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On the Convolution of the k-Lucas Sequences

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we study the iterated convolution of the k-Lucas sequences in a form similar to the iterated convolution of the k-Fibonacci sequences [1].
A particular case is for the self-convolution of these sequences. Moreover, the generating functions of all these convolved sequences, we find the recurrence relation between the terms of the resulting sequences.

Keywords: K-Fibonacci and k-Lucas numbers; convolution; generating function; recurrence relation.

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1 Introduction

In Hoggat et al. [2] the convolved Fibonacci sequences are defined in the form $F_n^{(r)} = \sum_{j=0}^{n} F_j F_{n-j}^{(r-1)}$ with initial condition $F_n^{(0)} = F_n$ and where F_n are the classical Fibonacci numbers.

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The aim of this paper consists of extending this concept to case of the k-Lucas numbers.

Definition 1. *For any integer* $k \geq 1$ *, the k-Fibonacci sequence, say* $\{F_{k,n}\}$ *is defined recurrently by:* $F_{k,0} = 0,\ F_{k,1} = 1,\ F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ *for n ≥ 1.*

The characteristic equation from the definition is $r^2 = k\,r + 1$ whose solutions are $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$ that verify $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 + \sigma_2 = k$, $\sigma_1 - \sigma_2 = \sqrt{k^2+4}$, $\sigma_1 > 0,\ \ \sigma_2 < 0$, $\sigma^2 = k\,\sigma + 1$. For the properties of the k-Fibonacci numbers, see Falcon and Plaza [3,4]. In particular, the Binet Identity is $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$.

The generating function of the -Fibonacci numbers is $f(k,x) = \frac{x}{1-k\,x-x^2}$. Finally, we define the k-Fibonacci numbers of negative index as $F_{k,-n} = (-1)^{n+1} F_{k,n}$

Definition 2. *For any integer number* $k \geq 1$, *the k-Lucas sequence, say* $\{L_{k,n}\}$ *is defined recurrently by:* $L_{k,0} = 2,\ L_{k,1} = k,\ L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ *for n ≥ 1.*

The Binet Identity for the k-Lucas numbers is $L_{k,n} = \sigma_1^n + \sigma_2^n$.

The k-Lucas numbers are related to the k-Fibonacci numbers by the relation $L_{k,n} = F_{k,n-1} + F_{k,n+1}$. From this relation it is easy to prove that $F_{k,n} = \frac{L_{k,n-1} + L_{k,n+1}}{k^2+4}$.

Since the recurrence relation indicated in the definition is the same for the k-Fibonacci numbers as for the k-Lucas numbers, the denominator of the generating function is the same for both. The numerator does vary since it depends on the initial conditions: (0, 1) for the k-Fibonacci numbers and (2, k) for the k-Lucas. Then the generating function of the k-Lucas numbers is Benjamin [5] $l(x) = \frac{2-k\,x}{1-k\,x-x^2}$. Moreover, $L_{k,-n} = (-1)^n L_{k,n}$

1.1 Convolved K-fibonacci numbers

A convolved k–Fibonacci sequence [1] is obtained by applying a convolution operation to the k–Fibonacci sequence one or more times. Specifically, define $F_{k,n}^{(0)} = F_{k,n}$ and

$$F_{k,n}^{(r)} = \sum_{j=0}^{n} F_{k,j} F_{k,n-j}^{(r-1)} \tag{1}$$

In particular, for k = 1, the classical numbers are $F_{k,n}^{(r)} = F_n^{(r)}$. For k = 2, we have the Pell numbers. For k = 2, 3, 4… no convolved k-Fibonacci sequence is indexed in OEIS [6] except for r = 0.

By induction [1] we have proven the following identities for the elements $F_{k,n}^{(r)}$ of the convolved k-Fibonacci sequence: $F_{k,n}^{(r)} = 0$ for $0 \le n \le r$ and $F_{k,\,r+p+1}^{(r)} = \sum_{j=0}^{n} \binom{p-j}{j}\binom{r+p-j}{p-j} k^{p-2j}$

If r = 0 this formula becomes Formula (1) to obtain the k-Fibonacci numbers.

1.2 Recurrence relation

The convolved k-Fibonacci sequences verify the following recurrence relation: for n, r ≥ 1, $F_{k,n+1}^{(r)} = k\,F_{k,n}^{(r)} + F_{k,n-1}^{(r)} + F_{k,n}^{(r-1)}$ and this formula was proven by induction [1].

1.1 Convolved k-Fibonacci numbers and the Fibonacci polynomials

The sequences $\left\{F_{k,n}^{(r)}\right\}$ are related to the k-Fibonacci polynomials by the relation $F_{k,n}^{(r)} = \frac{1}{r!}\frac{d^r F_{k,n}}{dk^r}$ where $\frac{d^r F_{k,n}}{dk^r}$ is the derivative of order "r" with respect to "k" of the k-Fibonacci numbers of Definition 1.

1.3 Generating function

In Herbert [7] Formula (2.2.3) the following formula is proven: *If f(x) and g(x) are the respective generating functions of the sequences* $\left\{u_n\right\}$ and $\left\{v_n\right\}$ *then f(x) · g(x) is the generating function of the convolution of these sequences.*

So, and taking into account that the generating function of the k-Fibonacci numbers is $f(k,x) = \frac{x}{1-k\,x-x^2}$,

the generating function of the convolved k-Fibonacci sequences is $f(x,k,r) = \left(\frac{x}{1-k\,x-x^2}\right)^{r+1}$ (2)

As a special case $F_{k,n}^{(1)}$ is the self convolution of the k-Fibonacci numbers.

Sometimes, the convolution of the sequences $U = \left\{u_n\right\}$ and $V = \left\{v_n\right\}$ is represented as $U \otimes V = \left\{u_n \otimes v_n\right\}$ so $F_{k,n}^{(1)} = \left\{F_{k,n} \otimes F_{k,n}\right\}$

Theorem 1. *The elements of the self convolution of the k-Fibonacci numbers verifies the formula*

$$F_{k,n}^{(1)} = \sum_{j=0}^{n} F_{k,j} F_{k,n-j} = \frac{n\,L_{k,n} - k\,F_{k,n}}{k^2+4} \tag{3}$$

For instance, for the Pell numbers, and after simplifying it is $P_n^{(1)} = \sum_{j=0}^{n} P_j P_{n-j} = \frac{(n-1)P_n + n\,P_{n-1}}{4}$

Theorem 2. *The elements of the self convolution of the k-Fibonacci numbers verifies the recurrence relation*

$$F_{k,n+1}^{(1)} = 2k\,F_{k,n}^{(1)} - (k^2-2)F_{k,n-1}^{(1)} - 2k\,F_{k,n-2}^{(1)} - F_{k,n-3}^{(1)} \tag{4}$$

A way to find these recurrences if we take into account that the denominator of the generating function corresponds to the recurrence relation between the terms of the sequence.

The expansion of the respective denominators leads us to the recurrence relation of the corresponding sequence by simply changing x^p by $F_{k,n-p}^{(r)}$. So, from Equation (2), and taking into account that $1 = x^0 = F_{k,n}^{(0)}$:

$$r = 0 \to 1 - k\,x - x^2 = 0 \to 1 = k\,x + x^2 \to F_{k,n}^{(0)} = k\,F_{k,n-1}^{(0)} + F_{k,n-2}^{(0)} \quad \left(F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}\right)$$

$$r = 1 \to \left(1 - k\,x - x^2\right)^2 = 0 \to (1-2k) + (k^2-2)x^2 + 2k\,x^3 + x^4 = 0 \to$$
$$\to 1 = 2k - (k^2-2)x^2 - 2k\,x^3 - x^4 = 0 \to$$
$$\to F_{k,n}^{(1)} = 2k\,F_{k,n-1}^{(1)} - (k^2-2)F_{k,n-2}^{(1)} - 2k\,F_{k,n-3}^{(1)} - F_{k,n-4}^{(1)}$$

$$r = 2 \to \left(1 - k\,x - x^2\right)^3 = 0 \to \cdots \to$$
$$\to 1 = 3k\,x + (3-3k^2)x^2 - (6k-k^3)x^3 - (3-3k^2)x^4 + 3k\,x^5 - x^6 \to$$
$$\to F_{k,n}^{(2)} = 3k\,F_{k,n-1}^{(2)} + (3-3k^2)F_{k,n-2}^{(2)} - (6k-k^3)F_{k,n-3}^{(2)} - (3-3k^2)F_{k,n-4}^{(2)} + 3k\,F_{k,n-5}^{(2)} - F_{k,n-6}^{(2)}$$

There necessarily 2(r + 1) initial conditions for these relations.

Corollary 1. *For the classical Fibonacci numbers (k = 1), the respective relations are*

$$F_n^0 = F_n = F_{n-1} + F_{n-2}$$
$$F_n^{(1)} = F_n \otimes F_n = 2F_{n-1}^{(1)} + F_{n-2}^{(1)} - 2F_{n-3}^{(1)} - F_{n-4}^{(1)}$$
$$F_n^{(2)} = F_n \otimes F_n^{(1)} = 3F_{n-1}^{(2)} - 5F_{n-3}^{(2)} + 3F_{n-5}^{(2)} - F_{n-6}^{(2)}$$

2 On the Convolution of the K-Lucas Sequences

The convolution of Fibonacci and Lucas numbers has been studied by many authors. The convolution of the k-Fibonacci numbers has also been studied, some of the results of which have been presented in the previous section. We want to apply the results obtained in that case to the k-Lucas sequences.
The definition of the convolved k-Lucas sequences is similar to definition of the convolution of the k-Fibonacci numbers (1.1).

Definition 3. *With the initial condition* $L_{k,n}^{(0)} = L_{k,n}$ *, the convolved k-Lucas sequences is defined as* $L_{k,n}^{(r)} = \sum_{j=0}^{n} L_{k,j} L_{k,n-j}^{(r-1)}$

Taking into account that $L_{k,-n} = (-1)^n L_{k,n}$ it i also $L_{k,-n}^{(r)} = (-1)^n L_{k,n}^{(r)}$. From the definition, we can obtain, $L_{k,0}^{(r)} = 2^{r+1}$, $L_{k,1}^{(r)} = 2^r(r+1)k$, $L_{k,2}^{(r)} = 2^{r-2}(r+1)(r+4)k^2 + 2^{r+1}(r+1)$.

Moreover

$$L_k^{(0)} = L_k = \left\{2, k, k^2+2, k^3+3k, k^4+4k^2+2, \ldots\right\}$$
$$L_k^{(1)} = L_k \otimes L_k = \left\{L_{k,n}^{(1)}\right\} = \left\{4, 4k, 5k^2+8, 6k^3+16k, 7k^4+26k^2+12, \ldots\right\}$$
$$L_k^{(2)} = \left\{L_{k,n}^{(2)}\right\} = \left\{8, 12k, 18k^2+24, 25k^3+60k, 33k^4+114k^2+48, \ldots\right\}$$
$$L_k^{(3)} = \left\{L_{k,n}^{(3)}\right\} = \left\{16, 32k, 56k^2+64, 88k^3+192k, 129k^4+416k^2+160, \ldots\right\}$$

For k = 1 and the classical Fibonacci sequences [8] it is

$$L^{(0)} = L = \{L_n\} = \{2,1,3,4,7,11,18,29,...\}$$
$$L^{(1)} = L \otimes L = \{L_n^{(1)}\} = \{4,4,13,22,45,82,152,274,...\}$$
$$L^{(2)} = L \otimes L^{(1)} = \{L_n^{(2)}\} = \{8,12,42,85,195,399,816,1611,...\}$$
$$L^{(3)} = L \otimes L^{(2)} = \{L_n^{(3)}\} = \{16,32,120,280,705,1588,3526,7520,...\}$$

The classical Lucas sequence $L = \{L_n\} = \{2,1,3,4,7,11,18,29,...\}$ is indexed in the OEIS [6] as A000032 and the self-convolution $L^{(1)} = L \otimes L$ as A099924. No other of these convolutions is indexed in the OEIS.

Theorem 3. *The first convolution of the k-Lucas numbers is called the self convolution of these numbers and verify the formula* [9]

$$L_{k,n} \otimes L_{k,n} = \sum_{j=0}^{n} L_{k,j} L_{k,n-j} = (n+1)L_{k,n} + 2F_{k,n+1} \tag{5}$$

Proof.

Applying the Binet Identity

$$\begin{aligned}
L_{k,n} \otimes L_{k,n} &= \sum_{j=0}^{n} L_{k,j} L_{k,n-j} = \sum_{j=0}^{n} \left(\sigma_1^j + \sigma_2^j\right)\left(\sigma_1^{n-j} + \sigma_2^{n-j}\right) \\
&= \sum_{j=0}^{n} \left(\sigma_1^n + \sigma_2^n + \sigma_1^n \left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n \left(\frac{\sigma_1}{\sigma_2}\right)^j \right) \\
&= (n+1)L_{k,n} + \sigma_1^n \sum_{j=0}^{n} \left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n \sum_{j=0}^{n} \left(\frac{\sigma_1}{\sigma_2}\right)^j \\
&= (n+1)L_{k,n} + \sigma_1^n \frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{n+1} - 1}{\frac{\sigma_2}{\sigma_1} - 1} + \sigma_2^n \frac{\left(\frac{\sigma_1}{\sigma_2}\right)^{n+1} - 1}{\frac{\sigma_1}{\sigma_2} - 1} \\
&= (n+1)L_{k,n} + \frac{\sigma_2^{n+1} - \sigma_1^{n+1}}{\sigma_2 - \sigma_1} + \frac{\sigma_1^{n+1} - \sigma_2^{n+1}}{\sigma_1 - \sigma_2} = (n+1L_{k,n} + 2F_{k,n}
\end{aligned}$$

Because the denominator of the generating function of the k-Lucas numbers and the k-Fibonacci numbers is the same, the recurrence relation between the terms of the sequences $F_{k,n}^{(r)}$ and $L_{k,n}^{(r)}$ is the same for both convolved sequences.

For instance, the terms of the self-convolution of the k-Lucas numbers verify the relation $L_{k,n+1}^{(1)} = 2k\,L_{k,n}^{(1)} - (k^2-2)L_{k,n-1}^{(1)} - 2k\,L_{k,n-2}^{(1)} - L_{k,n-3}^{(1)}$ that, in the classical case (k = 1) takes the form $L_{n+1}^{(1)} = 2L_n^{(1)} + L_{n-1}^{(1)} - 2L_{n-2}^{(1)} - L_{n-3}^{(1)}$ with initial conditions $L_0^{(1)} = 4$, $L_1^{(1)} = 4$, $L_2^{(1)} = 13$ and $L_3^{(1)} = 22$

Theorem 4 *The convolved k-Lucas numbers verify the recurrence relation*

$$L_{k,n+1}^{(r)} = k\, L_{k,n}^{(r)} + L_{k,n-1}^{(r)} + \sum_{i=0}^{r-1}\left(L_{k,n+1}^{(i)} + L_{k,n-1}^{(i)}\right)$$

Proof.

For r = 1 and from Equation (5)

$$\begin{aligned} L_{k,n+1}^{(r)} &= (n+2)L_{k,n+1} + 2F_{k,n+2} = (n+2)\left(k\,L_{k,n} + L_{k,n-1}\right) + 2\left(k\,F_{k,n} + F_{k,n}\right) \\ &= k\left((n+1)\,L_{k,n} + 2F_{k,n+1}\right) + \left(n\,L_{k,n-1} + 2F_{k,n}\right) + \left(k\,L_{k,n} + L_{k,n-1}\right) \\ &= k\,L_{k,n}^{(1)} + L_{k,n-1}^{(1)} + L_{k,n+1}^{(0)} + L_{k,n-1}^{(0)} \end{aligned}$$

Let us suppose this formula is true until *r*. Then

$$\begin{aligned} L_{k,n+1}^{(r+1)} &= \sum_{j=0}^{n+1} L_{k,j} L_{k,n+1-j}^{(r)} = \sum_{j=0}^{n+1} L_{k,j}\left[k\,L_{k,n-j}^{(r)} + L_{k,n-1-j}^{(r)} + \sum_{i=0}^{r-1}\left(L_{k,n+1-j}^{(i)} + L_{k,n-1-j}^{(i)}\right)\right] \\ &= k\sum_{j=0}^{n+1} L_{k,j} L_{k,n-j}^{(r)} + \sum_{j=0}^{n+1} L_{k,j} L_{k,n-1-j}^{(r)} + \sum_{i=0}^{r-1}\sum_{j=0}^{n+1} L_{k,j} L_{k,n+1-j}^{(i)} + \sum_{i=0}^{r-1}\sum_{j=0}^{n+1} L_{k,j} L_{k,n-1-j}^{(i)} \\ &= k\sum_{j=0}^{n} L_{k,j} L_{k,n-j}^{(r)} + \sum_{j=0}^{n-1} L_{k,j} L_{k,n-1-j}^{(r)} + \sum_{i=0}^{r-1} L_{k,n+1}^{(i+1)} + \sum_{i=0}^{r-1} L_{k,n-1}^{(i+1)} \\ &= k\,L_{k,n}^{(i+1)} + L_{k,n+1}^{(i+1)} + \sum_{i=0}^{r}\left(L_{k,n+1}^{(i)} + L_{k,n-1}^{(i)}\right) \end{aligned}$$

Both self-convolutions of the k-Fibonacci and k-Lucas numbers are related to each other by mean of the equation $L_k^{(1)} + (k^2+4)F_k^{(1)} = 2(n+1)L_{k,n}$. That is $\sum_{j=0}^{n} L_{k,j}L_{k,n-j} + (k^2+4)\sum_{j=0}^{n} F_{k,j}F_{k,n-j} = 2(n+1)L_{k,n}$

The terms of the sequences of the convolved k-Lucas sequence verify the recurrence relation

$L_{k,n}^{(1)} = 2k\,L_{k,n-1}^{(1)} + (2-k^2)L_{k,n-2}^{(1)} - 2k\,L_{k,n-3}^{(1)} - L_{k,n-4}^{(1)}$ that for the classical Lucas numbers is $L_n \otimes L_n = 2L_{n-1}^{(1)} + L_{n-2}^{(1)} - 2L_{n-3}^{(1)} - L_{n-4}^{(1)}$

3 Second Convolved k-Lucas Sequence

After studying the self-convolution of the k-Lucas sequence, this section is dedicated to studying the second convolution, that is, $L_k^{(2)} = \left\{L_{k,n}^{(2)}\right\}$

Theorem 5. *The second convolved k-Lucas verifies the relation* $L_{k,n}^{(2)} = \frac{n+2}{2}\left((n+1)L_{k,n} + 6F_{k,n+1}\right)$

Proof.

The self-convolution of the k-Lucas sequence is $L_k^{(1)} = L_k \otimes L_k = \left\{L_{k,n} \otimes L_{k,n}\right\}$ being $L_{k,n} \otimes L_{k,n} = \sum_{j=0}^{n} L_{k,j} L_{k,n-j} = (n+1)L_{k,n} + 2F_{k,n+1}$.

Besides $\sum_{j=0}^{n} F_{k,j} L_{k,n-j} = (n+1)F_{k,n}$ and therefore $\sum_{j=0}^{n} F_{k,j} L_{k,n+1-j} = (n+2)F_{k,n+1}$.

Finally $\sum_{j=0}^{n} j\, r^{j} = \frac{r^{n+1}\left(n(r-1)-1\right)+r}{\left(r-1\right)^{2}}$

Then

$$
\begin{aligned}
L_{k,n}^{(2)} &= \sum_{j=0}^{n} L_{k,j} L_{k,n-j}^{(1)} = \sum_{j=0}^{n} L_{k,j} \left((n+1-j)L_{k,n-j} + 2F_{k,n+1-j}\right) \\
&= (n+1)\sum_{j=0}^{n} L_{k,j} L_{k,n-j} - \sum_{j=0}^{n} j\, L_{k,j} L_{k,n-j} + 2\sum_{j=0}^{n} L_{k,j} L_{k,n+1-j} \\
&= (n+1)\left((n+1)L_{k,n} + 2F_{k,n+1}\right) + 2(n+2)F_{k,n+1} - C
\end{aligned}
$$

Where

$$
C = \sum_{j=0}^{n} j\left(\sigma_1^{j}+\sigma_2^{j}\right)\left(\sigma_1^{n-j}+\sigma_2^{n-j}\right) = \sum_{j=0}^{n} j\left(\sigma_1^{n}+\sigma_2^{j}+\sigma_1^{n}\left(\frac{\sigma_2}{\sigma_1}\right)^{j}+\sigma_2^{n}\left(\frac{\sigma_1}{\sigma_2}\right)^{j}\right)
$$

$$
= L_{k,n}\sum_{j=0}^{n} j + \sigma_1^{n}\sum_{j=0}^{n} j\left(\frac{\sigma_2}{\sigma_1}\right)^{j} + \sigma_2^{n}\sum_{j=0}^{n} j\left(\frac{\sigma_1}{\sigma_2}\right)^{j}
$$

$$
= \frac{n(n+1)}{2} L_{k,n} + \sigma_1^{n}\,\frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{n+1}\left(n\left(\frac{\sigma_2}{\sigma_1}\right)-n-1\right)+\left(\frac{\sigma_2}{\sigma_1}\right)}{\left(\frac{\sigma_2}{\sigma_1}-1\right)^{2}} + \sigma_2^{n}\,\frac{\left(\frac{\sigma_1}{\sigma_2}\right)^{n+1}\left(n\left(\frac{\sigma_1}{\sigma_2}\right)-n-1\right)+\left(\frac{\sigma_1}{\sigma_2}\right)}{\left(\frac{\sigma_1}{\sigma_2}-1\right)^{2}}
$$

$$
\begin{aligned}
&= \frac{n(n+1)}{2} L_{k,n} - \frac{1}{\left(\sigma_1-\sigma_2\right)^{2}}\left(\frac{n\,\sigma_2^{n+1}+n\,\sigma_1^{n-1}+\sigma_2^{n-1}}{\sigma_1}-\sigma_1^{n}\right) - \frac{1}{\left(\sigma_2-\sigma_1\right)^{2}}\left(\frac{n\,\sigma_1^{n+1}+n\,\sigma_2^{n-1}+\sigma_1^{n-1}}{\sigma_2}-\sigma_2^{n}\right) \\
&= \frac{n(n+1)}{2} L_{k,n} - \frac{1}{\left(\sigma_1-\sigma_2\right)^{2}}\left(-n\,\sigma_2^{n+1}-\sigma_1^{n}-(n+1)\sigma_2^{n-1}\right) - \frac{1}{\left(\sigma_2-\sigma_1\right)^{2}}\left(-n\,\sigma_1^{n+1}-\sigma_2^{n}-(n+1)\sigma_1^{n-1}\right) \\
&= \frac{n(n+1)}{2} L_{k,n} + \frac{1}{k^2+4}\left(n\,L_{k,n+2}+n\,L_{k,n}\right) = \frac{n(n+1)}{2} L_{k,n} + n\,F_{k,n+1} = \frac{n(n+1)}{2} L_{k,n} + (3n+6)F_{k,n+1}
\end{aligned}
$$

And substituting in the initial formula, the desired result is obtained: $L_{k,n}^{(2)} = \frac{n+2}{2}\left((n+1)L_{k,n} + 6F_{k,n+1}\right)$

The second convolved classical Lucas numbers verify the recurrence relation $L_{k,n}^{(2)} = 3L_{n-1}^{(2)} - 5L_{n-3}^{(2)} + 3L_{n-5}^{(2)} + L_{n-6}^{(2)}$ with the initial conditions $L_0^{(2)} = 8,\ L_1^{(2)} = 12,\ L_2^{(2)} = 42,\ L_3^{(2)} = 85,\ L_4^{(2)} = 195,\ L_5^{(2)} = 399$

3.1 Generating function

It is well known that if f(x) and g(x) are the generating functions of the numerical sequences $A=\{a_n\}$ and $B=\{b_n\}$ respectively, then $f(x)\,g(x)$ is the generating function of the convolution of both sequences $A\otimes B$. Therefore, since $l_k(x)=\dfrac{2-k\,x}{1-k\,x-x^2}$ is the generating function of the k-Lucas sequence, the generating function of the convolution $L_k^{(r)}$ is the function $l_k^r(x)$. Therefore $l_k^{(1)}(x)=\left(\dfrac{2-k\,x}{1-k\,x-x^2}\right)^2$ is the generating function of their self-convolution $L_k^{(1)}=L_k\otimes L_k$ and $l_k^{(2)}(x)=\left(\dfrac{2-k\,x}{1-k\,x-x^2}\right)^3$ that of the second convolution $L_k^{(2)}=L_k\otimes L_k\otimes L_k$ [10]

It is evident that developing this function in series becomes more and more complicated as the value of "r" increases. The solution is to use a Mathematics program that solves the problem. For example, if using Mathematica©, a small program for the self-convolution could have the following form:

- $l[k\_,x\_]:=\dfrac{2-k\,x}{1-k\,x-x^2}$
- r = 2
- Expand[CoefficientList[Series[f[k,x]^r, {x, 0, 10}], x]

When executing the program, the coefficients of the series expansion of l(x, k, r) are obtained, that is, the numerical sequence $L_{k,n}^{(r)}$ dependent on the value of "k".

And by order "%/. k→a" the corresponding numerical sequence is obtained, being "a = 1, 2, 3…".
You can also directly obtain the numerical sequences for "k = 1, 2, 3" using the command
Table[CoefficientList[Series[f[k, x]^r , {x, 0, 10}], x]],{k,3}] instead of the previous order.

4 Conclusions

In this paper, the iterated convolution of the k-Lucas numbers has been studied in a general way and then the first and second convolutions have been studied more specifically. Subsequently, a special dedication has been made to the case of classical Lucas numbers as well as to those of Lucas-Pell. More information on convolutions of sets of sequences defined by linear recurrence relations such as those of the Fibonacci or Lucas form can be found in Dresden and Wang [11,12,13].

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Competing Interests

Author has declared that no competing interests exist.

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