.

Similarly, multiplying the energy Equation (22) by ϑ and integrate by parts over R^3, we get:

$$\frac{1}{2}\frac{d}{dt}\|\vartheta\|_{L^2}^2+\alpha\int_{R^3}\vartheta\mathrm{div}\,\mathbf{v}dx-\beta^2\int_{R^3}\vartheta\Delta\vartheta dx=I_3, \quad (33)$$

where $I_3=\langle N_3,\vartheta\rangle$.

Finally, when the magnetic Equation (23) is multiplied by $\mathscr{H}$ and integrates over R^3, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\mathscr{H}\|_{L^2}^2+\beta\int_{R^3}\left(\overline{\mathbf{H}}\cdot\mathscr{H}\right)\operatorname{div}\mathbf{v}dx-\beta\int_{R^3}\left(\overline{\mathbf{H}}\cdot\mathscr{H}\right)\cdot\nabla\mathbf{v}dx$$
$$-\int_{R^3}(\mathscr{H}\cdot\Delta\mathscr{H})dx=I_4, \tag{34}$$

where $I_4=\langle N_4,\mathscr{H}\rangle$.

Now we add Equations (31)–(34) and then by integration by parts, we obtain the following result:

$$\frac{1}{2}\frac{d}{dt}\|\varrho,\mathbf{v},\vartheta\mathscr{H}\|_{L^2}^2+\mu\beta^2\|\partial_x\mathbf{v}\|_{L^2}^2$$
$$+\beta^2(\mu+\lambda)\|\operatorname{div}\mathbf{v}\|_{L^2}^2+\beta^2\|\partial_x\vartheta\|_{L^2}^2+\|\partial_x\mathscr{H}\|_{L^2}^2=\sum_{k=1}^{4}I_k. \tag{35}$$

Using Holder's and Sobolev's inequalities and Equation (29), we estimate the right-hand side of Equation (35) as follows:

$$I_1=-\alpha\beta^2\int_{R^3}\varrho\operatorname{div}(\varrho\mathbf{v})dx=\alpha\beta^2\int_{R^3}\varrho\mathbf{v}\cdot\partial_x\varrho dx$$
$$\le C\|\varrho\|_{L^3}\|\mathbf{v}\|_{L^6}\|\partial_x\varrho\|_{L^2}\le C\|\varrho\|_{H^1}\|\partial_x\mathbf{v}\|_{L^2}\|\partial_x\varrho\|_{L^2}$$
$$\le C\delta\|\partial_x\mathbf{v},\partial_x\varrho\|_{L^2}^2, \tag{36}$$

$$I_2=-\alpha\beta^2\int_{R^3}\mathbf{v}\cdot(\mathbf{v}\cdot\nabla\mathbf{v})dx-\alpha\int_{R^3}\frac{\mathbf{v}\cdot(\vartheta\cdot\nabla\varrho)}{\rho}dx-\frac{\alpha}{\beta^2}\int_{R^3}\mathbf{v}\cdot\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\nabla\varrho dx+\mu\int_{R^3}\mathbf{v}\cdot\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\Delta\mathbf{v}dx$$
$$+(\mu+\lambda)\int_{R^3}\mathbf{v}\cdot\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\nabla\operatorname{div}\mathbf{v}dx+\alpha\int_{R^3}\frac{\mathbf{v}\cdot(\mathscr{H}\cdot\nabla\mathscr{H})}{\rho}dx+\frac{1}{\beta}\int_{R^3}\mathbf{v}\cdot\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\left(\overline{\mathbf{H}}\cdot\nabla\mathscr{H}\right)dx$$
$$-\alpha\int_{R^3}\frac{\mathbf{v}\cdot(\mathscr{H}\cdot\nabla\mathscr{H}^T)}{\rho}dx-\frac{1}{\beta}\int_{R^3}\mathbf{v}\cdot\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}-\right)\left(\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T\right)dx$$
$$\le C\|\mathbf{v}\|_{H^1}[\|\partial_x\mathbf{v}\|_{L^2}\|\partial_x\mathbf{v}\|_{L^2}+\|\partial_x\vartheta\|_{L^2}\|\partial_x\varrho\|_{L^2}+\|\partial_x\varrho\|_{L^2}\|\partial_x\varrho\|_{L^2}$$
$$+\|\partial_x\varrho\|_{L^2}\|\partial_x\mathbf{v}\|_{L^2}+\|\partial_x\varrho\|_{L^2}\|\partial_x\operatorname{div}\mathbf{v}\|_{L^2}+\|\partial_x\mathscr{H}\|_{L^2}\|\partial_x\mathscr{H}\|_{L^2}$$
$$+\|\partial_x\varrho\|_{L^2}\|\partial_x\mathscr{H}\|_{L^2}+\|\partial_x\mathscr{H}\|_{L^2}\|\partial_x\mathscr{H}^T\|_{L^2}+\|\partial_x\varrho\|_{L^2}\|\partial_x\mathscr{H}^T\|_{L^2}]\le C\delta\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})\|_{L^2}^2, \tag{37}$$

where we have used the fact that $\frac{1}{\overline{\rho}}-\frac{1}{\rho}\sim\varrho$. Similarly, we find that:

$$I_3=-\alpha\beta^2\int_{R^3}\vartheta(\mathbf{v}\cdot\nabla\vartheta)dx-\alpha\beta^2\int_{R^3}\vartheta(\vartheta\operatorname{div}\mathbf{v})dx$$
$$+\int_{R^3}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)(\vartheta\Delta\vartheta)dx+\int_{R^3}\frac{\vartheta}{\rho}[|\nabla\mathscr{H}|^2-\nabla\mathscr{H}:\nabla\mathscr{H}^{\mathrm{T}}$$
$$+\mu\beta^2|\nabla\mathbf{v}|^2+\mu\beta^2\nabla\mathbf{v}:\nabla\mathbf{v}^T$$
$$+\lambda\beta^2|\operatorname{div}\mathbf{v}|^2]dx\le C\delta\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x\vartheta,\partial_x^2\vartheta,\partial_x\mathscr{H})\|_{L^2}^2, \tag{38}$$

and

$$I_4=\alpha\beta^2\int_{R^3}\mathscr{H}\cdot(\mathscr{H}\cdot\nabla\mathbf{v})dx-\alpha\beta^2\int_{R^3}\mathscr{H}\cdot(\mathbf{v}\cdot\nabla\mathscr{H})dx$$
$$-\alpha\beta^2\int_{R^3}\mathscr{H}\cdot(\mathscr{H}\cdot\operatorname{div}\mathbf{v})dx\le C\delta\|(\partial_x\mathbf{v},\partial_x\mathscr{H})\|_{L^2}^2. \tag{39}$$

Accordingly, from Equation (29) and Equations (35)–(39), we obtain:

$$\frac{d}{dt}\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{L^2}^2+C\|(\partial_x\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})\|_{L^2}^2$$
$$\le C\delta\|(\partial_x\varrho,\partial_x^2\mathbf{v},\partial_x^2\vartheta)\|_{L^2}^2. \tag{40}$$

B: L^2 norms of $\partial_x^3\varrho,\partial_x^3\mathbf{v},\partial_x^3\vartheta,\partial_x^3\mathscr{H}$:

Apply the differential operator ∂_{ijk} to the Equations (20)–(23), multiply the resulting equations by $\partial_{ijk}\varrho,\partial_{ijk}\mathbf{v}$, $\partial_{ijk}\vartheta$ and $\partial_{ijk}\mathscr{H}$, respectively, and integrating them over R^3, we obtain the following equations:

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}\|\partial_{ijk}\varrho\|^2dx+\alpha\int_{R^3}\partial_{ijk}\varrho\operatorname{div}\left(\partial_{ijk}\mathbf{v}\right)dx=J_1, \tag{41}$$

where $J_1=\langle\partial_{ijk}N_1,\partial_{ijk}\varrho\rangle$.

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}\|\partial_{ijk}\mathbf{v}\|^2dx+\alpha\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\nabla\partial_{ijk}\vartheta dx$$
$$+\alpha\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\nabla\partial_{ijk}\varrho dx-\mu\beta^2\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\Delta\partial_{ijk}\mathbf{v}dx$$
$$-\beta^2(\mu+\lambda)\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\nabla\operatorname{div}\left(\partial_{ijk}\mathbf{v}\right)dx$$
$$-\beta\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\left(\partial_{ijk}\overline{\mathbf{H}}\cdot\nabla\partial_{ijk}\mathscr{H}\right)dx$$
$$+\beta\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\left(\partial_{ijk}\overline{\mathbf{H}}\cdot\nabla\partial_{ijk}\mathscr{H}^T\right)=J_2, \tag{42}$$

where $J_2 = \langle \partial_{ijk} N_2, \partial_{ijk}\mathbf{v}\rangle$.

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} \|\partial_{ijk}\vartheta\|^2 dx + \alpha \int_{R^3} \partial_{ijk}\vartheta \operatorname{div}\left(\partial_{ijk}\mathbf{v}\right) dx - \beta^2 \int_{R^3} \partial_{ijk}\vartheta \cdot \Delta \partial_{ijk}\vartheta dx = J_3, \tag{43}$$

where $J_3 = \langle \partial_{ijk} N_3, \partial_{ijk}\vartheta\rangle$.
and

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} \|\partial_{ijk}\mathscr{H}\|^2 dx + \beta \int_{R^3} \left(\partial_{ijk}\mathscr{H} . \partial_{ijk}\overline{\mathbf{H}}\right) \operatorname{div}\left(\partial_{ijk}\mathbf{v}\right) dx - \int_{R^3} \left(\partial_{ijk}\mathscr{H} . \partial_{ijk}\overline{\mathbf{H}}\right) . \nabla \partial_{ijk}\mathbf{v} dx - \int_{R^3} \partial_{ijk}\mathscr{H} . \Delta \partial_{ijk}\mathscr{H} dx = J_4, \tag{44}$$

where $J_4 = \langle \partial_{ijk} N_4, \partial_{ijk}\mathscr{H}\rangle$.
We add Equations (41)–(44), obtaining:

$$\frac{1}{2}\frac{d}{dt}\|\partial_{ijk}(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^2}^2 + \mu\beta^2 \|\partial_x^4 \mathbf{v}\|_{L^2}^2 + \beta^2 \|\partial_x^4 \vartheta\|_{L^2}^2 + \|\partial_x^4 \mathscr{H}\|_{L^2}^2 = \sum_{k=1}^{4} J_k. \tag{45}$$

Now, we use Equation (16) and Cauchy's inequality to estimate the terms in the summation:

$$\begin{aligned} J_1 &= -\alpha\beta^2 \int_{R^3} \partial_{ijk}\varrho \cdot \partial_{ijk}[\operatorname{div}(\varrho \mathbf{v})] dx \\ &= -\alpha\beta^2 \int_{R^3} \left[\partial_{ijk}(\varrho \operatorname{div}\mathbf{v}) \cdot \partial_{ijk}\varrho + \partial_{ijk}(\mathbf{v}\cdot\nabla\varrho)\cdot\partial_{ijk}\varrho\right] dx \\ &\le C\|\partial_x^3(\varrho\operatorname{div}\mathbf{v})\|_{L^2}\|\partial_x^3\varrho\|_{L^2} + C\|\partial_x^2(\partial_x\mathbf{v}\cdot\nabla\varrho)\|_{L^2}\|\partial_x^3\varrho\|_{L^2} \\ &\quad + C\int_{R^3}(\mathbf{v}\cdot\nabla\partial_x^3\varrho)\cdot\partial_x^3\varrho dx \le C\|\partial_x^3\varrho\|_{L^2}[\|\varrho\|_{L^\infty}\|\partial_x^4\mathbf{v}\|_{L^2} \\ &\quad + \|\partial_x^3\varrho\|_{L^2}\|\partial_x\mathbf{v}\|_{L^\infty}] + C\|\partial_x^3\varrho\|_{L^2} \times [\|\partial_x\mathbf{v}\|_{L^\infty}\|\partial_x^3\varrho\|_{L^2} \\ &\quad + \|\partial_x^3\mathbf{v}\|_{L^2}\|\partial_x\varrho\|_{L^\infty}] + C\int_{R^3}(\mathbf{v}\cdot\nabla\partial_x^3\varrho)\cdot\partial_x^3\varrho dx \\ &\le C\delta\|(\partial_x^3\varrho, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2, \end{aligned} \tag{46}$$

where $(\mathbf{v}\cdot\partial_x^3\varrho\nabla\varrho)\cdot\partial_x^3\varrho = \mathbf{v}\cdot\nabla\frac{|\partial_x^3\varrho|^2}{2}$.
We next estimate J_2 in detail:

(i) $-\alpha\beta^2\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}(\mathbf{v}\cdot\nabla\mathbf{v})dx \le C\|\partial_x^3\mathbf{v}\|_{L^2}\|\partial_x^3(\mathbf{v}\cdot\nabla\mathbf{v})\|_{L^2} \le C\|\partial_x^3\mathbf{v}\|_{L^2}[\|\mathbf{v}\|_{L^\infty}\|\partial_x^4\mathbf{v}\|_{L^2} + \|\partial_x^3\mathbf{v}\|_{L^2}\|\partial_x\mathbf{v}\|_{L^\infty}] \le C\delta\|(\partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2$,

(ii) $-\alpha\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left(\frac{\vartheta\cdot\nabla\varrho}{\rho}\right)dx = \alpha\int_{R^3}\partial_{ijkk}\mathbf{v}\cdot\partial_{ij}\left(\frac{\vartheta\cdot\nabla\varrho}{\rho}\right)dx \le C\|\partial_x^4\mathbf{v}\|_{L^2}\|\partial_x^2\left(\frac{\vartheta\cdot\nabla\varrho}{\rho}\right)\|_{L^2} \le C\|\partial_x^4\mathbf{v}\|_{L^2}\left[\|\frac{1}{\rho}\|_{L^\infty}\|\partial_x^2(\vartheta\cdot\nabla\varrho)\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho}\right)\|_{L^2}\|(\vartheta\cdot\nabla\varrho)\|_{L^\infty}\right] \le C\|\partial_x^4\mathbf{v}\|_{L^2}[\|\partial_x^2(\vartheta\cdot\nabla\varrho)\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho}\right)\|_{L^2}] \le C\|\partial_x^4\mathbf{v}\|_{L^2}[\|\vartheta\|_{L^\infty}\|\partial_x^3\varrho\|_{L^2} + \|\partial_x^2\vartheta\|_{L^2}\|\partial_x\varrho\|_{L^\infty} + \|\frac{2}{\rho^3}\partial_x\varrho\cdot\partial_x\varrho\|_{L^2} + \|\frac{1}{\rho^2}\partial_x^2\varrho\|_{L^2}] \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^4\mathbf{v}, \partial_x^2\vartheta)\|_{L^2}^2$,

(iii) $-\frac{\alpha}{\beta^2}\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\cdot\nabla\varrho\right]dx = \frac{\alpha}{\beta^2}\int_{R^3}\partial_{ijkk}\mathbf{v}\cdot\partial_{ij}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\cdot\nabla\varrho\right]dx \le C\|\partial_x^4\mathbf{v}\|_{L^2}\|\partial_x^2\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\cdot\nabla\varrho\right]\|_{L^2} \le C\|\partial_x^4\mathbf{v}\|_{L^2}\left[\|\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\|_{L^\infty}\|\partial_x^3\varrho\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\|_{L^2}\|\partial_x\varrho\|_{L^\infty}\right] \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^4\mathbf{v})\|_{L^2}^2$,

(iv) $\mu\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\cdot\Delta\mathbf{v}\right]dx = -\mu\int_{R^3}\partial_{ijkk}\mathbf{v}\cdot\partial_{ij}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\cdot\Delta\mathbf{v}\right]dx \le C\|\partial_x^4\mathbf{v}\|_{L^2}\|\partial_x^2\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\cdot\Delta\mathbf{v}\right]\|_{L^2} \le C\|\partial_x^4\mathbf{v}\|_{L^2}\left[\|\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\|_{L^\infty}\|\partial_x^4\mathbf{v}\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\|_{L^2}\|\partial_{ii}\mathbf{v}\|_{L^\infty}\right] \le C\delta\|\partial_x^4\mathbf{v}\|_{L^2}[\|\partial_x^4\mathbf{v}\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\|_{L^2}\|\partial_{ii}\mathbf{v}\|_{H^2}] \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2$,

(v) $(\mu+\lambda)\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)\nabla\operatorname{div}\mathbf{v}\right]dx \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2$,

(vi) $\alpha\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left(\frac{\mathscr{H}\cdot\nabla\mathscr{H}}{\rho}\right)dx \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathbf{v}, \partial_x^3\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2$,

(vii) $\frac{1}{\beta}\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)(\overline{\mathbf{H}}\cdot\nabla\mathscr{H})\right]dx \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathbf{v}, \partial_x^4\mathscr{H})\|_{L^2}^2$,

(viii) $-\alpha\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left(\frac{\mathscr{H}\cdot\nabla\mathscr{H}^T}{\rho}\right)dx \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathbf{v}, \partial_x^3\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2$,

(ix) $-\frac{1}{\beta}\int_{R^3}\partial_{ijk}\mathbf{v}\cdot\partial_{ijk}\left[\left(\frac{1}{\rho}-\frac{1}{\bar\rho}\right)(\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T)\right]dx \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathbf{v}, \partial_x^4\mathscr{H})\|_{L^2}^2$.

Combining the above, we conclude that:

$$J_2 \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v}, \partial_x^2\vartheta, \partial_x^3\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2, \tag{47}$$

Similarly, we find that:

$$J_3 \le C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v}, \partial_x^2\vartheta, \partial_x^3\vartheta, \partial_x^4\vartheta, \partial_x^3\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2, \tag{48}$$

and

$$J_4 \le C\delta\|(\partial_x^3\mathbf{v}, \partial_x^4\mathbf{v}, \partial_x^3\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2. \tag{49}$$

Accordingly, from Equations (45)–(49), we obtain:

$$\begin{aligned} &\frac{d}{dt}\|\partial_x^3(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^2}^2 + C\|(\partial_x^4\mathbf{v}, \partial_x^4\vartheta, \partial_x^4\mathscr{H})\|_{L^2}^2 \\ &\le C\delta\|(\partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^2\vartheta, \partial_x^3\vartheta, \partial_x^3\mathscr{H})\|_{L^2}^2. \end{aligned} \tag{50}$$

Hence, the lowest and highest estimate inequalities Equations (40) and (50), with the help of Equation (14), yield:

$$\begin{aligned}&\frac{d}{dt}\|(\varrho,\partial_x^3\varrho,\mathbf{v},\partial_x^3\mathbf{v},\vartheta,\partial_x^3\vartheta,\mathscr{H},,\partial_x^3\mathscr{H})\|_{L^2}^2\\&+C\|(\partial_x\mathbf{v},\partial_x^4\mathbf{v},\partial_x\vartheta,\partial_x^4\vartheta,\partial_x\mathscr{H},\partial_x^4\mathscr{H})\|_{L^2}^2\le C\delta\|(\partial_x\varrho,\partial_x^3\varrho)\|_{L^2}^2.\end{aligned}\tag{51}$$

C: L^2-norms of $\partial_x\varrho,\partial_x^3\varrho$:

First, we estimate $\partial_x\varrho$. To this purpose, we calculated as follows:

$$\begin{aligned}&\int_{R^3}(\beta|\nabla\varrho|^2+\alpha\nabla\varrho\cdot\mathbf{v})_t dx\\&=\int_{R^3}(2\beta\nabla\varrho\cdot\nabla\varrho_t+\alpha\nabla\varrho_t\cdot\mathbf{v}+\alpha\nabla\varrho\cdot\mathbf{v}_t)dx.\end{aligned}\tag{52}$$

From Equation (20), we estimate the first term on the right-hand side of Equation (52) as follows:

$$\begin{aligned}&\int_{R^3}2\beta\nabla\varrho\cdot\nabla\varrho_t dx=\int_{R^3}2\beta\nabla\varrho\cdot\nabla[N_1-\alpha\operatorname{div}\mathbf{v}]dx\\&=\int_{R^3}2\beta\nabla\varrho\cdot\nabla[-\alpha\beta^2\operatorname{div}(\varrho\mathbf{v})-\alpha\operatorname{div}\mathbf{v}]dx\\&=-\int_{R^3}2\beta\nabla\varrho\cdot\nabla[\alpha\beta^2(\mathbf{v}\cdot\nabla\varrho+\varrho\operatorname{div}\mathbf{v})\\&\quad-\alpha\operatorname{div}\mathbf{v}]dx\le C\delta\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v})\|_{L^2}^2\\&\quad-\int_{R^3}2\alpha\beta\nabla\varrho\cdot\nabla\operatorname{div}\mathbf{v}dx\le C\delta\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v})\|_{L^2}^2\\&\quad-2\alpha\beta\int_{R^3}\nabla\varrho\cdot\nabla\operatorname{div}\mathbf{v}dx,\end{aligned}\tag{53}$$

where we have used:

$$\nabla\varrho\cdot(\nabla^2\varrho\cdot\mathbf{v})=\partial_i\varrho\partial_{ij}\varrho\mathbf{v}^j=\nabla\frac{|\nabla\varrho|^2}{2}\cdot\mathbf{v}.\tag{54}$$

We estimate the second term as follows:

$$\begin{aligned}&\int_{R^3}\alpha\nabla\varrho_t\cdot\mathbf{v}dx=\int_{R^3}\alpha\nabla(N_1-\alpha\operatorname{div}\mathbf{v})\cdot\mathbf{v}dx\\&=-\int_{R^3}\alpha(N_1-\alpha\operatorname{div}\mathbf{v})\operatorname{div}\mathbf{v}dx\le C\delta\|\partial_x\varrho,\partial_x\mathbf{v}\|_{L^2}^2\\&\quad+\alpha^2\int_{R^3}(\operatorname{div}\mathbf{v})^2dx.\end{aligned}\tag{55}$$

From Equation (21), we can estimate the last term:

$$\begin{aligned}&\int_{R^3}\alpha\nabla\varrho\cdot\mathbf{v}_t dx=\int_{R^3}\alpha\nabla\varrho\cdot[N_2-\alpha\nabla\vartheta-\alpha\nabla\varrho\\&\quad+\mu\beta^2\Delta\mathbf{v}+\beta^2(\mu+\lambda)\nabla\operatorname{div}\mathbf{v}+\beta\overline{\mathbf{H}}\cdot\nabla\mathscr{H}\\&\quad-\beta\overline{\mathbf{H}}\cdot\nabla^T\mathscr{H}\big]dx\le C\delta\\&\quad\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})\|_{L^2}^2\\&\quad-\int_{R^3}\alpha^2[|\nabla\varrho|^2+\nabla\varrho\cdot\nabla\vartheta]dx\\&\quad+\int_{R^3}\alpha\beta^2[\mu\nabla\varrho\cdot\Delta\mathbf{v}+(\mu+\lambda)\nabla\varrho\\&\quad\cdot\nabla\operatorname{div}\mathbf{v}]dx+\int_{R^3}\alpha\beta\nabla\varrho\\&\quad\cdot\big[\overline{\mathbf{H}}\cdot\nabla\mathscr{H}-\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T\big]dx.\end{aligned}\tag{56}$$

Then, estimate the right-hand side of the above inequalities, the detail of which is the following:

(i) $-\int_{R^3}\alpha^2[|\nabla\varrho|^2+\nabla\varrho\cdot\nabla\vartheta]dx\le-\int_{R^3}\alpha^2|\nabla\varrho|^2dx+\frac{1}{2}\int_{R^3}\alpha^2|\nabla\varrho|^2dx+\frac{1}{2}\int_{R^3}\alpha^2|\nabla\vartheta|^2dx\le-\frac{\alpha^2}{2}\|\partial_x\varrho\|_{L^2}^2+\frac{\alpha^2}{2}\|\partial_x\vartheta\|_{L^2}^2$,

(ii) $\int_{R^3}\alpha\beta^2[\mu\nabla\varrho\cdot\Delta\mathbf{v}+(\mu+\lambda)\nabla\varrho\cdot\nabla\operatorname{div}\mathbf{v}]dx-2\alpha\beta\int_{R^3}\nabla\varrho\cdot\nabla\operatorname{div}\mathbf{v}dx=\int_{R^3}\alpha\beta\nabla\varrho\cdot(\Delta\mathbf{v}-\nabla\operatorname{div}\mathbf{v})dx\le\frac{\alpha^2}{4}\|\partial_x\varrho\|_{L^2}^2+4\beta^2\|\partial_x\mathbf{v}\|_{L^2}^2$,

(iii) $\int_{R^3}\alpha\beta\nabla\varrho\cdot\big[\overline{\mathbf{H}}\cdot\nabla\mathscr{H}-\overline{\mathbf{H}}\cdot\nabla\mathscr{H}^T\big]dx\le\frac{\alpha^2}{8}\|\partial_x\varrho\|_{L^2}^2+C\|\partial_x\mathscr{H}\|_{L^2}^2$

Hence, from Equations (52)–(56) and with help of Equation (14), we obtain:

$$\begin{aligned}&\int_{R^3}(\beta|\nabla\varrho|^2+\alpha\nabla\varrho\cdot\mathbf{v})_t dx+\frac{\alpha^2}{8}\|\partial_x\varrho\|_{L^2}^2\\&\le C\delta\|\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H}\|_{L^2}^2+\alpha^2\|\operatorname{div}\mathbf{v}\|_{L^2}^2+4\beta^2\|\partial_x^2\mathbf{v}\|_{L^2}^2\\&\quad+\frac{\alpha^2}{2}\|\partial_x\vartheta\|_{L^2}^2+C\|\partial_x\mathscr{H}\|_{L^2}^2.\end{aligned}\tag{57}$$

We now turn to estimate $\partial_x^3\varrho$.

As in Equation (52), by direct calculation, we have:

$$\begin{aligned}&\int_{R^3}\big[\beta|\partial_{ijk}\varrho|^2+\alpha\partial_{ijk}\varrho\cdot\partial_{ij}\mathbf{v}^k\big]_t dx\\&=\int_{R^3}2\beta\partial_{ijk}\varrho\cdot\partial_{ijk}\varrho_t+\alpha\partial_{ijk}\varrho_t\cdot\partial_{ij}\mathbf{v}^k+\alpha\partial_{ijk}\varrho\cdot\partial_{ij}\mathbf{v}_t^k dx.\end{aligned}\tag{58}$$

We estimate the first term on the right side of Equation (58) given as follows:

$$\int_{R^3} 2\beta\partial_{ijk}\varrho \cdot \partial_{ijk}\varrho_t dx = \int_{R^3} 2\beta\partial_{ijk}\varrho \cdot \partial_{ijk}[N_1 - \alpha \text{div}\, \mathbf{v}]dx$$
$$= \int_{R^3} 2\beta\partial_{ijk}\varrho \cdot \partial_{ijk}[-\alpha\beta^2 \text{div}\,(\varrho\mathbf{v}) - \alpha \text{div}\, \mathbf{v}]dx$$
$$= \int_{R^3} 2\beta\partial_{ijk}\varrho \cdot \partial_{ijk}[-\alpha\beta^2 \nabla\varrho \cdot \mathbf{v} - \alpha\beta^2\varrho \text{div}\, \mathbf{v} - \alpha \text{div}\, \mathbf{v}]dx$$
$$\leq C\delta\|(\partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2 - \int_{R^3} \alpha\beta^3\nabla\left|\partial_{ijk}\varrho\right|^2 \cdot \mathbf{v}$$
$$+ 2\alpha\beta\partial_{ijk}\varrho\partial_{ijk}\text{div}\, \mathbf{v}dx \leq C\delta\|(\partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2$$
$$- 2\alpha\beta\int_{R^3} \partial_{ijk}\varrho\partial_{ijk}\text{div}\, \mathbf{v}dx. \tag{59}$$

Similarly, we estimate the second term:

$$\int_{R^3} \alpha\partial_{ijk}\varrho_t \cdot \partial_{ij}\mathbf{v}^k dx = \int_{R^3} \alpha\partial_{ijk}[N_1 - \alpha \text{div}\, \mathbf{v}] \cdot \partial_{ij}\mathbf{v}^k dx$$
$$= -\int_{R^3} \alpha\partial_{ij}[N_1 -- \alpha \text{div}\, \mathbf{v}]\partial_{ij}\text{div}\, \mathbf{v}dx$$
$$\leq C\delta\|(\partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v})\|_{L^2}^2 + \alpha^2\int_{R^3} \left(\partial_{ij}\text{div}\, \mathbf{v}\right)^2 dx. \tag{60}$$

And we estimate the last term as follows:

First, we detail the estimate of the integral $\int_{R^3}\alpha\partial_{ijk}\varrho \cdot \partial_{ij}\left[N_2^k\right]dx$

(i) $-\int_{R^3}\alpha^2\beta^2\partial_{ijk}\varrho \cdot \partial_{ij}(\mathbf{v} \cdot \nabla\mathbf{v})dx \leq C \| \partial_x^3\varrho\|_{L^2} \| (\mathbf{v} \cdot \nabla\mathbf{v})\|_{L^2} \leq C \| \partial_x^3\varrho\|_{L^2}[\|\mathbf{v}\|_{L^\infty} \| \partial_x^3\mathbf{v}\|_{L^2} + \|\partial_x^2\mathbf{v}\|_{L^2} \| \partial_x\mathbf{v}\|_{L^\infty}] \leq C\delta \| \partial_x^3\varrho\|_{L^2}[\|\partial_x^3\mathbf{v}\|_{L^2} + \|\partial_x^2\mathbf{v}\|_{L^2}] \leq C\delta \| (\partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v})\|_{L^2}^2$,

(ii) $-\int_{R^3}\alpha^2\partial_{ijk}\varrho \cdot \partial_{ij}\left(\frac{\vartheta\cdot\nabla\varrho}{\varrho}\right)dx \leq C\|\partial_x^3\varrho\|_{L^2}\|\partial_x^2\left(\frac{\vartheta\cdot\nabla\varrho}{\varrho}\right)\|_{L^2} \leq C\|\partial_x^3\varrho\|_{L^2}\left[\|\frac{1}{\rho}\|_{L^\infty}\|\partial_x^2(\vartheta \cdot \nabla\varrho)\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho}\right)\|_{L^2}\| (\vartheta \cdot \nabla\varrho)\|_{L^\infty}\right] \leq C\delta\|\partial_x^3\varrho\|_{L^2}[\|\vartheta\|_{L^\infty}\|\partial_x^3\varrho\|_{L^2} + \|\partial_x^2\vartheta\|_{L^2}\|\partial_x\varrho\|_{L^\infty} + \|\frac{2}{\rho^3}\partial_x\varrho \cdot \partial_x\varrho\|_{L^2} + \|\frac{1}{\rho^2}\partial_x^2\varrho\|_{L^2}] \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\vartheta)\|_{L^2}^2$,

(iii) $-\int_{R^3}\frac{\alpha^2}{\beta^2}\partial_{ijk}\varrho \cdot \partial_{ij}\left[\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right) \cdot \nabla\varrho\right]dx \leq C\|\partial_x^3\varrho\|_{L^2}\|\partial_x^2\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right) \cdot \nabla\varrho \|_{L^2} \leq C\|\partial_x^3\varrho\|_{L^2}\left[\|\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\|_{L^\infty}\|\partial_x^3\varrho\|_{L^2} + \|\partial_x^2\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\|_{L^2}\|\partial_x\varrho\|_{L^\infty}\right] \leq C\delta\|\partial_x^3\varrho\|_{L^2}[\|\partial_x^3\varrho\|_{L^2} + \|\frac{2}{\rho^3}\partial_x\varrho \cdot \partial_x\varrho\|_{L^2} + \|\frac{1}{\rho^2}\partial_x^2\varrho\|_{L^2}] \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho)\|_{L^2}^2$,

Similarly, we find that

(iv) $\int_{R^3}\mu\alpha\partial_{ijk}\varrho \cdot \partial_{ij}\left[\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\Delta\mathbf{v}\right]dx \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2$,

(v) $\int_{R^3}\alpha(\mu + \lambda)\partial_{ijk}\varrho \cdot \partial_{ij}\left[\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\nabla\text{div}\, \mathbf{v}\right]dx \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v})\|_{L^2}^2$,

(vi) $\int_{R^3}\alpha^2\partial_{ijk}\varrho \cdot \partial_{ij}\left[\frac{1}{\rho}(\mathscr{H} \cdot \nabla\mathscr{H})\right]dx \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathscr{H}, \partial_x^3\mathscr{H})\|_{L^2}^2$,

(vii) $\int_{R^3}\frac{\alpha}{\beta}\partial_{ijk}\varrho \cdot \partial_{ij}\left[\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\left(\overline{\mathbf{H}} \cdot \nabla\mathscr{H}\right)\right]dx \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathscr{H})\|_{L^2}^2$,

(viii) $-\int_{R^3}\alpha^2\partial_{ijk}\varrho \cdot \partial_{ij}\left[\frac{1}{\rho}(\mathscr{H} \cdot \nabla\mathscr{H}^T)\right]dx \leq C\delta \| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathscr{H}, \partial_x^3\mathscr{H})\|_{L^2}^2$,

(ix) $-\int_{R^3}\frac{\alpha}{\beta}\partial_{ijk}\varrho \cdot \partial_{ij}\left[\left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)\left(\overline{\mathbf{H}} \cdot \nabla\mathscr{H}^T\right)\right]dx \leq C\delta\| (\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^3\mathscr{H})\|_{L^2}^2$.

Hence, from (i) to (ix), we derive the following estimate:

$$\int_{R^3}\alpha\partial_{ijk}\varrho \cdot \partial_{ij}\mathbf{v}_t^k dx = \int_{R^3}\alpha\partial_{ijk}\varrho \cdot \partial_{ij}\left[N_2^k - \alpha\partial_k\vartheta - \alpha\partial_k\varrho\right.$$
$$+ \mu\beta^2\Delta\mathbf{v}^k + \beta^2(\mu + \lambda)\partial_k\text{div}\, \mathbf{v}^k$$
$$\left.+ \beta\overline{\mathbf{H}} \cdot \partial_k\mathscr{H} - \beta\overline{\mathbf{H}} \cdot \partial_k\mathscr{H}^T\right]dx$$
$$\leq C\delta\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v},$$
$$\partial_x^2\vartheta, \partial_x^2\mathscr{H}, \partial_x^3\mathscr{H})\|_{L^2}^2 - \int_{R^3}\alpha^2\left[\left|\partial_{ijk}\varrho\right|^2\right.$$
$$\left.+ \partial_{ijk}\varrho \cdot \partial_{ijk}\vartheta\right]dx + \int_{R^3}\alpha\beta\partial_{ijk}\varrho$$
$$\cdot\left[\partial_{ij}\Delta\mathbf{v}^k + \partial_{ijk}\text{div}\, \mathbf{v}^k\right]dx$$
$$+ \int_{R^3}\alpha\beta\partial_{ijk}\varrho \cdot \left[\overline{\mathbf{H}} \cdot \partial_k\mathscr{H} - \overline{\mathbf{H}} \cdot \partial_k\mathscr{H}^T\right]dx, \tag{61}$$

where

$$-\int_{R^3}\alpha^2\left[\left|\partial_{ijk}\varrho\right|^2 + \partial_{ijk}\varrho \cdot \partial_{ijk}\vartheta\right]dx \leq -\tfrac{\alpha^2}{2}\|\partial_x^3\varrho\|_{L^2}^2 + \tfrac{\alpha^2}{2}\|\partial_x^3\vartheta\|_{L^2}^2, \tag{62}$$

$$\int_{R^3}\alpha\beta\partial_{ijk}\varrho \cdot \left[\partial_{ij}\Delta\mathbf{v}^k - \partial_{ijk}\text{div}\, \mathbf{v}^k\right]dx \leq \frac{\alpha^2}{4}\|\partial_x^3\varrho\|_{L^2}^2 + 4\beta^2\|\partial_x^4\mathbf{v}\|_{L^2}^2, \tag{63}$$

and

$$\int_{R^3}\alpha\beta\partial_{ijk}\varrho \cdot \left[\overline{\mathbf{H}} \cdot \partial_k\mathscr{H} - \overline{\mathbf{H}} \cdot \partial_k\mathscr{H}^T\right]dx \leq \frac{\alpha^2}{8}\|\partial_x^3\varrho\|_{L^2}^2 + C\|\partial_x^3\mathscr{H}\|_{L^2}^2. \tag{64}$$

This together with Equations (58)–(61) and by inequality Equation (14), we obtain:

$$\int_{R^3}\left[\beta\left|\partial_{ijk}\varrho\right|^2 + \alpha\partial_{ijk}\varrho\partial_{ij}\mathbf{v}^k\right]_t dx + \frac{\alpha^2}{8}\|\partial_x^3\varrho\|_{L^2}^2$$
$$\leq C\delta\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x^4\mathbf{v}, \partial_x\vartheta, \partial_x^4\vartheta, \partial_x\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2$$
$$+ \alpha^2\left\|\partial_{ij}\text{div}\, \mathbf{v}\right\|_{L^2}^2 + 4\beta^2\|\partial_x^4\mathbf{v}\|_{L^2}^2 + \frac{\alpha^2}{2}\|\partial_x^3\vartheta\|_{L^2}^2 + C\|\partial_x^3\mathscr{H}\|_{L^2}^2. \tag{65}$$

D: Conclusion consequently, as in [4], adding Equations (57) and (65), and using Equation (14), we have:

$$\int_{R^3}(\beta|\nabla\varrho|^2+\alpha\nabla\varrho\cdot\mathbf{v})_t dx+\frac{\alpha^2}{8}\|\partial_x\varrho\|_{L^2}^2 \\ +\int_{R^3}\left[\beta\left|\partial_{ijk}\varrho\right|^2+\alpha\partial_{ijk}\varrho\partial_{ij}\mathbf{v}^k\right]_t dx+\frac{\alpha^2}{8}\|\partial_x^3\varrho\|_{L^2}^2 \\ \leq C\delta\|(\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})\|_{L^2}^2+\alpha^2\|\mathrm{div}\,\mathbf{v}\|_{L^2}^2 \\ +4\beta^2\|\partial_x^2\mathbf{v}\|_{L^2}^2+\frac{\alpha^2}{2}\|\partial_x\vartheta\|_{L^2}^2+C\|\partial_x\mathscr{H}\|_{L^2}^2 \\ +C\delta\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^4\mathbf{v},\partial_x\vartheta,\partial_x^4\vartheta,\partial_x\mathscr{H},\partial_x^4\mathscr{H})\|_{L^2}^2 \\ +\alpha^2\left\|\partial_{ij}\mathrm{div}\,\mathbf{v}\right\|_{L^2}^2+4\beta^2\|\partial_x^4\mathbf{v}\|_{L^2}^2+\frac{\alpha^2}{2}\|\partial_x^3\vartheta\|_{L^2}^2 \\ +C\|\partial_x^3\mathscr{H}\|_{L^2}^2\leq C\delta\|\partial_x\varrho\|_{L^2}^2 \\ +C\|(\partial_x\mathbf{v},\partial_x^4\mathbf{v},\partial_x\vartheta,\partial_x^4\vartheta,\partial_x\mathscr{H},\partial_x^4\mathscr{H})\|_{L^2}^2. \tag{66}$$

Multiplying the above estimate by $\zeta\in(0,1/2)$, and adding the resulting inequality and Equation (51), we obtain:

$$\zeta\left[\begin{array}{c}\int_{R^3}(\beta|\nabla\varrho|^2+\alpha\nabla\varrho\cdot\mathbf{v})_t dx+\frac{\alpha^2}{8}\|\partial_x\varrho\|_{L^2}^2+\int_{R^3}(\beta|\partial_{ijk}\varrho|^2 \\ +\alpha\partial_{ijk}\partial_{ij}\varrho\mathbf{v}^k)_t dx+\frac{\alpha^2}{8}\|\partial_x^3\varrho\|_{L^2}^2\end{array}\right] \\ +\frac{d}{dt}\|\varrho,\partial_x^3\varrho,\mathbf{v},\partial_x^3\mathbf{v},\vartheta,\partial_x^3\vartheta,\mathscr{H},\partial_x^3\mathscr{H}\|_{L^2}^2 \\ +\frac{C}{2}\|(\partial_x\mathbf{v},\partial_x^4\mathbf{v},\partial_x\vartheta,\partial_x^4\vartheta,\partial_x\mathscr{H},\partial_x^4\mathscr{H})\|_{L^2}^2\leq C\delta\zeta\|\partial_x\varrho\|_{L^2}^2 \\ +C\delta\|(\partial_x\varrho,\partial_x^3\varrho)\|_{L^2}^2. \tag{67}$$

Now, letting $\delta\in(0,\zeta\alpha^2/(16C))$, we further have:

$$\frac{d}{dt}\{\|(\varrho,\partial_x^3\varrho,\mathbf{v},\partial_x^3\mathbf{v},\vartheta,\partial_x^3\vartheta,\mathscr{H},\partial_x^3\mathscr{H})\|_{L^2}^2 \\ +\int_{R^3}\zeta\left[\beta|\nabla\varrho|^2+\alpha\nabla\varrho\cdot\mathbf{v}+\beta\left|\partial_{ijk}\varrho\right|^2 \\ +\alpha\partial_{ijk}\varrho\partial_{ij}\mathbf{v}^k\right]dx\}+C_1(\zeta) \\ \|(\partial_x\varrho,\partial_x^3\varrho,\partial_x\mathbf{v},\partial_x^4\mathbf{v},\partial_x\vartheta,\partial_x^4\vartheta,\partial_x\mathscr{H},\partial_x^4\mathscr{H})\|_{L^2}^2\leq 0, \tag{68}$$

where $C_1(\zeta)\coloneqq\min\{C/2,\zeta\alpha^2/16\}$. By Cauchy's inequality, we have:

$$\int_{R^3}\zeta\left[\beta|\nabla\varrho|^2+\alpha\nabla\varrho\cdot\mathbf{v}+\beta\left|\partial_{ijk}\varrho\right|^2 \\ +\alpha\partial_{ijk}\varrho\partial_{ij}\mathbf{v}^k\right]dx\geq\frac{\zeta\beta}{2}\left(\|\nabla\varrho\|_{L^2}^2+\left\|\partial_{ijk}\varrho\right\|_{L^2}^2\right) \\ -\frac{\zeta\alpha^2}{2\beta}\left(\|\mathbf{v}\|_{L^2}^2+\left\|\partial_{ij}\mathbf{v}^k\right\|_{L^2}^2\right). \tag{69}$$

Noting that α and β are finite, integrating Equation (68) directly with respect to time, and using Equations (14), (19), (29), and (30), we can obtain the a priori estimates of the Equations (20)–(24) by choosing sufficiently small ζ. We can finish the proof of Theorem 4.

2.2. Global Existence Proof. In this subsection, the existence part has been proofed from Theorem 2. There is no need to prove the local existence because it is already proved in [6] and found in [7, 8]:

Theorem 5. *Under the assumptions of Theorem 2, there exists a positive constant T such that the initial value problem Equations (5)–(6) has a unique solution $(\rho,\mathbf{u},\theta,\mathbf{H})$, which continuous in $[0,T]\times R^3$ together with its derivatives of first order in t and of second order in x, and there exists a constant $C_2>1$ such that the following inequality is satisfied:*

$$\left\|\left(\rho-\bar{\rho},\mathbf{v},\theta-\bar{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3}^2 \\ +\int_0^t\|\partial_x\rho(\cdot,s)\|_{H^2}^2+\|(\partial_x\mathbf{v},\partial_x\theta,\partial_x\mathbf{H})(\cdot,s)\|_{H^3}^2 ds \\ \leq C_2\left\|\left(\rho_0-\bar{\rho},\mathbf{v}_0,\mathbf{H}_0-\overline{\mathbf{H}}\right)\right\|_{H^3}^2, \tag{70}$$

for any $t\in[0,T]$.

The global existence of smooth solutions is confirmed by a continued argument that combines the local existence theorem and the theorem of a priori estimates.

Postulate

$$E_0=\left\|\left(\rho_0-\bar{\rho},\mathbf{v}_0,\theta_0-\bar{\theta},\mathbf{H}_0-\overline{\mathbf{H}}\right)\right\|_{H^3}<\delta/\sqrt{C_1C_2}, \tag{71}$$

where δ is defined in Theorem 4. Because the initial data satisfy $E_0<\delta/\sqrt{C_2}$, then by Theorem 5, there exists a positive constant $T_1>0$, such that the smooth solution of Equations (5) and (6) on $[0,T_1]$ exists and has the following estimate:

$$\left\|\left(\rho-\bar{\rho},\mathbf{v},\theta-\bar{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3}^2+\int_0^t\|\partial_x\rho(\cdot,s)\|_{H^2}^2 \\ +\|(\partial_x\mathbf{v},\partial_x\theta,\partial_x\mathbf{H})(\cdot,s)\|_{H^3}^2 ds\leq C_2E_0^2, \tag{72}$$

for $0\leq t\leq T_1$. It implies:

$$E_1=\sup_{0\leq t\leq T_1}\left\|\left(\rho-\bar{\rho},\mathbf{v},\theta-\bar{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3}\leq\sqrt{C_2}E_0<\delta. \tag{73}$$

Thus, the solution satisfies the a priori estimate Equation (17), by Theorem 4 and Equation (71), we have:

$$E_1\leq\sqrt{C_1}E_0<\frac{\delta}{\sqrt{C_2}}. \tag{74}$$

Thus, by Theorem 5, the initial problem (5) for $t\geq T_1$, with initial data $(\rho,\mathbf{v},\theta,\mathbf{H})(x,T_1)$ again has a unique local solution$(\rho,\mathbf{v},\theta,\mathbf{H})$ that satisfies:

$$\begin{aligned}&\left\|\left(\rho-\overline{\rho},\mathbf{v},\theta-\overline{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3}^2\\&+\int_0^t\|\partial_x\rho(\cdot,s)\|_{H^2}^2+\|(\partial_x\mathbf{v},\partial_x\theta,\partial_x\mathbf{H})(\cdot,s)\|_{H^3}^2ds\\&\leq C_2\left\|\left(\rho-\overline{\rho},\mathbf{v},\theta-\overline{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,T_1)\right\|_{H^3}^2,\end{aligned}\tag{75}$$

for $T_1<t<2T_1$. This together with Equations (71) and (74) yields:

$$\begin{aligned}&\sup_{T_1\leq t\leq T_2}\left\|\left(\rho-\overline{\rho},\mathbf{v},\theta-\overline{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3}\\&\leq\sqrt{C_1}E_1\leq\sqrt{C_1C_2}E_0<\delta.\end{aligned}\tag{76}$$

Then by Equations (73) and (76) and Theorem 4, we have:

$$\begin{aligned}E_2=&\sup_{0\leq t\leq 2T_1}\left\|\left(\rho-\overline{\rho},\mathbf{v},\theta-\overline{\theta},\mathbf{H}-\overline{\mathbf{H}}\right)(\cdot,t)\right\|_{H^3(R^3)}^2\\&\leq\sqrt{C_1}E_0\leq\delta/\sqrt{C_2}.\end{aligned}\tag{77}$$

We are able to perform similar procedure with $0\leq t\leq nT_1, n=3,4,5,\cdots$ and eventually obtain the global solution and the estimate Equation (8). It is easy to prove the uniqueness of the solutions, although the proof is omitted here.

3. Decay Rate of the Solution

In this section, we will prove the rate of convergence of the solution to complete the proof of Theorem 3. In subsection 1, we give some elementary conclusions about the estimates of the decay-in-time for the linearized system Equations (20)–(24) and a useful inequality. In subsection 2, we first obtain the energy inequality for the derivatives of order one through two, and then point out a decay-in-time estimate for the first-order derivatives, where the error is on the higher-order derivatives. Finally, we determine the optimal decay rates by bringing together these two estimates.

3.1. Some Elementary Decay-in-Time Estimates. We consider the rate of convergence of the solution $(\varrho,\mathbf{v},\vartheta,\mathscr{H})$ for the linearization problem (20)–(24). For later use, the result on the global existence of solutions to Equations (20)–(24) is reformulated as follows:

Proposition 6. *Under the assumption of* $\operatorname{div}\mathbf{H}_0=0$ *and (7), there exists a unique globally smooth solution* $(\varrho,\mathbf{v},\vartheta,\mathscr{H})$ *of the Cauchy problem (20)–(24) satisfying for any* $t\in[0,\infty)$,

$$\begin{aligned}&\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})(\cdot,t)\|_{H^3}^2+\int_0^t\|\partial_x\varrho(\cdot,s)\|_{H^2}^2\\&+\|(\partial_x\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})(\cdot,s)\|_{H^3}^2ds\leq C\|(\varrho_0,\mathbf{v}_0,\vartheta_0,\mathscr{H}_0)\|_{H^3}^2.\end{aligned}\tag{78}$$

Moreover, $(\rho,\mathbf{v},\vartheta,\mathscr{H})$ which satisfies Equation (19) uniquely solves the initial problem Equations (5)–(6) for all time. To utilize the L^p-L^q estimates to the linear problem for the nonlinear problem (20)–(23), we rewrite the solution of Equations (20)–(23) as:

$$U(t)=E(t)U_0+\int_0^tE(t-s)F(U(s))ds,\tag{79}$$

where

$$\begin{aligned}&U=[\varrho,\mathbf{v},\vartheta,\mathscr{H}]^T,U_0=[\varrho_0,\mathbf{v}_0,\vartheta_0,\mathscr{H}_0]^T,\\&F=[N_1,N_2,N_3,N_4]^T,\end{aligned}\tag{80}$$

and $E(t)$ is the solution of the semigroup defined by $E(t)=e^{-tA}, t\geq 0,$ where A is a matrix-valued differential operator given by

$$A=\begin{bmatrix}0&\alpha\nabla^T&0&0\\\alpha\nabla&-\mu\beta^2I\Delta-\beta^2(\mu+\lambda)\nabla\nabla^T&\alpha\nabla&-\beta I\overline{\mathbf{H}}\cdot\nabla+\beta\nabla\overline{\mathbf{H}}^T\\0&\alpha\nabla^T&-\beta^2\Delta&0\\0&-\beta I\overline{\mathbf{H}}\cdot\nabla+\beta\overline{\mathbf{H}}\cdot\nabla^T&0&-I\Delta\end{bmatrix}.$$

The semigroup $E(t)$ has the following properties on the decay in time [9, 10].

Lemma 7. *Let* $k\geq 0$ *be an integer and* $1\leq p\leq 2\leq q<\infty$. *Then, for any* $t\geq 0$, *it holds that:*

$$\left\|\nabla^kE(t)U_0\right\|_{L^q}\leq C(1+t)^{-\sigma(p,q;k)}\|U_0\|_{L^p\cap H^k},\tag{81}$$

where $\sigma(p,q;k)$ *is defined by Equation (12) and* $\|\cdot\|_{L^p\cap H^k}=\|\cdot\|_{L^p}+\|\cdot\|_{H^k}$.

Lemma 8. *If* $k\geq 0$ *is an integer and* $1\leq p\leq 2$. *Then, for any* $t\geq 0$,

$$\left\|\nabla^kE(t)U_0\right\|_{L^2}\leq C(1+t)^{-\sigma(p,2;k)}\left(\|U_0\|_{L^p}+\left\|\partial_x^kU_0\right\|_{L^2}\right),\tag{82}$$

holds where $\sigma(p,q;k)$ *is defined by Equation (12) and* $\|\cdot\|_{L^p\cap H^k}=\|\cdot\|_{L^p}+\|\cdot\|_{H^k}$.

We end this subsection by listing an elementary but useful inequality [12]:

Lemma 9. *If* $r_1>1$ *and* $r_2\in[0,r_1]$, *then we have* $\int_0^t(1+t-\tau)^{-r_1}(1+\tau)^{-r_2}d\tau\leq C(r_1,r_2)(1+t)^{-r_2}$.

3.2. Convergence Rates. First, we will estimate the decay rates of the first-order derivatives.

Lemma 10. *Under the assumption of Proposition 6, let $(\varrho, \mathbf{v}, \vartheta, \mathscr{H})$ be the solution to the initial problem (20)–(24). Then, we have:*

$$\|\partial_x(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^2} \le CK_0(1+t)^{-\sigma(p,2;1)} + C\delta_0\int_0^t (1+t-s)^{-\sigma(p,2;1)}\|\partial_x(\varrho, \mathbf{v}, \vartheta, \mathscr{H})(\cdot, s)\|_{H^2}ds, \tag{83}$$

when $K_0 = \|(\varrho_0, \mathbf{v_0}, \vartheta_0, \mathscr{H}_0)\|_{L^p\cap H^3}$ is finite from Equations (7) and (9). Here, $1 \le p < \frac{6}{5}$ and σ is defined by Equation (13).

Proof. From the integral formula (79) and Lemma 7, we have:

$$\|\partial_x(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^2} \le CK_0(1+t)^{-\sigma(p,2;1)} + C\int_0^t (1+t-s)^{-\sigma(p,2;1)}\|(N_1, N_2, N_3, N_4)\|_{L^p\cap H^1}(\cdot, s)ds, \tag{84}$$

where $1 \le p < \frac{6}{5}$. To derive (83), we need to control $\|(N_1, N_2, N_3, N_4)\|_{L^p\cap H^1}$ by the L^2 norm of derivatives of $\varrho, \mathbf{v}, \vartheta, \mathscr{H}$. From Proposition 6:

$$\|N_1\|_{L^p} = \|\alpha\beta^2(\nabla\varrho.\mathbf{v} + \varrho.\mathrm{div}\,\mathbf{v})\|_{L^2}, \tag{85}$$

$$\le C\|(\varrho, \mathbf{v})\|_{L^{\frac{2p}{2-p}}}\|(\partial_x\varrho, \partial_x\mathbf{v})\|_{L^2}, \tag{86}$$

$$\le \|(\varrho, \mathbf{v})\|_{H^1}\|(\partial_x\varrho, \partial_x\mathbf{v})\|_{L^2}, \tag{87}$$

$$\le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v})\|_{L^2}, \tag{88}$$

$$\|N_1\|_{L^2} \le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v})\|_{L^2}, \tag{89}$$

and

$$\|\partial_x N_1\|_{L^2} \le C\delta_0\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x\mathbf{v}, \partial_x^2\mathbf{v})\|_{L^2}. \tag{90}$$

Thus, we achieve:

$$\|N_1\|_{L^p\cap H^1} \le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v})\|_{H^1}, \tag{91}$$

$$\|N_2\|_{L^p} \le C\|(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^{\frac{2p}{2-p}}}\left\|\left(\partial_x\varrho, \partial_x\mathbf{v}, \partial_{ii}\mathbf{v}, \partial_{ij}\mathbf{v}, \partial_x\mathscr{H}\right)\right\|_{L^2} \le C\|(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^{\frac{2p}{2-p}}}\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\mathscr{H})\|_{L^2}, \tag{92}$$

$$\le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\mathscr{H})\|_{L^2}, \tag{93}$$

$$\|N_2\|_{L^2} \le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\mathscr{H})\|_{L^2}, \tag{94}$$

and

$$\|\partial_x N_2\|_{L^2} \le C\|(\varrho, \partial_x\varrho, \mathbf{v}, \partial_x\mathbf{v}, \vartheta, \partial_x\vartheta, \mathscr{H}, \partial_x\mathscr{H})\|_{L^\infty} \|(\partial_x\varrho, \partial_x^2\varrho, \partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x^4\mathbf{v}, \partial_x\mathscr{H}, \partial_x^2\mathscr{H},)\|_{L^2} \le C\delta_0\|(\partial_x\varrho, \partial_x^2\varrho, \partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x\mathscr{H}, \partial_x^2\mathscr{H},)\|_{L^2}. \tag{95}$$

Hence

$$\|N_2\|_{L^p\cap H^1} \le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\mathscr{H})\|_{H^2}, \tag{96}$$

$$\|N_3\|_{L^p} \le C\|(\varrho, \mathbf{v}, \partial_x\mathbf{v}, \vartheta, \partial_x\mathscr{H})\|_{L^{\frac{2p}{2-p}}} \|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\vartheta, \partial_x^2\vartheta, \partial_x\mathscr{H})\|_{L^2} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\vartheta, \partial_x^2\vartheta, \partial_x\mathscr{H})\|_{L^2}, \tag{97}$$

$$\|N_3\|_{L^2} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\vartheta, \partial_x^2\vartheta, \partial_x\mathscr{H})\|_{L^2}, \tag{98}$$

and

$$\|\partial_x N_3\|_{L^2} \le C\|(\varrho, \partial_x\varrho, \mathbf{v}, \partial_x\mathbf{v}, \vartheta, \partial_x\vartheta, \partial_x\mathscr{H})\|_{L^\infty} \|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\vartheta, \partial_x^2\vartheta, \partial_x^3\vartheta, \partial_x\mathscr{H})\|_{L^2} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\vartheta, \partial_x^2\vartheta, \partial_x^3\vartheta, \partial_x\mathscr{H})\|_{L^2}. \tag{99}$$

We further obtain:

$$\|N_3\|_{L^p\cap H^1} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{H^2}, \tag{100}$$

$$\|N_4\|_{L^p} \le C\|(\mathbf{v}, \mathscr{H})\|_{L^{\frac{2p}{2-p}}}\|(\partial_x\mathbf{v}, \partial_x\mathscr{H})\|_{L^2} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x\mathscr{H})\|_{L^2}, \tag{101}$$

$$\|N_4\|_{L^2} \le C\|(\mathbf{v}, \mathscr{H})\|_{L^\infty}\|(\partial_x\mathbf{v}, \partial_x\mathscr{H})\|_{L^2} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x\mathscr{H})\|_{L^2}, \tag{102}$$

and

$$\|\partial_x N_4\|_{L^2} \le C\|(\mathbf{v}, \partial_x\mathbf{v}, \mathscr{H}, \partial_x\mathscr{H})\|_{L^\infty}\|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\mathscr{H}, \partial_x^2\mathscr{H})\|_{L^2} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x^2\mathbf{v}, \partial_x\mathscr{H}, \partial_x^2\mathscr{H})\|_{L^2}. \tag{103}$$

Finally, we obtain:

$$\|N_4\|_{L^p\cap H^1} \le C\delta_0\|(\partial_x\mathbf{v}, \partial_x\mathscr{H})\|_{H^2}. \tag{104}$$

Hence, from Equations (91)–(104), we find:

$$\|(N_1, N_2, N_3, N_4)\|_{L^p\cap H^1} \le C\delta_0\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{H^2}. \tag{105}$$

Then, we can derive Equation (83) from Equations (84) and (105). □

We now show the energy equation is an equality as follows:

Lemma 11. *Under the assumption of Proposition 6, let* $(\varrho, \mathbf{v}, \vartheta, \mathscr{H})$ *be the solution to the initial problem (20)–(24), and* $(\varrho, \mathbf{u}, \theta, \mathbf{H})$ *satisfies Equation (19), then there are two constants* $C>0, D>0$ *such that if* $\delta_0>0$ *in Equation (7) is small enough, the following holds:*

$$\frac{dM(t)}{dt} + DM(t) \le C\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{L^2}^2, \tag{106}$$

where $M(t)$ *defined by Equation (115) is equivalent to* $\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{H^2}^2$*; that is, there exists a constant* $C_3>0$ *such that:*

$$\begin{aligned}\frac{1}{C_3}\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{H^2}^2 &\le M(t)\\ &\le C_3\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{H^2}^2.\end{aligned} \tag{107}$$

Proof. Considering ∂_x to Equations (20)–(23), multiplying by $\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta$, and $\partial_x\mathscr{H}$, respectively, and integrating them over R^3, and adding the results, we have:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{L^2}^2 + \mu\beta^2\|\partial_x^2\mathbf{v}\|_{L^2}^2\\ &\quad + \beta^2\|\partial_x^2\vartheta\|_{L^2}^2 + \|\partial_x^2\mathscr{H}\|_{L^2}^2 = \langle\partial_x N_1, \partial_x\varrho\rangle\\ &\quad + \langle\partial_x N_2, \partial_x\mathbf{v}\rangle + \langle\partial_x N_3, \partial_x\vartheta\rangle + \langle\partial_x N_4, \partial_x\mathscr{H}\rangle,\end{aligned} \tag{108}$$

We then estimate the right-hand side of Equation (108), and the details are as follows:

$$\langle\partial_x N_1, \partial_x\varrho\rangle \le \|N_1\|_{L^2}\|\partial_x^2\varrho\|_{L^2} \le C\delta_0\|\partial_x\varrho, \partial_x\mathbf{v}\|_{H^1}^2, \tag{109}$$

$$\langle\partial_x N_2, \partial_x\mathbf{v}\rangle \le \|N_2\|_{L^2}\|\partial_x^2\mathbf{v}\|_{L^2} \le C\delta_0\|\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\mathscr{H}\|_{H^1}^2, \tag{110}$$

$$\langle\partial_x N_3, \partial_x\vartheta\rangle \le \|N_3\|_{L^2}\|\partial_x^2\vartheta\|_{L^2} \le C\delta_0\|\partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H}\|_{H^1}^2, \tag{111}$$

and

$$\langle\partial_x N_4, \partial_x\mathscr{H}\rangle \le \|N_4\|_{L^2}\|\partial_x^2\mathscr{H}\|_{L^2} \le C\delta_0\|\partial_x\mathbf{v}, \partial_x\mathscr{H}\|_{H^1}^2, \tag{112}$$

Hence

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})\|_{L^2}^2 + \mu\beta^2\|\partial_x^2\mathbf{v}\|_{L^2}^2\\ &+ \beta^2\|\partial_x^2\vartheta\|_{L^2}^2 + \|\partial_x^2\mathscr{H}\|_{L^2}^2 \le C\delta_0\|\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H}\|_{H^1}^2.\end{aligned} \tag{113}$$

□

From this together with Equations (52) and (68), we can derive the following inequality with the aid of Proposition 6 and inequality (14):

$$\begin{aligned}&\frac{d}{dt}\Big\{\int_{R^3}|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x\vartheta, \partial_x^3\vartheta, \partial_x\mathscr{H}, \partial_x^3\mathscr{H})|^2\\ &\quad + \epsilon\big(\beta|\partial_{ijk}\varrho|^2 + \alpha\partial_{ijk}\varrho\partial_{ij}\mathbf{v}^k\big)dx\Big\}\\ &\quad + C\|(\partial_x^3\varrho, \partial_x^2\mathbf{v}, \partial_x^4\mathbf{v}, \partial_x^2\vartheta, \partial_x^4\vartheta, \partial_x^2\mathscr{H}, \partial_x^4\mathscr{H})\|_{L^2}^2\\ &\le C(\epsilon)\delta_0\|\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H}\|_{L^2}^2,\end{aligned} \tag{114}$$

where ϵ is sufficiently small. We define the temporal energy functional as:

$$\begin{aligned}M(t) = &\int_{R^3}|((\partial_x\varrho, \partial_x\mathbf{v}, \partial_x^3\mathbf{v}, \partial_x\vartheta, \partial_x^3\vartheta, \partial_x\mathscr{H}, \partial_x^3\mathscr{H}))|^2\\ &\quad + \epsilon\big(\beta|\partial_{ijk}\varrho|^2 + \alpha\partial_{ijk}\varrho\partial_{ij}\mathbf{v}^k\big)dx,\end{aligned} \tag{115}$$

where we note that $M(t)$ is equivalent to $\|\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H}\|_{H^2}^2$. By selecting a sufficiently large constant $D_1>0$, and adding $D_1\|\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H}\|_{H^2}^2$ to both sides of Equation (114), we derive Equation (106). Now, we are in a position to prove Equations (9) and (10). Next, we shall state the following estimates for $(\varrho, \mathbf{v}, \vartheta, \mathscr{H})$.

Proposition 12. *Under the assumption of Proposition 6, let* $(\varrho, \mathbf{v}, \vartheta, \mathscr{H})$ *be the solution to the initial problem (79) and* $(\varrho, \mathbf{v}, \vartheta, \mathscr{H})$ *satisfies (78). Then, for* $p \in [1, \frac{6}{5})$*, there exists a constant C such that:*

$$\left\|\partial_x^k(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\right\|_{L^2} \le C(1+t)^{-\sigma(p,2;1)}, k = 0, 1, 2, \tag{116}$$

and

$$\|(\varrho, \mathbf{v}, \vartheta, \mathscr{H})\|_{L^q} \le C(1+t)^{-\sigma(p,q;0)}, \tag{117}$$

for any $t \ge 0$ *as well as* σ *is defined by Equation (12).*

Proof. Define

$$h(t) = \sup_{0 \le s \le t} M(s)(1+s)^{2\sigma(p,2;1)}, \tag{118}$$

Note that $h(t)$ is nondecreasing, and

$$\begin{aligned}&\|(\partial_x\varrho, \partial_x\mathbf{v}, \partial_x\vartheta, \partial_x\mathscr{H})(;s)\|_{H^2}\\ &\le C\sqrt{M(s)} \le C(1+s)^{-\sigma(p,2;1)}\sqrt{h(t)},\end{aligned} \tag{119}$$

for $0 \le s \le t$. Then, it follows from Equation (83) and lemma (81) that:

$$\begin{aligned}&\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})\|_{L^2}\le CK_0(1+t)^{-\sigma(p,2;1)}\\&\quad+C\delta_0\int_0^t(1+t-s)^{-\sigma(p,2;1)}(1+s)^{-\sigma(p,2;1)}ds\sqrt{h(t)}\\&\quad\le C(1+t)^{-\sigma(p,2;1)}\left[K_0+\delta_0\sqrt{h(t)}\right],\end{aligned}\tag{120}$$

□

Thus, from Gronwall's inequality and Equations (106) and (120), we obtain:

$$\begin{aligned}&M(t)\le M(0)e^{-Dt}+C\int_0^t e^{-D(t-s)}\\&\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})(;s)\|_{L^2}^2ds\le CM(0)(1+t)^{-2\sigma(p,2;1)}\\&\quad+C\int_0^t(1+t-s)^{-2\sigma(p,2;1)}(1+s)^{-2\sigma(p,2;1)}ds[K_0^2+\delta_0^2h(t)]\\&\quad\le C(1+t)^{-2\sigma(p,2;1)}[M(0)+K_0^2+\delta_0^2h(t)].\end{aligned}\tag{121}$$

Because $h(t)$ is nondecreasing, then from Equation (118), we have that $M(s)(1+s)^{2\sigma(p,2;1)}\le C[M(0)+K_0^2]+C\delta_0^2h(t)$, for $0\le s\le t$, which implies that:

$$\sup_{0\le s\le t}M(s)(1+s)^{2\sigma(p,2;1)}\le C[M(0)+K_0^2+\delta_0^2h(t)].\tag{122}$$

Then by the smallness of δ, we have

$$h(t)\le C[M(0)+K_0^2].\tag{123}$$

This gives Equation (116). Now we turn to estimate for $\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{L^q}$.

In a similar manner to Equation (84), we have from Equation (79) that

$$\begin{aligned}&\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{L^q}\le CK_0(1+t)^{-\sigma(p,q;0)}\\&\quad+C\int_0^t(1+t-s)^{-\sigma(p,q;0)}\|(N_1,N_2,N_3,N_4)(;s)\|_{L^p\cap L^2}ds.\end{aligned}\tag{124}$$

Thus, using Equations (105), (119), and (123), we find:

$$\begin{aligned}&\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{L^q}\le CK_0(1+t)^{-\sigma(p,q;0)}\\&\quad+C\delta_0\int_0^t(1+t-s)^{-\sigma(p,q;0)}\|((\partial_x\varrho,\partial_x\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H}))\|_{H^2}ds\\&\quad\le CK_0(1+t)^{-\sigma(p,q;0)}+C\delta_0\int_0^t(1+t-s)^{-\sigma(p,q;0)}\\&\quad(1+s)^{-\sigma(p,2;1)}dsh(t)\le CK_0(1+t)^{-\sigma(p,q;0)}\\&\quad+C\delta_0[M(0)+K_0^2]^{\frac12}\int_0^t(1+t-s)^{-\sigma(p,q;0)}(1+s)^{-\sigma(p,2;1)}ds\end{aligned}\tag{125}$$

$$\le C(1+t)^{-\sigma(p,q;0)},\tag{126}$$

where $2\le q\le 6$. Equations (116) and (117) imply Equations (10) and (11) with the help of Equation (19). Finally, we improve the decay rates of the second derivatives by modifying the condition on the initial data. First, applying Lemma 8 and formula (79), we can immediately deduce the following lemma:

Lemma 13. *If $k\ge 0$ is an integer and $1\le p\le 2\le q\le\infty$, then for any $t>0$,*

$$\begin{aligned}&\left\|\partial_x^kU(t)\right\|_{L^2}\le C(1+t)^{-\sigma(p,2;k)}(\|U_0(t)\|_{L^p}\\&\quad+\left\|\partial_x^kU_0(t)\right\|_{L^2})+C\int_0^t(1+t-s)^{-\sigma(p,2;k)}(\|F(U(s))\|_{L^p}\\&\quad+\left\|\partial_x^kF(U(s))\right\|_{L^2}).\end{aligned}\tag{127}$$

where σ is defined by Equation (12).

Next, we state the following proposition, which together with Equation (19) yields Equation (13). This completes the proof of Equations (10), (11), and (13) in Theorem 3.

Proposition 14. *Under the assumption of Proposition 6, let $(\varrho,\mathbf{v},\vartheta,\mathscr{H})$ be the solution to the initial problem (79) and $(\varrho,\mathbf{v},\vartheta,\mathscr{H})$ satisfies Equation (78), if in addition the initial data $(\varrho_0,\mathbf{v_0},\vartheta_0,\mathscr{H}_0)\in H^4$ and $\|(\varrho_0,\mathbf{v_0},\vartheta_0,\mathscr{H}_0)\|_{H^4}$ are small, then there exists a constant C such that:*

$$\|\partial_x^2(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{L^2}\le C(1+t)^{-\sigma(p,2;2)}.\tag{128}$$

As in the proof of the Theorem 5, we have the global solution$(\varrho,\mathbf{v},\vartheta,\mathscr{H})\in H^4(R^3)$ by the smallness of $\|(\varrho_0,\mathbf{v_0},\vartheta_0,\mathscr{H}_0)\|_{H^4}$. Moreover, we determine the decay in-time estimate (116) for $k=0,1,2,3$ (117). Now, we shall estimate for N_1, N_2,N_3 and N_4 in Equation (128):

$$\begin{aligned}&\|(N_1,N_2,N_3,N_4)\|_{L^p}\le C\|(\varrho,\mathbf{v},\partial_x\mathbf{v},\vartheta,\mathscr{H},\partial_x\mathscr{H})\|_{L^{2p}}\\&\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x^2\vartheta,\partial_x\mathscr{H})\|_{L^{2p}}\\&\le\|(\varrho,\mathbf{v},\partial_x\mathbf{v},\vartheta,\mathscr{H},\partial_x\mathscr{H})\|_{L^2}^{\epsilon}\|(\varrho,\mathbf{v},\partial_x\mathbf{v},\vartheta,\mathscr{H},\partial_x\mathscr{H})\|_{L^6}^{1-\epsilon}\\&\times\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x^2\vartheta,\partial_x\mathscr{H})\|_{L^2}^{\epsilon}\\&\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x^2\vartheta,\partial_x\mathscr{H})\|_{L^6}^{1-\epsilon}\\&\le C\|(\varrho,\mathbf{v},\partial_x\mathbf{v},\vartheta,\mathscr{H},\partial_x\mathscr{H})\|_{L^2}^{\epsilon}\|(\partial_x\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x^2\vartheta,\partial_x\mathscr{H})\|_{L^2}\\&\times\|(\partial_x^2\varrho,\partial_x^2\mathbf{v},\partial_x^3\mathbf{v},\partial_x^2\vartheta,\partial_x^3\vartheta,\partial_x^2\mathscr{H})\|_{L^2}^{1-\epsilon},\end{aligned}\tag{129}$$

where $\epsilon=\frac{3}{2p}-\frac12$.

We shall now estimate $\partial_x^2(N_1,N_2,N_3,N_4)$ in detail:

$$\begin{aligned}&\|\partial_x^2N_1\|_{L^2}\le C\|\varrho,\partial_x\varrho,\mathbf{v},\partial_x\mathbf{v}\|_{L^4}\|\partial_x^2\varrho,\partial_x^3\varrho,\partial_x^2\mathbf{v},\partial_x^3\mathbf{v}\|_{L^4}\\&\quad\le C\|(\varrho,\mathbf{v})\|_{H^2}\|(\partial_x^2\varrho,\partial_x^2\mathbf{v})\|_{H^2},\end{aligned}\tag{130}$$

$$\begin{aligned}\|\partial_x^2 N_2\|_{L^2} &\le C\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{L^\infty}\|(\partial_x^3\varrho,\partial_x^3\mathbf{v},\partial_x^4\mathbf{v},\partial_x^3\mathscr{H})\|_{L^2}\\ &+C\|(\partial_x\varrho,\partial_x^2\varrho,\partial_x\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H})\|_{L^4}\|(\partial_x^2\varrho,\partial_x^2\mathbf{v},\partial_x^3\mathbf{v},\partial_x^2\vartheta,\partial_x^2\mathscr{H})\|_{L^4}\\ &\le C\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{H^2}\|(\partial_x^2\varrho,\partial_x^2\mathbf{v},\partial_x^2\vartheta,\partial_x^2\mathscr{H})\|_{H^2},\end{aligned} \tag{131}$$

$$\begin{aligned}\|\partial_x^2 N_3\|_{L^2} &\le C\|(\varrho,\mathbf{v},\vartheta)\|_{L^\infty}\|(\partial_x^3\mathbf{v},\partial_x^3\vartheta,\partial_x^4\vartheta)\|_{L^2}\\ &+C\|(\partial_x\varrho,\partial_x^2\varrho,\partial_x\mathbf{v},\partial_x^2\mathbf{v},\partial_x\vartheta,\partial_x\mathscr{H},\partial_x^2\mathscr{H})\|_{L^4}\\ &\|(\partial_x^2\mathbf{v},\partial_x^3\mathbf{v},\partial_x^2\vartheta,\partial_x^3\vartheta,\partial_x^2\mathscr{H},\partial_x^3\mathscr{H})\|_{L^4}\le C\|(\varrho,\mathbf{v},\vartheta,\mathscr{H})\|_{H^2}\\ &\|(\partial_x^2\varrho,\partial_x^2\mathbf{v},\partial_x^2\vartheta,\partial_x^2\mathscr{H})\|_{H^2},\end{aligned} \tag{132}$$

$$\begin{aligned}\|\partial_x^2 N_4\|_{L^2} &\le C\|(\mathbf{v},\partial_x\mathbf{v},\mathscr{H},\partial_x\mathscr{H})\|_{L^4}\|(\partial_x^2\mathbf{v},\partial_x^3\mathbf{v},\partial_x^2\mathscr{H},\partial_x^3\mathscr{H})\|_{L^4}\\ &\le C\|(\mathbf{v},\mathscr{H})\|_{H^2}\|(\partial_x^2\mathbf{v},\partial_x^2\mathscr{H})\|_{H^2}.\end{aligned} \tag{133}$$

Therefore, by Lemma 13, the decay-in-time estimates Equations (116) and (117), and the above estimates for N_1, $N_2, N_3,$ and N_4, we obtain:

$$\begin{aligned}\|\partial_x^2 U(t)\|_{L^2} &\le C(1+t)^{-\sigma(p,2;2)}K_0\\ &+C\int_0^t(1+t-s)^{-\sigma(p,2;2)}(\|F(U(s))\|_{L^p})\\ &+\partial_x^2(\|F(U(s))\|_{L^2})ds\le C(1+t)^{-\sigma(p,2;2)}\\ &+\int_0^t(1+t-s)^{-\sigma(p,2;2)}(1+s)^{-5/4}ds\le C(1+t)^{-\sigma(p,2;2)}.\end{aligned} \tag{134}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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