Hindawi
Abstract and Applied Analysis
Volume 2023, Article ID 9939530, 17 pages
https://doi.org/10.1155/2023/9939530

Research Article

Study of the Stability Properties for a General Shape of Damped Euler–Bernoulli Beams under Linear Boundary Conditions

Teya Kouakou Kra Isaac[1] **Bomisso Gossrin Jean-Marc,**[2] **Touré Kidjegbo Augustin,**[1] **and Coulibaly Adama**[3]

[1]*Felix Houphouet Boigny National Polytechnic Institute, Yamoussoukro, BP 1093 Yamoussoukro, Côte d'Ivoire*
[2]*Nangui Abrogoua University, Abidjan, BP 801 Abidjan 02, Côte d'Ivoire*
[3]*Felix Houphouet Boigny University, Abidjan, BP 582 Abidjan 22, Côte d'Ivoire*

Correspondence should be addressed to Teya Kouakou Kra Isaac; isaacteya@gmail.com

Received 16 June 2023; Revised 5 September 2023; Accepted 4 October 2023; Published 14 November 2023

Academic Editor: Jozef Banas

We study in this paper a general shape of damped Euler–Bernoulli beams with variable coefficients. Our main goal is to generalize several works already done on damped Euler–Bernoulli beams. We start by studying the spectral properties of a particular case of the system, and then we determine asymptotic expressions that generalize those obtained by other authors. At last, by adopting well-known techniques, we establish the Riesz basis property of the system in the general case, and the exponential stability of the system is obtained under certain conditions relating to the feedback coefficients and the sign of the internal damping on the interval studied of length 1.

1. Introduction

The stabilization of damped beams submitted to vibrations has been one of the main research topics in smart materials and structures. A beam has two spatially nonhomogeneous damping terms. The first is often called structural damping, while the second acts opposite to the velocity and is called viscous damping. When these damping coefficients are constant, it is known that viscous damping causes a constant attenuation rate for all frequencies of vibrations. In this paper, we study a general shape of Euler–Bernoulli beams with variable coefficients under the influence of a viscous damping. This beam is clamped at one end and controlled at its free end in force by a linear combination of the velocity, rotation, and velocity of rotation and in a moment by a linear combination of the velocity, velocity of rotation, and the position term. Let $y(x,t)$ the transversal deviation at position x and time t. The equations are defined by the following:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) + \gamma(x)y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ -EI(1)y_{xx}(1,t) = (2\zeta_{11}y_t + \zeta_{12}y_{xt} + \alpha y_x)(1,t), & t>0, \\ (EI(.)y_{xx})_x(1,t) = (\zeta_{21}y_t + 2\zeta_{22}y_{xt} + \beta y)(1,t), & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \quad (1)$$

The terms $\alpha, \beta, \zeta_{11}, \zeta_{22}, \zeta_{12}$, and ζ_{21} are positive given constants. The function $m(.)$ is the mass density of the beam and the function $EI(.)$ is its flexural rigidity satisfying $m(.) > 0$ and $EI(.) > 0$. Throughout this paper, we assume that

$$\zeta_{12} \geq 0, \zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2 \text{and} \, (m(.), EI(.)) \in [C^4(0,1)]^2. \tag{2}$$

Moreover, $\gamma(.)$ is a continuous coefficient function of feedback damping that is assumed to satisfy the condition

$$\int_0^1 \left(\frac{\gamma(x)}{m(x)}\right)\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx > 0. \tag{3}$$

Our interest in this work is to generalize the works carried out by several authors on damped Euler–Bernoulli beams.

Aouragh and Yebari [1] studied the system (1) without damping ($\gamma = 0$) and with the condition $\zeta_{12} > 0$. In their article, the authors showed that there is a sequence of generalized eigenfunctions that form a Riesz basis of the appropriate Hilbert space and that there is exponential stability under certain conditions relating to the feedback coefficients.

For our part, we study the system (1) with the presence of a viscous damping $\gamma(x) > 0$ and the condition $\zeta_{12} \geq 0$. To do this, we analyze two situations as follows:

Case 1: We consider the case where $\zeta_{12} = 0$, which implies with the condition (2) that $\zeta_{11} = \zeta_{22} = 0$. Our investigation of this case allows us to deduce the results of asymptotic expressions by Guo [2], Koffi et al. [3], Touré et al. [4], and Wang et al. [5]. In these cases, the authors have concluded for the exponential stability.

Case 2: We study the case where $\zeta_{12} > 0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2$. Here, first, we find a general asymptotic expression of the eigenvalues of the system where we could deduce those found by Jean-Marc et al. [6] and Aouragh and Yebari [1] in the constant case. Next, by proceeding by Wang [7], the exponential stability of the system (1) is obtained when γ is non-negative in [0,1], and we discuss the case that γ is continuous and indefinite on the interval [0, 1].

In the second case, we consider non-zero all the coefficients in the boundary conditions, which made the calculations very complex. The calculations have been carried out meticulously to obtain the expected result.

This present work reveals capital importance because it is a general case of the results obtained in several works [1–6].

During our analysis, we used the asymptotic method of Wang, which is essential for our study. This method comes from the work of Birkhoff [8, 9]. We also find in the literature several authors who have used this approach to study Euler–Bernoulli beam equations with variable coefficients (see Wang et al. [5], Wang [7], Guo [10], Guo and Wang [11]).

The contents of the paper are arranged as follows: in Section 2, we formulate the system (1) into an abstract Cauchy problem in Hilbert state space and discuss some basic properties of the system. In Section 3, we study the spectrum properties for each case. The Riesz basis property and the exponential stability in the general case are concluded in Section 4.

2. Semigroup Formulation

In this part, we study the well-posedness of the system (1) and deduce some properties of the operator of this system. We introduce this Hilbert spaces as follows:

$$\mathbb{V} = \{u \in H^2(0,1) \,|\, u(0) = u_x(0) = 0\}, \tag{4}$$

$$\mathscr{H} = \mathbb{V} \times L^2(0,1), \tag{5}$$

with the inner product

$$\langle z_1, z_2 \rangle_{\mathscr{H}} = \int_0^1 m(x) v_1 \overline{v_2} d(x) + \int_0^1 EI(x)(u_1)_{xx}(\overline{u_2})_{xx} d(x) + \alpha (u_1)_x(1)(\overline{u_2})_x(1) + \beta u_1(1)\overline{u_2}(1), \tag{6}$$

where $z_1 = (u_1, v_1)^T \in \mathscr{H}$ and $z_2 = (u_2, v_2)^T \in \mathscr{H}$. Here, $\|.\|_{\mathscr{H}}$ denotes the corresponding norm. The spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as follows:

$$L^2(0,1) = \left\{u : [0,1] \longrightarrow \mathbb{C}\left\{\int_0^1 |u|^2 dx < \infty\right\}, \tag{7}$$

$$H^k(0,1) = \left\{u : [0,1] \longrightarrow \mathbb{C}\{\, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)} \in L^2(0,1)\right\}. \tag{8}$$

Next, we define an unbounded linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \longrightarrow \mathscr{H}$ as follows:

$$\mathscr{A}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v(.) \\ -\dfrac{1}{m(.)}((EI(.)u_{xx}(.))_{xx} + \gamma v(.)) \end{bmatrix}, \tag{9}$$

with the domain

$$D(\mathscr{A}) = \left\{ \begin{array}{l} (u,v)^T \in (H^4(0,1) \cap \mathbb{V}) \times \mathbb{V} : \\ -EI(1)u_{xx}(1) = 2\zeta_{11} v(1) + \zeta_{12} v_x(1) + \alpha u_x(1) \\ (EI(.)u_{xx})_x(1) = \zeta_{21} v(1) + 2\zeta_{22} v_x(1) + \beta u(1) \end{array} \right\}. \tag{10}$$

The system (1) can be formally written as a first-order evolution problem

$$\begin{cases} \frac{dY(t)}{dt} = \mathcal{A}Y(t) \\ Y(0) = Y_0 \in \mathcal{H}, \end{cases} \tag{11}$$

where $Y(t) = (u(.,t), u_t(.,t))^T$ and $Y(0) = (u_0, u_1)^T$. Furthermore, we introduce the linear operator $\mathcal{B}$ defined on $D(\mathcal{A})$ by the following:

$$\mathcal{B}\begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} - A_0\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\gamma(x)v(x)}{m(x)} \end{bmatrix}, \forall (u,v) \in D(\mathcal{A}). \tag{12}$$

Notice that A_0 denotes the operator where $\gamma(x) = 0$. In the study of Aouragh and Yebari [1], it is shown that the linear operator A_0 generates a C_0-semigroup of contractions on $\mathcal{H}$. Moreover, the linear operator $\mathcal{B}$ is bounded on $\mathcal{H}$. In fact,

$$|\langle \mathcal{B}(u,v)^T, (u,v)^T \rangle_{\mathcal{H}}| \le R\|(u,v)^T\|^2, \tag{13}$$

where

$$R = \sup_{x\in[0,1]} \left\{ \frac{|\gamma(x)|}{m(x)} \right\}. \tag{14}$$

Now, we deduce the well-posedness of the system through the following theorem.

Theorem 2.1. *The operator $\mathcal{A}$ defined by Equations (9) and (10) generates a C_0-semigroup of contraction $\{e^{\mathcal{A}t}\}_{t\ge 0}$ on $\mathcal{H}$ and has compact resolvent and $0 \in \rho(\mathcal{A})$. Therefore, the spectrum $\sigma(\mathcal{A})$ consists entirely of isolated eigenvalues.*

Proof. Let $U = (y, y_t)^T \in D(\mathcal{A})$. Integrating twice by parts and taking real part, we obtain the following: □

$$Re \ \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -(\zeta_{21}|y_t(1)|^2 + 2(\zeta_{22} + \zeta_{11})|y_{xt}(1)y_t(1)| + \zeta_{12}|y_{xt}(1)|^2 - \int_0^1 \gamma(x)|y_t|^2 d(x) \le 0. \tag{15}$$

Thus, $\mathcal{A}$ is a dissipative operator.

Moreover, the linear operator $\mathcal{B}$ is bounded on $\mathcal{H}$ and A_0 generates a C_0-semigroup of contractions. Then, according to perturbation theory by a bounded linear operator (see Pazy [12] P.76), the operator $\mathcal{A} = \mathcal{B} + A_0$ is an infinitesimal generator of a C_0-semigroup of contractions $\{T(t)\}_{t\ge 0} = \{e^{\mathcal{A}t}\}_{t\ge 0}$, satisfying to $\|T(t)\|_{L(\mathcal{H})} \le Me^{(\omega + M\|\mathcal{B}\|)t}$ for $M, \omega \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$.

Now, we show that the operator $\mathcal{A}^{-1}$ exists. For any $G = (u, v) \in \mathcal{H}$, we must find a unique $F = (f, g) \in D(\mathcal{A})$ such that $\mathcal{A}F = G$. This leads us to the following system:

$$\begin{cases} g(x) = u(x), u \in \mathbb{V} \\ (EI(x)f_{xx})_x(x) = -m(x)v(x) - \gamma(x)g(x), v \in L^2(0,1) \\ -EI(1)f_{xx}(1) = 2\zeta_{11}u(1) + \zeta_{12}u_x(1) + \alpha f_x(1) \\ (EI(1)f_{xx})_x(1) = \zeta_{21}u(1) + 2\zeta_{22}u_x(1) + \beta f(1) \\ f(0) = f_x(0) = 0. \end{cases} \tag{16}$$

A direct calculation shows that

$$\begin{cases} g(x) = u(x) \\ f(x) = \int_0^x \int_0^s [-\frac{1}{EI(\delta)} \int_\delta^1 \int_\eta^1 [m(r)v(r) + \gamma(r)u(r)]drd\eta \\ -\frac{1}{EI(\delta)}(2\zeta_{11}u + \zeta_{12}u_x)(1) + \frac{\delta - 1}{EI(\delta)}(\zeta_{21}u + 2\zeta_{22}u_x + \beta f)(1) - \frac{\alpha f_x(1)}{EI(\delta)}]d\delta ds, \end{cases} \tag{17}$$

where $f(1)$ and $f_x(1)$ are defined as follows:

$$\begin{cases} f(1) = \frac{-1}{1 + \beta \int_0^1 \int_0^s \frac{1-\delta}{EI(\delta)} d\delta ds} \int_0^1 \int_0^s (\frac{1}{EI(\delta)} \int_\delta^1 \int_\eta^1 [m(r)v(r) + \gamma(r)u(r)]drd\eta \\ \quad + \frac{1}{EI(\delta)}((2\zeta_{11}u + \zeta_{12}u_x)(1) + \alpha f_x(1)) + (\zeta_{21}u + 2\zeta_{22}u_x)(1)\frac{1-\delta}{EI(\delta)})d\delta ds \\ f_x(1) = \frac{-1}{1 + \alpha \int_0^1 \frac{1}{EI(\delta)} d\delta} \int_0^1 \left(\frac{1}{EI(\delta)} \int_\delta^1 \int_\eta^1 [m(r)v(r) + \gamma(r)u(r)]drd\eta \right. \\ \quad \left. + \frac{1}{EI(\delta)}(2\zeta_{11}u + \zeta_{12}u_x)(1) + (\zeta_{21}u + 2\zeta_{22}u_x + \beta f)(1)\frac{1-\delta}{EI(\delta)} \right) d\delta. \end{cases} \tag{18}$$

Obviously, we have $F = (f, g) \in D(\mathscr{A})$. Therefore, we get the following:

$$F = (f, g) = \mathscr{A}^{-1} G. \tag{19}$$

Thus, $\mathscr{A}^{-1}$ exists and consequently we have $0 \in \rho(\mathscr{A})$ and Sobolev's embedding theorem implies that $\mathscr{A}^{-1}$ is a compact operator on $\mathscr{H}$.

3. General Spectral Analysis

3.1. Study of Particular Case: $\zeta_{12} = \zeta_{22} = \zeta_{11} = 0$. In this part, we study the eigenvalue problem of the operator in order to determine the behavior of these eigenvalues. The system (1) becomes

$$\begin{cases} m(x) y_{tt}(x,t) + (EI(x) y_{xx})_{xx}(x,t) + \gamma(x) y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ -EI(1) y_{xx}(1,t) = \alpha y_x(1,t), & t>0, \\ (EI(.) y_{xx})_x(1,t) = (\zeta_{21} y_t + \beta y)(1,t), & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1, \end{cases} \tag{20}$$

and

$$D(\mathscr{A}) = \left\{ \begin{array}{c} (u,v)^T \in (H^4(0,1) \cap \mathbb{V}) \times \mathbb{V}: \\ -EI(1) u_{xx}(1) = \alpha u_x(1) \\ (EI(.) u_{xx})_x(1) = \zeta_{21} v(1) + \beta u(1) \end{array} \right\}. \tag{21}$$

Our work shall follow the results from Wang [5]. Let $\lambda \in \sigma(\mathscr{A})$ be an eigenvalue of the operator $\mathscr{A}$ and let $Y = (\phi, \psi)^T \in D(\mathscr{A})$, the corresponding eigenfunction.

$$\mathscr{A} Y = \lambda Y \Longleftrightarrow \begin{cases} \lambda^2 m(x) \phi(x) + (EI(x) \phi''(x))'' + \gamma(x) \lambda \phi(x) = 0, & 0<x<1, \\ \phi(0) = \phi'(0) = 0, \\ -EI(1) \phi''(1) = \alpha \phi'(1), \\ (EI(.) \phi''(.))'(1) = (\zeta_{21} \lambda + \beta) \phi(1), \end{cases} \tag{22}$$

where the prime refers to the derivative with respect to space variables. This equation leads to the following system:

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)} \phi'''(x) + \dfrac{EI''(x)}{EI(x)} \phi''(x) + \dfrac{\lambda^2 m(x)}{EI(x)} \phi(x) + \dfrac{\lambda \gamma(x)}{EI(x)} \phi(x) = 0, \\ \phi(0) = \phi'(0) = 0, \\ -EI(1) \phi''(1) = \alpha \phi'(1), \\ (EI(.) \phi''(.))'(1) = (\zeta_{21} \lambda + \beta) \phi(1). \end{cases} \tag{23}$$

First, we use a space transformation to transform the coefficient function appearing with ϕ in the first expression of Equation (23) into a constant (see Guo [10]).

$$f(z) = \phi(x), z = z(x) = \frac{1}{h} \int_0^x \left(\frac{m(\xi)}{EI(\xi)} \right)^{1/4} d\xi, \quad \text{where } h = \int_0^1 \left(\frac{m(\xi)}{EI(\xi)} \right)^{1/4} d\xi. \tag{24}$$

A direct computation gives this system as follows:

$$\begin{cases} f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \lambda^2 h^4 f(z) \\ \qquad + \lambda h^4 d(z) f(z) = 0, \\ f(0) = f'(0) = 0, \\ f''(1) + \left[\dfrac{z_{xx}(1)}{z_x^2(1)} + \dfrac{\alpha}{z_x(1)EI(1)}\right] f'(1) = 0, \\ f'''(1) + \left[\dfrac{3z_{xx}(1)}{z_x^2(1)} + \dfrac{EI'(1)}{z_x(1)EI(1)}\right] f''(1) \\ \quad + \left[\dfrac{z_{xxx}(1)}{z_x^3(1)} + \dfrac{EI'(1)z_{xx}(1)}{z_x^3(1)EI(1)}\right] f'(1) + \left[-\dfrac{\zeta_{21}\lambda + \beta}{z_x^3(1)EI(1)}\right] f(1) = 0, \end{cases} \tag{25}$$

where

$$\begin{aligned} a(z) &= \frac{6z_{xx}}{z_x^2} + \frac{2EI'(x)}{z_x EI(x)} \\ b(z) &= \frac{3z_{xx}^2}{z_x^4} + \frac{4z_{xxx}}{z_x^3} + \frac{6z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI''(x)}{z_x^2 EI(x)} \\ c(z) &= \frac{z_{xxxx}}{z_x^4} + \frac{2z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{EI''(x) z_{xx}}{z_x^4 EI(x)}, \end{aligned} \tag{26}$$

$$d(z) = \frac{\gamma(x)}{m(x)}, \quad z_x = \frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{1/4}, \quad z_x^4 = \frac{m(x)}{h^4 EI(x)}. \tag{27}$$

Next, we make an invertible state transformation by Naimark [13].

$$f(z) = g(z) e^{\left(-\frac{1}{4}\int_0^z a(\xi) d\xi\right)}, \quad 0 < z < 1. \tag{28}$$

The function g satisfies the following:

$$\begin{cases} g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \lambda^2 h^4 g(z) \\ + \lambda h^4 d(z) g(z) = 0, \\ g(0) = g'(0) = 0, \\ g''(1) + c_{11}(1)g'(1) + c_{12}(1)g(1) = 0, \\ g'''(1) + c_{21}(1)g''(1) + c_{22}(1)g'(1) + c_{23}(1)g(1) = 0, \end{cases} \tag{29}$$

with

$$\begin{aligned} b_1(z) &= (a(z), b(z)), c_1(z) = (a(z), b(z), c(z)), d_1(z) = (a(z), b(z), c(z)) \\ c_{11}(1) &= -\frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)} + \frac{\alpha}{z_x(1)EI(1)} \\ c_{12}(1) &= -\frac{1}{4}a'(1) + \frac{1}{16}a^2(1) - \frac{z_{xx}(1)a(1)}{4z_x^2(1)} - \frac{\alpha a(1)}{4z_x(1)EI(1)} \\ c_{21}(1) &= -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)} + \frac{EI'(1)}{z_x(1)EI(1)}, \end{aligned} \tag{30}$$

$$\begin{aligned} c_{22}(1) &= -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{a(1)EI'(1)}{2z_x(1)EI(1)} - \frac{3a(1)z_{xx}(1)}{2z_x^2(1)} \\ &\quad + \frac{z_{xxx}(1)}{z_x^3(1)} + \frac{z_{xx}(1)EI'(1)}{z_x^3(1)EI(1)} \\ c_{23}(1) &= -\frac{1}{4}a''(1) + \frac{3}{16}a(1)a'(1) - \frac{1}{64}a^3(1) - \frac{a'(1)EI'(1)}{4z_x(1)EI(1)} \\ &\quad - \frac{3a'(1)z_{xx}(1)}{4z_x^2(1)} + \frac{a^2(1)EI'(1)}{16z_x(1)EI(1)} + \frac{3a^2(1)z_{xx}(1)}{16z_x^2(1)} \\ &\quad - \frac{a(1)z_{xx}(1)EI'(1)}{4z_x^3(1)EI(1)} - \frac{a(1)z_{xxx}(1)}{4z_x^3(1)} - \frac{\zeta_{21}\lambda + \beta}{z_x^3(1)EI(1)}. \end{aligned} \tag{31}$$

By setting $\lambda = \rho^2/h^2$, the first equation of the system (29) becomes the following:

$$\begin{aligned} & g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) \\ & + \rho^4 g(z) + \rho^2 h^2 d(z) g(z) = 0, \; 0 < z < 1. \end{aligned} \tag{32}$$

Now, to solve the eigenvalue problem (29), we divide the complex plane into eight distinct sectors by Naimark [13].

$$S_k = \left\{ z \in \mathbb{C} : \frac{k\pi}{4} \le \arg(z) \le \frac{(k+1)\pi}{4} \right\}, \quad k = 0, 1, \ldots, 7. \tag{33}$$

Let $\omega_1, \omega_2, \omega_3$, and ω_4 be the roots of characteristic equation $\omega^4 + 1 = 0$, with

$$Re(\rho\omega_1) \le Re(\rho\omega_2) \le Re(\rho\omega_3) \le Re(\rho\omega_4), \forall \rho \in S_k. \tag{34}$$

In sector S_1, we name the roots as follows:

$$\begin{aligned} \omega_1 &:= e^{i\frac{3}{4}\pi} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_2 := e^{i\frac{1}{4}\pi} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \\ \omega_3 &:= e^{i\frac{5}{4}\pi} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \omega_4 := e^{i\frac{7}{4}\pi} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \end{aligned} \tag{35}$$

that satisfy the inequalities (34). Similarly, the choices can also be made for other sectors. In the following, we study the asymptotic behavior of the eigenvalues specific to sectors S_1 and S_2 because this will be verified in the other sectors with similar arguments. For this, we will use the following lemma:

Lemma 3.1. *For $\rho \in S_k$, with $|\rho|$ large enough, the equation*

$$g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \rho^4 g(z) + \rho^2 h^2 d(z)g(z) = 0, \quad 0<z<1. \tag{36}$$

has four linearly independent asymptotic fundamental solutions

$$\phi_s(z,\rho) = e^{\rho\omega_s z}\left(1 + \frac{\phi_{s,1}(z)}{\rho} + \mathcal{O}(\rho^{-2})\right), \quad s = 1,2,3,4, \tag{37}$$

and hence their derivatives for $s = 1,2,3,4$ and $j = 1,2,3$ are given by the following:

$$\phi_s^{(j)}(z,\rho) = (\rho\omega_s)^{(j)} e^{\rho\omega_s z}\left(1 + \frac{\phi_{s,1}(z)}{\rho} + \mathcal{O}(\rho^{-2})\right), \tag{38}$$

where for $s = 1,2,3,4$

$$\phi_{s,1}(z) = -\frac{1}{4\omega_s}\int_0^z b_1(\xi)d\xi - \frac{h^2}{4\omega_s^3}\int_0^z d(\xi)d\xi, \tag{39}$$

$$\phi_{s,1}(0) = 0, \quad \phi_{s,1}(1) = -\frac{1}{4\omega_s}\int_0^z b_1(\xi)d\xi - \frac{h^2}{4\omega_s^3}\int_0^z d(\xi)d\xi = \frac{\omega_s^2\mu_1 + \mu_2}{\omega_s^3}, \tag{40}$$

with

$$\mu_1 = -\frac{1}{4}\int_0^z b_1(\xi)d\xi, \quad \mu_2 = -\frac{h^2}{4}\int_0^z d(\xi)d\xi\ . \tag{41}$$

Proof. The reader can refer to the study of Touré et al. [4]. □

For convenience, we introduce the notation $[a]_2 = a + \mathcal{O}(\rho^{-2})$.

Lemma 3.2. *For $\rho \in S_1$, If $\delta = \sin\pi/4 = \sqrt{2}/2$, then we have the following inequalities:*

$Re(\rho\omega_1) \le -|\rho|\delta$, $Re(\rho\omega_4) \ge -|\rho|\delta$, $e^{\rho\omega_1} = \mathcal{O}(\rho^{-2})$ when $|\rho| \to \infty$.

Furthermore, substituting Equations (37) and (38) into the boundary conditions (29), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$ as follows:

$$\begin{cases} U_4(\phi_s,\rho) = \phi_s(0,\rho) = 1 + \mathcal{O}(\rho^{-2}) = [1]_2, \\ U_3(\phi_s,\rho) = \phi_s'(0,\rho) = \rho\omega_s(1 + \mathcal{O}(\rho^{-2})) = \rho\omega_s[1]_2, \\ U_2(\phi_s,\rho) = (\rho\omega_s)^2 e^{\rho\omega_s}[1 + (\mu_2 + m_{22}\omega_s^2)\rho^{-1}\omega_s^{-3}]_2, \\ U_1(\phi_s,\rho) = (\rho\omega_s)^3 e^{\rho\omega_s}[1 + (\mu_1 + c_{21}(1))\rho^{-1}\omega_s^{-1} + m_{11}\rho^{-1}\omega_s^{-3}]_2, \end{cases} \tag{42}$$

where

$$m_{11} = \mu_2 - \frac{\zeta_{21}}{h^2 z_x^3(1) EI(1)}, \qquad m_{22} = \mu_1 - \frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)} + \frac{\alpha}{z_x(1)EI(1)}. \tag{43}$$

Notice that $\lambda = \rho^2/h^2 \ne 0$, is the eigenvalue of Equation (29) if and only if ρ satisfies the characteristic equation as follows:

$$\Delta(\rho) = \begin{vmatrix} U_4(\phi_1,\rho) & U_4(\phi_2,\rho) & U_4(\phi_3,\rho) & U_4(\phi_4,\rho) \\ U_3(\phi_1,\rho) & U_3(\phi_2,\rho) & U_3(\phi_3,\rho) & U_3(\phi_4,\rho) \\ U_2(\phi_1,\rho) & U_2(\phi_2,\rho) & U_2(\phi_3,\rho) & U_2(\phi_4,\rho) \\ U_1(\phi_1,\rho) & U_1(\phi_2,\rho) & U_1(\phi_3,\rho) & U_1(\phi_4,\rho) \end{vmatrix} = 0. \tag{44}$$

By substitution, the following expression is obtained:

$$\Delta(\rho) = \begin{vmatrix} [1]_2 & [1]_2 \\ \rho\omega_1[1]_2 & \rho\omega_2[1]_2 \\ 0 & (\rho\omega_2)^2 e^{\rho\omega_2}[1 + (\mu_2 + m_{22}\omega_2^2)\rho^{-1}\omega_2^{-3}]_2 \\ 0 & (\rho\omega_2)^3 e^{\rho\omega_2}[1 + (\mu_1 + c_{21}(1))\rho^{-1}\omega_2^{-1} + m_{11}\rho^{-1}\omega_2^{-3}]_2, \end{vmatrix} \tag{45}$$

$$\begin{array}{c} [1]_2 \\ \rho\omega_3[1]_2 \\ (\rho\omega_3)^2 e^{\rho\omega_3}[1+(\mu_2+m_{22}\omega_3^2)\rho^{-1}\omega_3^{-3}]_2 \\ (\rho\omega_3)^3 e^{\rho\omega_3}[1+(\mu_1+c_{21}(1))\rho^{-1}\omega_3^{-1}+m_{11}\rho^{-1}\omega_3^{-3}]_2, \end{array} \tag{46}$$

$$\left.\begin{array}{c} 0 \\ 0 \\ (\rho\omega_4)^2 e^{\rho\omega_4}[1+(\mu_2+m_{22}\omega_4^2)\rho^{-1}\omega_4^{-3}]_2 \\ (\rho\omega_4)^3 e^{\rho\omega_4}[1+(\mu_1+c_{21}(1))\rho^{-1}\omega_4^{-1}+m_{11}\rho^{-1}\omega_4^{-3}]_2 \end{array}\right|. \tag{47}$$

In sector S_1, the choices are such that

$$\begin{array}{l} \omega_1^2=-i, \omega_2^2=i, \omega_3^2=i, \omega_4^2=-i, \omega_2^2\omega_4^2=1, \omega_3^2\omega_4^2=1 \\ \omega_2-\omega_4=i\sqrt{2}, \omega_4-\omega_3=\sqrt{2}, \omega_3^{-1}\omega_4=i, \omega_3^{-2}\omega_4=i, \omega_3=-\omega_2 \\ \omega_1-\omega_3=i\sqrt{2}, \omega_2-\omega_1=\sqrt{2}, \omega_2^{-2}-\omega_4^{-2}=-2i, \omega_3^{-2}-\omega_4^{-2}=-2i. \end{array} \tag{48}$$

Developing the determinant, and after a straightforward computation, we obtain the following:

$$\begin{aligned} \Delta(\rho)= & 2\rho^6 e^{\rho\omega_4}\left\{\left[1+\sqrt{2}(m_{22}-m_{11})\rho^{-1}\right]e^{\rho\omega_2}\right. \\ & \left.+\left[1+\sqrt{2}(m_{22}+m_{11})i\rho^{-1}\right]e^{-\rho\omega_2}\right\}+\mathcal{O}(\rho^{-2}). \end{aligned} \tag{49}$$

Noting that

$$\begin{aligned} \mu_3 &= \sqrt{2}(m_{22}-m_{11}) \\ \mu_4 &= \sqrt{2}(m_{22}+m_{11}), \end{aligned} \tag{50}$$

the characteristic determinant in sector S_1 becomes the following:

$$\begin{aligned} \Delta(\rho)= & 2\rho^6 e^{\rho\omega_4}\{e^{\rho\omega_2}+e^{-\rho\omega_2}+[\mu_3 e^{\rho\omega_2} \\ & +\mu_4 i e^{-\rho\omega_2}]\rho^{-1}+\mathcal{O}(\rho^{-2})\}. \end{aligned} \tag{51}$$

Then, we have $\Theta_{-10}=2$, $\Theta_{10}=2$ and $\Theta_{00}=0$, (see Jean-Marc et al. [6]).

We notice that $\Theta_{00}^2-4\Theta_{-10}\Theta_{10}\neq 0$. Then, the following theorem follows directly.

Theorem 3.3. *The boundary conditions of the eigenvalue problem (29) are strongly regular. Therefore, the eigenvalues are asymptotically simple and separated.*

Now, we study the asymptotic behavior for the eigenvalues λ_n of problem (29). The equation $\Delta(\rho)=0$ implies

$$e^{\rho\omega_2}+e^{-\rho\omega_2}+[\mu_3 e^{\rho\omega_2}+\mu_4 i e^{\rho\omega_2}]\rho^{-1}+\mathcal{O}(\rho^{-2})=0, \tag{52}$$

which can also be rewritten as follows:

$$e^{\rho\omega_2}+e^{-\rho\omega_2}+\mathcal{O}(\rho^{-2})=0. \tag{53}$$

The equation below:

$$e^{\rho\omega_2}+e^{-\rho\omega_2}=0, \tag{54}$$

has the solutions given by the following:

$$\rho_k=\left(k+\frac{1}{2}\right)\frac{\pi i}{\omega_2}, k=1,2,\cdots. \tag{55}$$

Let $\tilde{\rho}_k$ be the solution of Equation (53). According to Rouché's theorem (see Naimark [13]), we obtain the following expression:

$$\begin{aligned} & \widetilde{\rho}_k=\rho_k+\alpha_k=\left(k+\frac{1}{2}\right)\frac{\pi i}{\omega_2}+\alpha_k, \quad \alpha_k=\mathcal{O}(k^{-1}), \\ & k=K, K+1, \cdots, \end{aligned} \tag{56}$$

where K is a sufficiently large positive integer.

By substituting it in Equation (52) and using the equality $e^{-\rho_k\omega_2}=-e^{\rho_k\omega_2}$, we get the following:

$$\begin{aligned} & e^{\alpha_k\omega_2}-e^{-\alpha_k\omega_2}+\mu_3\widetilde{\rho}_k^{-1}e^{\alpha_k\omega_2}+\mu_4 i\widetilde{\rho}_k^{-1}e^{-\alpha_k\omega_2}\rho^{-1} \\ & +\mathcal{O}\left(\widetilde{\rho}_k^{-2}\right)=0. \end{aligned} \tag{57}$$

Moreover, expanding the exponential function according to its Taylor series, we get the following:

$$\begin{aligned} & \alpha_k=-\frac{\mu_3}{2\rho_k\omega_2}+\frac{\mu_4}{2\rho_k\omega_2}i+\mathcal{O}(k^{-2}), \quad k=K, K+1, \cdots, \\ & \widetilde{\rho}_k^2=\left(k+\frac{1}{2}\right)^2\pi^2 i-\frac{\mu_3}{\omega_2}+\frac{\mu_4}{\omega_2}i+\mathcal{O}(k^{-1}), \\ & k=K, K+1, \cdots. \end{aligned} \tag{58}$$

Note that $\lambda_k=\tilde{\rho}_k^2/h^2$ and in sector S_1, $\omega_2=e^{i\frac{1}{4}\pi}$ and $\omega_2^2=i$. So, we have the following:

$$\lambda_k = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) + \frac{1}{h^2}\left[\left(k+\frac{1}{2}\right)^2\pi^2 + \frac{\sqrt{2}}{2}(\mu_4+\mu_3)\right]i + \mathcal{O}(k^{-1}), \tag{59}$$

where $k = K, K+1, \cdots$, with K large enough.

Now, in sector S_2, the eigenvalues of the problem (29) can be obtained by a similar computation with the choices.

$$\omega_1 := e^{i\frac{1}{4}\pi} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_2 := e^{i\frac{3}{4}\pi} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_3 := e^{i\frac{7}{4}\pi} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \omega_4 := e^{i\frac{5}{4}\pi} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \tag{60}$$

such that: $Re(\rho\omega_1) \le Re(\rho\omega_2) \le Re(\rho\omega_3) \le Re(\rho\omega_4), \forall \rho \in S_2$.

Hence, in the sector S_2, the characteristic determinant $\Delta(\rho)$ is as follows:

$$\Delta(\rho) = 2\rho^6 e^{\rho\omega_4}\{e^{\rho\omega_2} + e^{-\rho\omega_2} + [-\mu_3 e^{\rho\omega_2} + \mu_4 i e^{-\rho\omega_2}]\rho^{-1} + \mathcal{O}(\rho^{-2})\}. \tag{61}$$

By a similar calculation to the one done in sector S_1, we have the following:

$$\widetilde{\rho}_k = \left(k+\frac{1}{2}\right)\frac{\pi i}{\omega_2} - \frac{1}{2}\frac{\mu_3}{\left(k+\frac{1}{2}\right)\pi}i + \frac{1}{2}\frac{\mu_4}{\left(k+\frac{1}{2}\right)\pi} + \mathcal{O}(k^{-2}), \quad k = K, K+1, \ldots, \tag{62}$$

with K a large enough integer. In the sector S_2, using $\omega_2 = e^{i\frac{3}{4}\pi}$ and $\omega_2^2 = -i$, we obtain for $k = K, K+1, \ldots,$

$$\lambda_k = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) - \frac{1}{h^2}\left[\left(k+\frac{1}{2}\right)^2\pi^2 + \frac{\sqrt{2}}{2}(\mu_4+\mu_3)\right]i + \mathcal{O}(k^{-1}). \tag{63}$$

Remark 3.4. By referring to Naimark [13], we can say that the eigenvalues generated by the sectors S_n coincide with those determined in the sectors S_1 and S_2.

Combining with Equations (59) and (63), we obtain the asymptotic expression

Theorem 3.5. *An asymptotic expression of the eigenvalues of the problem (29) is given by the following:*

$$\lambda_k = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) \pm \frac{1}{h^2}\left[\left(k+\frac{1}{2}\right)^2\pi^2 + \frac{\sqrt{2}}{2}(\mu_4+\mu_3)\right]i + \mathcal{O}(k^{-1}), \tag{64}$$

where $k = K, K+1, \cdots$, with K a large enough integer, and

$$\mu_4 - \mu_3 = 2\sqrt{2}\mu_2 - \frac{2\sqrt{2}\zeta_{21}}{h^2 z_x^3(1)EI(1)}, \quad \mu_4 + \mu_3 = 2\sqrt{2}m_{22}, \tag{65}$$

$$\mu_2 = -\frac{h^2}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}dx = -\frac{h}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}dx. \tag{66}$$

Moreover, $\lambda_k (k = K, K+1, \cdots)$ with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

$$\lim_{k\to+\infty} Re(\lambda_k) = -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}dx - \frac{2\zeta_{21}}{hEI(1)}\left(\frac{m(1)}{EI(1)}\right)^{-\frac{3}{4}}. \tag{67}$$

From Equation (67), we find these results already known. The following examples illustrate the special cases carried out in certain works:

Example 1. For $\zeta_{21} = 0$ and $\alpha = \beta = 0$ and from Equation (20), we get the system as follows:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) + \gamma(x)y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ -EI(1)y_{xx}(1,t) = 0, & t>0, \\ (EI(.)y_{xx})_x(1,t) = 0, & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \tag{68}$$

The asymptotic expression (67) becomes the following:

$$\lim_{k\to+\infty} Re\,(\lambda_k) = -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx. \tag{69}$$

This case corresponds to the one studied by Wang et al. [5], where the result (69) is obtained.

Example 2. Suppose that $\alpha = 0$. Then, the system (20) becomes the following:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) + \gamma(x)y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ y_{xx}(1,t) = 0, & t>0, \\ (EI(.)y_{xx})_x(1,t) = (\zeta_{21}y_t + \beta y)(1,t),, & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \tag{70}$$

This system has been studied by Touré et al. [4]. The same asymptotic expression (67) is obtained.

Example 3. Suppose that $\alpha = \beta = 0$. Equation (20) is equivalent to the following:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) + \gamma(x)y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ y_{xx}(1,t) = 0, & t>0, \\ (EI(.)y_{xx})_x(1,t) = \zeta_{21}y_t(1,t), & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \tag{71}$$

Guo [2] studied this system. His work yielded a similar asymptotic expression as Equation (67) in the uniform case $m(x) = EI(x) = 1$.

Example 4. For $\zeta_{21} > 0$ and $\alpha, \beta \geq 0$, we get the following system:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) + \gamma(x)y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ EI(1)y_{xx}(1,t) = -\alpha y_x(1,t), & t>0, \\ (EI(.)y_{xx})_x(1,t) = \zeta_{21}y_t(1,t) + \beta y(1,t), & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \tag{72}$$

This case was investigated by Koffi et al. [3], where the asymptotic expression (67) was found.

Note that in the cited examples, the authors proved that the systems are exponentially stable.

3.2. Study of General Case: $\zeta_{12} > 0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2$.

In this subsection, we study the system (1) with the conditions $\zeta_{12} > 0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2$. Following the previous approach, we determine a general asymptotic expression of the eigenvalues of the system (1) and deduce those established by Aouragh and Yebari [1] and Touré et al. [4].

Let $\lambda \in \sigma(A)$ be an eigenvalue of the operator $\mathscr{A}$ of the system (1).

$$\mathscr{A}Y = \lambda Y \Longleftrightarrow \begin{cases} \lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' + \gamma(x)\lambda\phi(x) = 0, & 0<x<1, \\ \phi(0) = \phi'(0) = 0 \\ -EI(1)\phi''(1) = 2\zeta_{11}\lambda\phi(1) + (\zeta_{12}\lambda + \alpha)\phi'(1), \\ (EI(.)\phi''(.))'(1) = (\zeta_{21}\lambda + \beta)\phi(1) + 2\zeta_{22}\lambda\phi'(1). \end{cases} \tag{73}$$

The system (73) is rewritten as follows:

$$
\begin{cases}
\phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \dfrac{\lambda^2 m(x)}{EI(x)}\phi(x) + \dfrac{\lambda\gamma(x)}{EI(x)}\phi(x) = 0, \\
\phi(0) = \phi'(0) = 0, \\
\phi''(1) = -\dfrac{\zeta_{12}\lambda + \alpha}{EI(1)}\phi'(1) - \dfrac{2\zeta_{11}\lambda}{EI(1)}\phi(1), \\
\phi'''(1) = -\dfrac{EI'(1)}{EI(1)}\phi''(1) + \dfrac{2\zeta_{22}\lambda}{EI(1)}\phi'(1) + \dfrac{\zeta_{21}\lambda + \beta}{EI(1)}\phi(1),
\end{cases}
\tag{74}
$$

First, by using the space transformation as follows:

$$
\begin{aligned}
&f(z) = \phi(x), \quad z = z(x) = \frac{1}{h}\int_0^x \left(\frac{m(\xi)}{EI(\xi)}\right)^{1/4} d\xi, \\
&h = \int_0^1 \left(\frac{m(\xi)}{EI(\xi)}\right)^{1/4} d\xi,
\end{aligned}
\tag{75}
$$

we have

$$
\begin{cases}
f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \lambda^2 h^4 f(z) \\
+\lambda h^4 d(z) f(z) = 0, \qquad 0<z<1, \\
f(0) = f'(0) = 0, \\
f''(1) + k_{11}(1)f'(1) + k_{12}(1)f(1) = 0, \\
f'''(1) + k_{21}(1)f''(1) + k_{22}(1)f'(1) + k_{23}(1)f(1) = 0,
\end{cases}
\tag{76}
$$

where $a(z), b(z), c(z)$ and $d(z), z_x, z_x^4$ are defined in Equations (26) and (27)

$$
\begin{aligned}
&k_{11}(1) = \frac{z_{xx}(1)}{z_x^2(1)} + \frac{\zeta_{12}\lambda + \alpha}{z_x(1)EI(1)}, \qquad k_{12}(1) = \frac{2\zeta_{11}\lambda}{z_x^2(1)EI(1)}, \\
&k_{21}(1) = \frac{3z_{xx}(1)}{z_x^2(1)} + \frac{EI'(1)}{z_x(1)EI(1)}, \qquad k_{23}(1) = -\frac{\zeta_{21}\lambda + \beta}{EI(1)z_x^3(1)}, \\
&k_{22}(1) = \frac{z_{xxx}(1)}{z_x^3(1)} + \frac{EI'(1)z_{xx}(1)}{EI(1)z_x^3(1)} - \frac{2\zeta_{22}\lambda}{EI(1)z_x^2(1)}.
\end{aligned}
\tag{77}
$$

Then, by making the invertible state transformation

$$
f(z) = g(z)e^{\left(-\frac{1}{4}\int_0^z a(\xi)d\xi\right)}, \quad 0<z<1,
\tag{78}
$$

the system (76) can be written as follows, for any $0<z<1$:

$$\begin{cases} g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \lambda^2 h^4 g(z) + \lambda h^4 d(z)g(z) = 0, \\ g(0) = g'(0) = 0, \\ g''(1) + b_{11}(1)g'(1) + b_{12}(1)g(1) = 0, \\ g'''(1) + b_{21}(1)g''(1) + b_{22}(1)g'(1) + b_{23}(1)g(1) = 0, \end{cases} \tag{79}$$

where

$$\begin{cases} b_1(z) = (a(z), b(z)),\ \ c_1(z) = (a(z), b(z), c(z)), d_1(z) = (a(z), b(z), c(z)) \\ b_{12}(1) = -\frac{1}{4}a'(1) + \frac{1}{16}a^2(1) - \frac{z_{xx}(1)a(1)}{4z_x^2(1)} - \frac{(\zeta_{12}\lambda + \alpha)a(1)}{4z_x(1)EI(1)} + \frac{2\zeta_{11}\lambda}{z_x^2(1)EI(1)}, \\ b_{21}(1) = -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)} + \frac{EI'(1)}{z_x(1)EI(1)}, \\ b_{22}(1) = -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{a(1)EI'(1)}{2z_x(1)EI(1)} - \frac{3a(1)z_{xx}(1)}{2z_x^2(1)} \\ + \frac{z_{xxx}(1)}{z_x^3(1)} - \frac{2\zeta_{22}\lambda}{z_x^2(1)EI(1)} + \frac{z_{xx}(1)EI'(1)}{z_x^3(1)EI(1)}, \\ b_{23}(1) = -\frac{1}{4}a''(1) + \frac{3}{16}a(1)a'(1) - \frac{1}{64}a^3(1) - \frac{a'(1)EI'(1)}{4z_x(1)EI(1)} - \frac{3a'(1)z_{xx}(1)}{4z_x^2(1)} \\ + \frac{a^2(1)EI'(1)}{16z_x(1)EI(1)} + \frac{3a^2(1)z_{xx}(1)}{16z_x^2(1)} - \frac{a(1)z_{xx}(1)EI'(1)}{4z_x^3(1)EI(1)} \\ - \frac{a(1)z_{xxx}(1)}{4z_x^3(1)} - \frac{\zeta_{21}\lambda + \beta}{z_x^3(1)EI(1)} + \frac{2\zeta_{22}\lambda a(1)}{4z_x^2(1)EI(1)}. \end{cases} \tag{80}$$

By setting $\lambda = \rho^2/h^2$, the first equation of the system (79) becomes the following:

$$g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \rho^4 g(z) + \rho^2 h^2 d(z)g(z) = 0, \quad 0<z<1, \tag{81}$$

which has four linearly independent asymptotic fundamental solutions given by Equations (37) and (38). To solve the eigenvalue problem (79), we make a study in the sector S_1 which is defined like this

$$S_1 = \left\{z \in \mathbb{C} : \frac{\pi}{4} \le \arg(z) \le \frac{\pi}{2}\right\}. \tag{82}$$

Let $\omega_1, \omega_2, \omega_3$, and ω_4 be the roots of characteristic equation $\omega^4 + 1 = 0$ that are arranged so that

$$Re\,(\rho\omega_1) \le Re(\rho\omega_2) \le Re(\rho\omega_3) \le Re(\rho\omega_4), \quad \forall \rho \in S_1. \tag{83}$$

In sector S_1, the choices are the same as in Equation (35). Substituting Equations (37) and (38) into the boundary conditions (79), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$:

$$\begin{aligned} U_4(\phi_s, \rho) &= \phi_s(0, \rho) = 1 + \mathcal{O}(\rho^{-2}) = [1]_2, \\ U_3(\phi_s, \rho) &= \phi_s'(0, \rho) = \rho\omega_s(1 + \mathcal{O}(\rho^{-2})) = \rho\omega_s[1]_2, \\ U_2(\phi_s, \rho) &= (\rho\omega_s)^2 e^{\rho\omega_s}[1 + (\tau_{21} + \tau_{22}\omega_s^{-2})\omega_s^{-2} + (\tau_{23}\omega_s^2 + \tau_{24} + \tau_{25}\omega_s^{-2})\rho^{-1}\omega_s^{-3} + \tau_{26}\rho\omega_s^{-1}]_2, \\ U_1(\phi_s, \rho) &= (\rho\omega_s)^3 e^{\rho\omega_s}[1 + \tau_{11}\omega_s^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_s^{-1} + (\tau_{12} + \tau_{13}\omega_s^{-2})\rho^{-1}\omega_s^{-3}]_2, \end{aligned} \tag{84}$$

with

$$
\begin{cases}
\tau_{21} = \dfrac{\zeta_{12}\mu_1}{h^2 z_x(1)EI(1)} - \dfrac{\zeta_{12}a(1)}{4h^2 z_x(1)EI(1)} + \dfrac{2\zeta_{11}}{h^2 z_x^2(1)EI(1)}, & \tau_{22} = \dfrac{\zeta_{12}\mu_2}{h^2 z_x(1)EI(1)}, \\
\tau_{23} = \mu_1 - \dfrac{1}{2}a(1) + \dfrac{z_{xx}(1)}{z_x^2(1)} + \dfrac{\alpha}{z_x(1)EI(1)}, & \tau_{26} = \dfrac{\zeta_{12}}{h^2 z_x(1)EI(1)}, \\
\tau_{25} = -\dfrac{\zeta_{12}a(1)\mu_2}{4h^2 z_x(1)EI(1)} + \dfrac{2\zeta_{11}\mu_2}{h^2 z_x^2(1)EI(1)}, & \tau_{11} = -\dfrac{2\zeta_{22}}{h^2 z_x^2(1)EI(1)}, \\
\tau_{24} = \mu_2 - \dfrac{\zeta_{12}a(1)\mu_1}{4h^2 z_x(1)EI(1)} + \dfrac{2\zeta_{11}\mu_1}{h^2 z_x^2(1)EI(1)}, & \tau_{13} = -\dfrac{2\zeta_{22}\mu_2}{h^2 z_x^2(1)EI(1)}, \\
\tau_{12} = \mu_2 + \dfrac{2\zeta_{22}a(1)}{4h^2 z_x^2(1)EI(1)} - \dfrac{\zeta_{21}}{h^2 z_x^3(1)EI(1)} - \dfrac{2\zeta_{22}\mu_1}{h^2 z_x^2(1)EI(1)}. &
\end{cases}
\tag{85}
$$

By substituting the expression (84) into (44), the characteristic determinant of the eigenvalue problem (79) is given by the following:

$$
\Delta(\rho) = \left| \begin{matrix} [1]_2 \\ \rho\omega_1[1]_2 \\ 0 \\ 0 \end{matrix} \right. \tag{86}
$$

$$
\begin{matrix}
[1]_2 \\
\rho\omega_2[1]_2 \\
(\rho\omega_2)^2 e^{\rho\omega_2}[1 + (\tau_{21} + \tau_{22}\omega_2^{-2})\omega_2^{-2} + (\tau_{23}\omega_2^2 + \tau_{24} + \tau_{25}\omega_2^{-2})\rho^{-1}\omega_2^{-3} + \tau_{26}\rho\omega_2^{-1}]_2 \\
(\rho\omega_2)^3 e^{\rho\omega_2}[1 + \tau_{11}\omega_2^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_2^{-1} + (\tau_{12} + \tau_{13}\omega_2^{-2})\rho^{-1}\omega_2^{-3}]_2,
\end{matrix}
\tag{87}
$$

$$
\begin{matrix}
[1]_2 \\
\rho\omega_3[1]_2 \\
(\rho\omega_3)^2 e^{\rho\omega_3}[1 + (\tau_{21} + \tau_{22}\omega_3^{-2})\omega_3^{-2} + (\tau_{23}\omega_3^2 + \tau_{24} + \tau_{25}\omega_3^{-2})\rho^{-1}\omega_3^{-3} + \tau_{26}\rho\omega_3^{-1}]_2 \\
(\rho\omega_3)^3 e^{\rho\omega_3}[1 + \tau_{11}\omega_3^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_3^{-1} + (\tau_{12} + \tau_{13}\omega_3^{-2})\rho^{-1}\omega_3^{-3}]_2,
\end{matrix}
\tag{88}
$$

$$
\left. \begin{matrix}
0 \\
0 \\
(\rho\omega_4)^2 e^{\rho\omega_4}[1 + (\tau_{21} + \tau_{22}\omega_4^{-2})\omega_4^{-2} + (\tau_{23}\omega_4^2 + \tau_{24} + \tau_{25}\omega_4^{-2})\rho^{-1}\omega_4^{-3} + \tau_{26}\rho\omega_4^{-1}]_2 \\
(\rho\omega_4)^3 e^{\rho\omega_4}[1 + \tau_{11}\omega_4^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_4^{-1} + (\tau_{12} + \tau_{13}\omega_4^{-2})\rho^{-1}\omega_4^{-3}]_2
\end{matrix} \right| .
\tag{89}
$$

Developing the determinant, and after a straightforward computation, we obtain the following:

$$\Delta(\rho)= 2\sqrt{2}\tau_{26}\rho^7 e^{\rho\omega_4}\{e^{\rho\omega_2}+ie^{-\rho\omega_2}+[\mu_3 e^{\rho\omega_2}+\mu_4 e^{-\rho\omega_2}]\rho^{-1}+\mathcal{O}(\rho^{-2})\}, \quad (90)$$

with

$$\begin{cases} \mu_3=\dfrac{1}{\tau_{26}\sqrt{2}}(1+\tau_{21}-\tau_{22}-\tau_{11}+\tau_{11}\tau_{21}+\tau_{11}\tau_{22}+\tau_{26}(\mu_1+b_{21})-\tau_{26}\tau_{12}-\tau_{26}\tau_{13}),\\ \mu_4=\dfrac{1}{\tau_{26}\sqrt{2}}(1-\tau_{21}-\tau_{22}+\tau_{11}+\tau_{11}\tau_{21}-\tau_{11}\tau_{22}-\tau_{26}(\mu_1+b_{21})-\tau_{26}\tau_{12}+\tau_{26}\tau_{13}). \end{cases} \quad (91)$$

Then, we have $\Theta_{-10}=2\sqrt{2}\tau_{26}i$, $\Theta_{10}=2\sqrt{2}\tau_{26}$, and $\Theta_{00}=0$.

We notice that $\Theta_{00}^2-4\Theta_{-10}\Theta_{10}\neq 0$. Then, the following theorem follows directly.

Theorem 3.6. *The boundary conditions of the eigenvalue problem (79) are strongly regular. Therefore, the eigenvalues are asymptotically simple and separated.*

Now, we study the asymptotic behavior for the eigenvalues λ_n of problem (79). The equation $\Delta(\rho)=0$ implies

$$e^{\rho\omega_2}+ie^{-\rho\omega_2}+[\mu_3 e^{\rho\omega_2}+\mu_4 e^{-\rho\omega_2}]\rho^{-1}+\mathcal{O}(\rho^{-2})=0, \quad (92)$$

which can also be rewritten as follows:

$$e^{\rho\omega_2}+ie^{-\rho\omega_2}+\mathcal{O}(\rho^{-1})=0. \quad (93)$$

The equation $e^{\rho\omega_2}+ie^{-\rho\omega_2}=0$, has the solutions given by the following:

$$\rho_k=\left(k+\frac{3}{4}\right)\frac{\pi i}{\omega_2}, \quad k=1,2,\cdots. \quad (94)$$

Let $\tilde{\rho}_k$ be the solution of Equation (93). According to Rouché's theorem (see Naimark [13]), we obtain the following expression:

$$\widetilde{\rho}_k=\rho_k+\alpha_k=\left(k+\frac{3}{4}\right)\frac{\pi i}{\omega_2}+\alpha_k, \alpha_k=\mathcal{O}(k^{-1}),\quad k=K,K+1,\cdots, \quad (95)$$

where K is a sufficiently large positive integer.

By substituting it in Equation (92) and using the equality $e^{-\rho_k\omega_2}=ie^{\rho_k\omega_2}$, we get the following:

$$e^{\alpha_k\omega_2}-e^{-\alpha_k\omega_2}+\mu_3\widetilde{\rho}_k^{-1}e^{\alpha_k\omega_2}+\mu_4 i\widetilde{\rho}_k^{-1}e^{-\alpha_k\omega_2}+\mathcal{O}\left(\widetilde{\rho}_k^{-2}\right)=0. \quad (96)$$

Moreover, expanding the exponential function according to its Taylor series, we get the following:

$$\alpha_k=-\frac{\mu_3}{2\rho_k\omega_2}-\frac{\mu_4}{2\rho_k\omega_2}i+\mathcal{O}(k^{-2}), k=K,K+1,\cdots,$$
$$\widetilde{\rho}_k=\left(k+\frac{3}{4}\right)\frac{\pi i}{\omega_2}+\frac{1}{2}\frac{\mu_3}{\left(k+\frac{3}{4}\right)\pi}i-\frac{1}{2}\frac{\mu_4}{\left(k+\frac{3}{4}\right)\pi}+\mathcal{O}(k^{-2}),\quad k=K,K+1,\cdots. \quad (97)$$

Note that $\lambda_k=\tilde{\rho}_k^2/h^2$ and in sector S_1, $\omega_2=e^{i\frac{1}{4}\pi}$ and $\omega_2^2=i$. So, we have the following:

$$\lambda_k= -\frac{\sqrt{2}}{2h^2}(\mu_4+\mu_3)+\frac{1}{h^2}\left[\left(k+\frac{3}{4}\right)^2\pi^2+\frac{\sqrt{2}}{2}(\mu_3-\mu_4)\right]i+\mathcal{O}(k^{-1}), \quad (98)$$

where $k=K,K+1,\ldots$, with K large enough.

By applying the same proof to the sector S_2, the eigenvalues of the problem (79) can be obtained by a similar computation with the same choices as in Equation (60). Hence, in the sector S_2, the characteristic determinant $\Delta(\rho)$ is as follows:

$$\Delta(\rho)= -2\sqrt{2}\tau_{26}\rho^7 e^{\rho\omega_4}\{e^{\rho\omega_2}-ie^{-\rho\omega_2}-[\mu_3 e^{\rho\omega_2}+\mu_4 e^{-\rho\omega_2}]\rho^{-1}+\mathcal{O}(\rho^{-2})\}. \quad (99)$$

By a calculation similar to the one done in sector S_1, we have the following:

$$\widetilde{\rho}_k=\left(k+\frac{3}{4}\right)\frac{\pi i}{\omega_2}-\frac{1}{2}\frac{\mu_3}{\left(k+\frac{3}{4}\right)\pi}i-\frac{1}{2}\frac{\mu_4}{\left(k+\frac{3}{4}\right)\pi}+\mathcal{O}(k^{-2}),\quad k=K,K+1,\ldots. \quad (100)$$

In the sector S_2, using $\omega_2 = e^{i\frac{3}{4}\pi}$ and $\omega_2^2 = -i$, we obtain the following:

$$\lambda_k = -\frac{\sqrt{2}}{2h^2}(\mu_4+\mu_3) - \frac{1}{h^2}\left[\left(k+\frac{3}{4}\right)^2\pi^2 + \frac{\sqrt{2}}{2}(\mu_3-\mu_4)\right]i + \mathcal{O}(k^{-1}), \tag{101}$$

where $k = K, K+1, \ldots$, with K a large enough integer.

Theorem 3.7. *Consider that $\zeta_{12}>0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11}+\zeta_{22})^2$, then an asymptotic expression of the eigenvalues of the problem (79) is given by*

$$\lambda_k = -\frac{\sqrt{2}}{2h^2}(\mu_4+\mu_3) \pm \frac{1}{h^2}\left[\left(k+\frac{3}{4}\right)^2\pi^2 + \frac{\sqrt{2}}{2}(\mu_3-\mu_4)\right]i + \mathcal{O}(k^{-1}), \tag{102}$$

where $k = K, K+1, \ldots$, with K a large enough integer, and

$$\mu_4+\mu_3 = -2\sqrt{2}\mu_2 + \frac{\sqrt{2}h}{\zeta_{12}EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{-\frac{3}{4}} (\zeta_{12}\zeta_{21} - 4\zeta_{22}\zeta_{11} + m(1)\mathrm{EI}(1)), \tag{103}$$

$$\mu_2 = -\frac{h^2}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx = -\frac{h}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx. \tag{104}$$

Moreover, $\lambda_k (k = K, K+1, \cdots)$ with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

$$\lim_{k\to+\infty} Re\,(\lambda_k) = -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{\ell}{h\zeta_{12}EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{-\frac{3}{4}}, \tag{105}$$

where $\ell = \zeta_{12}\zeta_{21} - 4\zeta_{22}\zeta_{11} + m(1)\mathrm{EI}(1)$.

According to the condition (2), the constant ℓ is positive. From Equation (105), we find the following results:

Example 1. Suppose that $\zeta_{12}>0$ and $\zeta_{11}=\zeta_{22}=\zeta_{21}=\beta=0$. Then, the system (1) becomes the following:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) + \gamma(x)y_t(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ (EI(.)y_{xx})_x(1,t) = 0 & t>0, \\ -EI(1)y_{xx}(1,t) = (\zeta_{12}y_{xt} + \alpha y_x)(1,t), & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \tag{106}$$

This system has been studied by Jean-Marc et al. [6]. If $\zeta_{11}=\zeta_{22}=\zeta_{21}=\beta=0$, then $\ell = m(1)EI(1)$ and Equation (105) becomes the following:

$$\lim_{k\to+\infty} Re\,(\lambda_k) = -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{1}{h\zeta_{12}}(m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}, \tag{107}$$

The asymptotic expression (107) is equivalent to the one in Theorem 8 by Jean-Marc et al. [6].

Example 2. Suppose that $\zeta_{12}>0$ and $\gamma=0$, the system (1) is equivalent to the following:

$$\begin{cases} m(x)y_{tt}(x,t) + (EI(x)y_{xx})_{xx}(x,t) = 0, & 0<x<1, t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ -EI(1)y_{xx}(1,t) = (2\zeta_{11}y_t + \zeta_{12}y_{xt} + \alpha y_x)(1,t), & t>0, \\ (EI(.)y_{xx})_x(1,t) = (\zeta_{21}y_t + 2\zeta_{22}y_{xt} + \beta y)(1,t), & t>0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0<x<1. \end{cases} \tag{108}$$

which has been studied by Aouragh and Yebari [1], where they get the same result as Equation (105) in the uniform case $m(x) = EI(x) = 1$.

Remark 3.8. In the cited examples, the authors proved that the systems are exponentially stable.

4. Riesz Basis Property and Exponential Stability of the System in the General Case

In this section, we consider the system (1) in the general case with the conditions $\zeta_{12} > 0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2$.

4.1. Riesz Basis Property of the System. Here, we discuss the Riesz basis property of the eigenfunctions of the operator $\mathscr{A}$ of the system by following an idea due to Wang et al. [5]. We begin by considering a bounded invertible operator defined on $\mathscr{H}$ by the following:

$$\mathscr{L}(f, g) = (\phi, \psi), \tag{109}$$

with

$$\begin{gathered}\phi(x) = f(z), \quad \psi(x) = g(z), \\ z = \frac{1}{h}\int_0^x \left(\frac{m(\xi)}{EI(\xi)}\right)^{1/4} d\xi, \qquad h = \int_0^1 \left(\frac{m(\xi)}{EI(\xi)}\right)^{1/4} d\xi .\end{gathered} \tag{110}$$

Then, we define the ordinary differential operator as follows:

$$\begin{cases} L(f) = f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z), \\ \mu(z) = h^2 d(z), \\ B_1(f) = f(0) = 0, B_2(f) = f'(0) = 0, \\ B_3(f) = f''(1) + k_{11}(1)f'(1) + k_{12}(1)f(1) = 0, \\ B_4(f) = f'''(1) + k_{21}(1)f''(1) + k_{22}(1)f'(1) + k_{23}(1)f(1) = 0, \end{cases} \tag{111}$$

Let $\mathbb{H}$ be the Hilbert space defined by Wang [7] and define the operator $\mathbb{A}$ in $\mathbb{H}$ by the following:

$$\begin{aligned} &\mathbb{A}(f, g) = (g, -L(f) - \mu(x)g) \\ &D(\mathbb{A}) = \left\{ (f, g) \in \mathbb{H} \mid A(f, g) \in \mathbb{H}, B_j(f) = 0, j = 1, .., 4 \right\}. \end{aligned} \tag{112}$$

Let $\eta \in \sigma(\mathbb{A})$ be an eigenvalue of $\mathbb{A}$ and (f, g) the corresponding eigenfunction. Then we obtain $g = \eta f$ and f satisfies as follows:

$$\begin{aligned} &f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) \\ &\quad + \eta\mu(z)f(z) + \eta^2 f(z) = 0. \end{aligned} \tag{113}$$

Now by taking

$$\lambda = \frac{\eta}{h^2} \quad \text{and} \quad \mathscr{L}(f, g) = (\phi(x), \psi(x)) \;, \tag{114}$$

we see that $\psi = \lambda\phi$ and ϕ satisfies

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \dfrac{\lambda^2 m(x)}{EI(x)}\phi(x) + \dfrac{\lambda\gamma(x)}{EI(x)}\phi(x) = 0, \\ \phi(0) = \phi'(0) = 0, \\ \phi''(1) = -\dfrac{\zeta_{12}\lambda + \alpha}{EI(1)}\phi'(1) - \dfrac{2\zeta_{11}\lambda}{EI(1)}\phi(1), \\ \phi'''(1) = -\dfrac{EI'(1)}{EI(1)}\phi''(1) + \dfrac{2\zeta_{22}\lambda}{EI(1)}\phi'(1) + \dfrac{\zeta_{21}\lambda + \beta}{EI(1)}\phi(1). \end{cases} \tag{115}$$

Hence, we have that: $\eta \in \sigma(\mathbb{A}) \Leftrightarrow \lambda \in \sigma(\mathscr{A})$.

Theorem 4.1. *Let operator $\mathscr{A}$ be defined by Equations (9) and (10).Then the eigenvalues of operator$\mathscr{A}$are all simple except for finitely many of them, and the generalized eigenfunctions of operator$\mathscr{A}$form a Riesz basis for the Hilbert state space$\mathscr{H}$.*

Proof. We have shown that the boundary problem (79) is strongly regular (see Theorem 3.6). Therefore, the eigenvalues are separated and simple except for finitely many of them. Thus, the first statement follows. Moreover, Theorem 4.1.1 in the study of [7] ensures that the strong regularity of the boundary problem leads to the sequence of generalized eigenfunctions $F_n = (f_n, \eta_n f_n)$ of operator $\mathbb{A}$ forms a Riesz basis for $\mathbb{H}$. Since $\mathscr{L}$ is bounded and invertible on $\mathbb{H}$, it follows that $\psi_n = (\phi_n, \lambda_n \phi_n) = \mathscr{L} F_n$ also forms a Riesz basis on $\mathscr{H}$. □

4.2. Exponential Stability of the System (1). Referring to the study of Curtain and Zwart [14], we note that the Riesz basis property implies the spectrum-determined growth condition and Equation (105) describes the asymptote of $\sigma(\mathscr{A})$, for any small $\varepsilon > 0$ there are only finitely many eigenvalues of $\mathscr{A}$ in the following half-plane:

$$\begin{aligned} \sum : Re\lambda > -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} \\ dx - \frac{\ell}{h\zeta_{12}EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{-\frac{3}{4}} + \varepsilon. \end{aligned} \tag{116}$$

Following Theorem 2.4 in the study of Gao [10], all the properties of operator $\mathscr{A}$ found above, allow us to claim that for the semigroup $\{e^{\mathscr{A}t}\}_{t\geq 0}$ generated by $\mathscr{A}$ the spectrum-determined growth condition holds the following:

$$\omega(\mathscr{A}) = s(\mathscr{A}), \tag{117}$$

with

$$\begin{aligned} \omega(\mathscr{A}) = \lim_{t\to\infty}\frac{1}{t}\|e^{\mathscr{A}t}\|_{\mathscr{H}} \quad \text{and} \\ s(\mathscr{A}) = \sup\{Re(\lambda) \mid \lambda \in \sigma(\mathscr{A})\}. \end{aligned} \tag{118}$$

By using Theorem 4.1, the exponential stability of system (1) can be concluded.

Theorem 4.2. *Consider that $\zeta_{12} > 0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2$. If $\gamma(x) > 0$, then the system (1) is exponential stable for any $\zeta_{11}, \zeta_{22}, \zeta_{21} \geq 0$ and $\alpha, \beta \geq 0$. That is, there are constants $M > 0$ and $w > 0$ such as the energy*

$$E(t) = \frac{1}{2}\left(\int_0^1 m(x)y_t^2 dx + \int_0^1 EI(x)y_{xx}^2 dx + \alpha y_x^2(1) + \beta y^2(1)\right), \tag{119}$$

of system (1) satisfies $E(t) \leq ME(0)e^{-wt}, \forall t \geq 0$, for any initial condition $(y(x,0), y_t(x,0))^T \in \mathscr{H}$.

Proof. The operator $\mathscr{A}$ is dissipative. In effect, For $\Psi = (\phi, \psi)^T \in D(\mathscr{A})$,

$$Re\left(\langle \mathscr{A}\Psi, \Psi\rangle_{\mathscr{H}}\right) = -(\zeta_{21}|\psi(1)|^2 + 2(\zeta_{22} + \zeta_{11})|\psi'(1)\psi(1)| + \zeta_{12}|\psi'(1)|^2) - \int_0^1 \gamma(x)|\psi(x)|^2 d(x) \leq 0. \tag{120}$$

We also know that $\mathscr{A}$ generates a semigroup of contractions $\{e^{\mathscr{A}t}\}_{t\geq 0}$ on $\mathscr{H}$. Moreover, the spectrum of the operator $\mathscr{A}$ has an asymptote as follows:

$$Re\lambda \backsim -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{\ell}{h\zeta_{12}EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{-\frac{3}{4}}. \tag{121}$$

Then, the study of the exponential stability is equivalent to verifying that $Re\lambda < 0$. Let $\lambda = ib$ with $b \in \mathbb{R}^*$ be an eigenvalue of operator $\mathscr{A}$ on the imaginary axis and $\Psi = (\phi, \psi)^T$ be the corresponding eigenfunction, then $\psi = \lambda\phi$.

$Re(\langle \mathscr{A}\Psi, \Psi\rangle_{\mathscr{H}}) = \|\Psi\|_{\mathscr{H}}^2 Re\lambda = 0$, then

$$\begin{aligned} -\zeta_{21}|\psi(1)|^2 - 2(\zeta_{22} + \zeta_{11})|\psi'(1)\psi(1)| - \zeta_{12}|\psi'(1)|^2 \\ -\int_0^1 \gamma(x)|\psi(x)|^2 d(x) = 0. \end{aligned} \tag{122}$$

Since $\gamma(x) > 0$ and $\psi(x)$ are continuous with $\zeta_{12} > 0, \zeta_{11}, \zeta_{22}, \zeta_{21} \geq 0$, we obtain the following:

$\psi'(1) = 0$ and

$$\gamma(x)|\psi(x)|^2 = 0, \forall x \in [0,1]. \tag{123}$$

Then,

$$\phi'(1) = 0 \text{ and } \psi \equiv 0, \forall x \in [0,1]. \tag{124}$$

Moreover $\psi = \lambda\phi$, then from Equations (74) and (124), the following differential equation is satisfied by $\phi(x)$:

$$\begin{cases} \lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' + \lambda\gamma(x)\phi(x) = 0, & 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi'(1) = \phi''(1) = \phi'''(0) = \phi(1) = 0. \end{cases} \tag{125}$$

We show with the help of Rolle's Theorem that the null function is the unique solution of Equation (125). For this, the reader is invited to follow a method used by Jean-Marc et al. [6].

From Theorem 4.1 and the spectrum-determined growth condition, the system is exponentially stable for any $\zeta_{12} > 0$ and $\zeta_{12}\zeta_{21} \geq (\zeta_{11} + \zeta_{22})^2$ with $\alpha, \beta \geq 0$. □

Now, using an idea of Wang [7], we study the situation where $\gamma(x)$ is continuous and indefinite in [0,1]. We have the following theorem.

Theorem 4.3. *Note* $\gamma^+(x) = \max(\gamma(x), 0), \gamma^-(x) = \max(-\gamma(x), 0)$ *and let*

$$\mathscr{A}^+(f, g)^T = \left(g(x), -\frac{1}{m(x)}(EI(x)f''(x))'' + \gamma^+(x)g(x)\right), \tag{126}$$

$$\forall (f, g)^T \in D(\mathscr{A}^+) = D(\mathscr{A}), \tag{127}$$

$$\mathscr{B}^-(f, g)^T = \left(0, \frac{\gamma^-(x)g(x)}{m(x)}\right), \forall (f, g)^T \in \mathscr{H}. \tag{128}$$

Hence, the operator $\mathscr{A}$ *can be written as* $\mathscr{A} = \mathscr{A}^+ + \mathscr{B}^-$. *Note* $s(\mathscr{A}^+) = \sup\{Re\lambda | \lambda \in \sigma(\mathscr{A}^+)\}$. *If*

$$\max_{x\in[0,1]} \left\{\frac{\gamma^-(x)}{m(x)}\right\} < |s(\mathscr{A}^+)|, \tag{129}$$

then the system (1) is exponentially stable.

Proof. According to Section 2 of this paper, the operator $\mathscr{B}^-$ is bounded and symmetric, so it is a self-adjoint operator (see Brezis [15]) and

$$\|\mathscr{B}^-\| = \max_{x\in[0,1]} \left\{\frac{\gamma^-(x)}{m(x)}\right\}. \tag{130}$$

By Theorem 4.2 and the definition of operator $\mathscr{A}^+$, $\{e^{\mathscr{A}^+ t}\}$ is a contraction semigroup and $s(\mathscr{A}^+) < 0$. Applying the perturbation theory of linear operator's semigroup (see Pazy [12]), we have $\lambda \in \sigma(\mathscr{A})$ whenever $Re\lambda > s(\mathscr{A}^+) + \|\mathscr{B}^-\|$. Furthermore, Theorem 4.1 ensures that

$$\omega(A) = s(A) \le s(\mathscr{A}^+) + \|\mathscr{B}^-\|. \tag{131}$$

Therefore, the system (1) is exponentially stable if $\|\mathscr{B}^-\| < |s(\mathscr{A}^+)|$. □

5. Conclusion

During our analysis, after showing the basic properties of the operator, we established an asymptotic expression of the operator, which generalizes several other results. Then, using the Riesz basis property, we have shown that the system is exponentially stable in the general case following the sign of γ. In addition, our results can also be extended to other Euler–Bernoulli beam problems.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] M. D. Aouragh and N. Yebari, "Stabilisation exponentielle d'une équation de poutres d'Euler-Bernoulli à coefficients variables," *Annales Mathématiques Blaise Pascal*, vol. 16, no. 2, pp. 483–510, 2009.

[2] B.-Z. Guo, "Riesz basis property and exponential stability of controlled Euler–Bernoulli beam equation with variable coefficients," *SIAM Journal on Control and Optimization*, vol. 40, no. 6, pp. 1905–1923, 2002.

[3] C. J. J. Koffi, G. J.-M. Bomisso, K. A. Touré, and A. Coulibaly, "Riesz basis property and exponential stability for a damped system and controlled dynamically," *International Journal of Numerical Methods and Applications*, vol. 22, 2022.

[4] K. A. Touré, A. Coulibaly, and A. A. Hermith Kouassi, "Riesz basis, exponential stability of variable coefficients Euler-Bernoulli beams with an indefinite damping under a force control in position and velocity," *Electronic Journal of Differential Equations*, vol. 2015, no. 54, pp. 1–20, 2015.

[5] J.-M. Wang, G.-Q. Xu, and S.-P. Yung, "Riesz basis property, exponential stability of variable coefficient Euler–Bernoulli beams with indefinite damping," *IMA Journal of Applied Mathematics*, vol. 70, no. 3, pp. 459–477, 2005.

[6] B. G. Jean-Marc, T. K. Augustin, and Y. Gozo, "Stabilization of variable coefficients euler-bernoulli beam with viscous damping under a force control in rotation and velocity rotation," *Journal of Mathematics Research*, vol. 9, no. 6, pp. 1–13, 2017.

[7] J.-M. Wang, "*Riesz basis property of some infinite-dimensional control problems and its applications*," Ph.D. ThesisThe University of Hong Kong, 2004.

[8] G. D. Birkhoff, "On the asymptotic character of the solutions of certain linear differential equations containing a parameter," *Transactions of the American Mathematical Society*, vol. 9, no. 2, pp. 219–231, 1908.

[9] G. D. Birkhoff, "Boundary value and expansion problems of ordinary linear differential equations," *Transactions of the American Mathematical Society*, vol. 9, no. 4, pp. 373–395, 1908.

[10] B. Z. Guo, "On the boundary control of a hybrid system with variable coefficients," *Journal of optimization Theory and Applications*, vol. 114, no. 2, pp. 373–395, 2002.

[11] B.-Z. Guo and J.-M. Wang, "Riesz basis generation of abstract second order partial differential equation systems with general non-separated boundary conditions," *Numerical Functional Analysis and Optimization*, vol. 27, no. 3-4, pp. 291–328, 2006.

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Vol. 44, Springer-Verlag, New York, 1983.

[13] M. A. Naimark, *Linear Differential Operators*, Vol. 1, F. Ungar, New York, 1967.

[14] R. F. Curtain and H. Zwart, "An introduction to infinite dimensional linear system theory," in *Texts in Applied Mathematics*, vol. 21, Springer-Verlag, Berlin, 1995.

[15] H. Brezis, "Analyse fonctionnelle," in *Théorie et Applications*, Masson, Paris, 1983.