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SOME GENERALIZATIONS OF REVERSE HARDY-TYPE INEQUALITIES VIA JENSEN INTEGRAL INEQUALITY ON TIME SCALES

Samer D. Makharesh, Hassan M. El-Owaidy, Ahmed A. El-Deeb *
Department of Mathematics, Faculty of Science (Boys), Al-Azhar University, Nasr City (11884), Cairo, Egypt.
*Corresponding author: ahmedeldeeb@azhar.edu.eg

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ABSTRACT

In this article, we will obtain some new dynamic inequalities of Hardy-type on time scales. Our results will be proved by using Hölder's inequality and Jensen's inequality. We will apply the main results to the continuous calculus and discrete calculus as special cases.

Keywords: Dynamic inequality; Hardy inequality; Time scale.

1. Introduction

In 1920, Hardy [1] proved the following result

Theorem 1.1. Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers. If $p > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\left(\sum_{m=1}^{n} a(m)\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a(n). \tag{1.1}$$

In 1925, the continuous analogous of inequality (1.1) was given by Hardy [2] in the following form.

Theorem 1.2. Let f be a nonnegative continuous function on $[0, \infty)$. If $p > 1$, then

$$\int_0^{\infty} \frac{1}{x^p}\left(\int_0^x f(s)\, ds\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx . \tag{1.2}$$

The constant $\left(\frac{p}{p-1}\right)^p$ in the two inequalities (1.1) and (1.2) are the best possible.

In 2012, Sulaimn [3] proved the following inequalities

Theorem 1.3. Let φ be nonnegative function defined on $[a, b]$ and $G(x) = \int_a^x \varphi(t)\, dt$, then

(i) for $p > 1$,

$$p\int_a^b \frac{G^p(x)}{x^p}\, dx \le (b-a)^p \int_a^b \frac{\varphi^p(x)}{x^p}\, dx - \int_a^b (x-a)^p\, \frac{\varphi^p(x)}{x^p}\, dx. \tag{1.3}$$

(ii) For $0 < p < 1$,

$$p\int_a^b \frac{G^p(x)}{x^p}\, dx \ge \left(\frac{b-a}{b}\right)^p \int_a^b \varphi^p(x)\, dx - \int_a^b (x-a)^p\, \varphi^p(x)\, dx. \tag{1.4}$$

In 2020, Benassia [4] gave a generalization of Theorem 1.3 as the following.

Theorem 1.4. Let φ and Ω be nonnegative functions defined on $[a, b]$ and $G(x) = \int_a^x \varphi(t)\, dt$. If Ω is nondecreasing, then

(i) for p > 1,

$$p\int_a^b \frac{G^p(x)}{\Omega^{\,p}(x)}\,dx \le (b-a)^p \int_a^b \frac{\varphi^p(x)}{\Omega^{\,p}(x)}\,dx - \int_a^b (x-a)^p\,\frac{\varphi^p(x)}{\Omega^{\,p}(x)}\,dx. \tag{1.5}$$

(ii) For $0 < p < 1$,

$$p\int_a^b \frac{G^p(x)}{\Omega^{\,p}(x)}\,dx \ge \left(\frac{b-a}{b}\right)^p \int_a^b \varphi^p(x)\,dx - \int_a^b (x-a)^p\,\varphi^p(x)dx. \tag{1.6}$$

In 2021, Benassia et al [5], gave a generalization of Hardy's integral inequalities (1.5) and (1.6) by using a weight μ function and a second q parameter as the following.

Theorem 1.5. Let φ, Ω be nonnegatives and integrable functions on $[a,b]$. Let μ be a weight function on $[a,b]$ and

$$G_{\mu}(x) = \int_a^b \varphi(x)\mu(x)dx\,.$$

If Ω is nondecreasing and μ is nonincresing, then

(i) if $1 \le p \le q$,

$$\int_a^b \frac{G_{\mu}{}^p(x)}{\Omega\,(x)}\,dx \le \left(\frac{\mu(a)}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}} \left\{(b-a)^q \int_a^b \frac{\varphi^q(x)}{\Omega^{\frac{p}{q}}(x)}\,dx - \int_a^b (x-a)^q\,\frac{\varphi^q(x)}{\Omega^{\frac{p}{q}}(x)}\,dx\right\}^{\frac{p}{q}}. \tag{1.7}$$

(ii) For $0 \le q \le p \le 1$,

$$\int_a^b \frac{G_{\mu}{}^p(x)}{\Omega\,(x)}\,dx \ge \frac{(b-a)^{1-\frac{p}{q}}\mu(b)}{q^{\frac{p}{q}}\,\Omega^{\,p}(b)} \left\{\int_a^b \varphi^q(x)\,dx - \int_a^b (x-a)^q\,\varphi^q(x)\,dx\right\}^{\frac{p}{q}}. \tag{1.8}$$

Now, we recall the following concepts related to the notion of time scales. In 1988, S. Hilger [6], presented time scales theory to unify continuous and discrete analysis. We will need the following important relations between calculus on time scales T and either continuous calculus ℝ or discrete calculus on ℤ . Note that:

(i) If $\mathrm{T} = \mathbb{R}$, then

$$\sigma(t) = t, \qquad \mu(t) = 0, \qquad f^{\Delta}(t) = f', \qquad \int_a^b f(t)\ \Delta t = \int_a^b f(t)dt. \tag{1.9}$$

(ii) If $\mathrm{T} = \mathbb{Z}$, then

$$\sigma(t) = t+1, \quad \mu(t) = 1, \quad f^{\Delta}(t) = f(t+1) - f(t), \qquad \int_a^b f(t)\Delta t = \sum_{t=1}^{b-1} f(t)\,.$$

Lemma 1.6 (see [7]). Let $0 < p \le q < \infty$, and φ, Ω are nonnegative and rd-continuous functions on $[a,b]_{\mathrm{T}}$ and suppose that $0 < \int_a^b \varphi(t)\,\Delta t < \infty$, then

$$\int_a^b \varphi^p(t)\Omega(t)\ \Delta t \ \leq \left(\int_a^b \Omega(t)\,\Delta t\right)^{\frac{q-p}{q}} \left(\int_a^b \Omega(t)\,\varphi^p(t)\,\Delta t\right)^{\frac{p}{q}}. \tag{1.10}$$

The inequality (1.10) hold for $-\infty < q \leq p < \infty$ and inverted for $0 < q \leq p < \infty$.

One of the forms of the chain rule on time scales is the following form.

Lemma 1.7 (Chain Rule on Time Scales, see [8]). Let $: \mathrm{T} \to \mathbb{R}$, be a delta differentiable function on T^{κ}, and $f: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then there is c in the interval $[t, \sigma(t)]$ such that

$$(f\ o\ g)^{\Delta}(t) = f'\big(g(c)\big)\, g^{\Delta}(t). \tag{1.11}$$

The following lemma is known as Keller's chain rule on time scales.

Lemma 1.8 (Chain Rule on Time Scales, see [9]). Assume $f: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and $g: \mathrm{T} \to \mathbb{R}$, be a delta differentiable function then

$$(f\ o\ g)^{\Delta}(t) = \left\{\int_0^1 f'\Big(g(t) + h\mu(t)\, g^{\Delta}(t)\Big)\, dh\right\}\ g^{\Delta}(t). \tag{1.12}$$

Next, we write Hölder's inequality and Jensen's inequality on time scales.

Lemma 1.9 (Dynamic Holder's Inequality [10]). Let $a, b \in \mathrm{T}$ and $f, g \in C_{rd}([a,b]_{\mathrm{T}}, [0,\infty))$. |If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b f(t)g(t)\Delta t \ \leq \left(\int_a^b f^p(t)\ \Delta t\right)^{\frac{1}{p}} \left(\int_a^b g^q(t)\ \Delta t\right)^{\frac{1}{q}}. \tag{1.13}$$

This inequality is reversed if $0 < p < 1$ and if $p < 0\ or\ q < 0$.

Lemma 1.10 (Dynamic Jensen's Inequality [10]). Let $a, b \in \mathrm{T}$ and $c, d \in \mathbb{R}$. Assume that $g \in C_{rd}([a,b]_{\mathrm{T}}, [c,d])$ and $r \in C_{rd}([a,b]_{\mathrm{T}}, \mathbb{R})$ are nonnegative with $\int_a^b r(t)\ \Delta t > 0$. If $\Phi \in C_{rd}([c,d]_{\mathrm{T}}, \mathbb{R})$ be a convex function, then

$$\Phi\left(\frac{\int_a^b g(t)r(t)\,\Delta t}{\int_a^b r(t)\,\Delta t}\right) \ \leq \ \frac{\int_a^b r(t)\Phi\big(g(t)\big)\Delta t}{\int_a^b r(t)\Delta t}. \tag{1.14}$$

This inequality is reversed if $\Phi \in C_{rd}([c,d]_{\mathrm{T}}, \mathbb{R})$ is concave.

After these initial results, many generalizations, extensions and refinements of dynamic Hardy inequality were made by various authors. For a comprehensive survey on the dynamic inequalities of Hardy-type on time scales, one can refer to the papers [10,11,12,13,14,15] and [16-22].

In this article, we will state and prove some reverse Hardy-type dynamic inequalities on time scales. The obtained Hardy-type dynamic inequalities are completely original, and thus, we get some new integral and discrete inequalities of Hardy-type. In addition to that, some of our results we generalize Theorem 1.5 on time scales.

The following section contains our main results:

2. Main results

Theorem 2.1. Let T be a time scale with $a, b \in \mathrm{T}$, and assume that φ, χ are nonnegative, rd-continuous and $\Delta-$integrable functions on $[a,b]_{\mathrm{T}}$. Let μ be a weight function on $(a,b)_{\mathrm{T}}$ and define

$$G_\mu(x) = \int_a^x \varphi(t)\mu(t)\Delta t.$$

If χ is nondecreasing and μ is nonincresing, then

(i) for $1 \leq p \leq q$,

$$\int_a^b \frac{G_\mu{}^p(x)}{\chi\,(x)}\,\Delta x \le \left(\frac{\mu(a)}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}} \left\{(b-a)^q \int_a^b \frac{\varphi^q(x)}{\chi^{\frac{p}{q}}(x)}\,\Delta x - \int_a^b (x-a)^q \frac{\varphi^q(x)}{\chi^{\frac{p}{q}}(x)}\,\Delta x\right\}^{\frac{p}{q}}. \quad (2.1)$$

(ii) For $0 \le q \le p \le 1$,

$$\int_a^b \frac{G_\mu{}^p(x)}{\chi\,(x)}\,\Delta x \ \ge\ \frac{(b-a)^{1-\frac{p}{q}}\mu(b)}{q^{\frac{p}{q}}\,\chi^{\ p}(b)} \left\{\int_a^b \varphi^q(x)\,\Delta x - \int_a^b (x-a)^q\ \varphi^q(x)\,\Delta x\right\}^{\frac{p}{q}}. \quad (2.2)$$

Proof . (i) By using Hölder's inequality (1.13), we see that

$$\int_a^b \frac{G_\mu{}^p(x)}{\chi\,(x)}\,\Delta x = \int_a^b \chi^{\ -1}(x) \left(\int_a^x \varphi(t)\ \mu(t)\Delta t\right)^p \Delta x$$

$$\le \int_a^b \chi^{-1}(x) \left\{\left(\int_a^x \varphi^p(t)\ \mu(t)\Delta t\right)^{\frac{1}{p}} \left(\int_a^x \mu(t)\,\Delta t\right)^{\frac{1}{q}}\right\}^p \Delta x$$

$$= \int_a^b \chi^{-1}(x) \left\{\left(\int_a^x \varphi^p(t)\ \mu(t)\Delta t\right) \left(\int_a^x \mu(t)\,\Delta t\right)^{p-1}\right\} \Delta x. \quad (2.3)$$

We use Lemma (1.6), then from (2.3), we get that

$$\int_a^b \frac{G_\mu{}^p(x)}{\chi\,(x)}\,\Delta x \le \int_a^b \chi^{-1}(x) \left(\int_a^x \mu(t)\,\Delta t\right)^{p-\frac{p}{q}} \left(\int_a^x \varphi^q(t)\ \mu(t)\Delta t\right)^{\frac{p}{q}} \Delta x$$

$$\le \int_a^b \chi^{-1}\ (x)\mu^p(a)(x-a)^{\frac{p}{q}(q-1)} \left(\int_a^x \varphi^q(t)\ \Delta t\right)^{\frac{p}{q}} \Delta x$$

$$= \mu^p(a) \int_a^b (H(x))^{\frac{p}{q}}\ \Delta x, \quad (2.4)$$

where

$$H(x) = \int_a^x \chi^{\frac{-q}{p}}(x)(x-a)^{q-1}\ \varphi^q(t)\Delta t.$$

Let $\phi(x) = x^{\frac{p}{q}}$, be a concave function and χ be nondecreasing function, by Jensen's inequality, we get that

$$\int_a^b \left(H(x)\right)^{\frac{p}{q}} \Delta x = \int_a^b \phi\left(H(x)\right) \Delta x$$

$$\le (b-a)\,\phi\left(\frac{1}{b-a} \int_a^b H(x)\,\Delta x\right)$$

$$= (b-a)^{1-\frac{p}{q}} \left(\int_a^b \int_a^x \chi^{-1}(x)(x-a)^{q-1}\varphi^q(t)\,\Delta t\Delta x\right)^{\frac{p}{q}}$$

$$= (b-a)^{1-\frac{p}{q}} \left(\int_a^b \varphi^q(t) \int_t^b \chi^{\frac{-q}{p}}(x)(x-a)^{q-1}\Delta t\, \Delta x\right)^{\frac{p}{q}}$$

$$\leq \quad (b-a)^{1-\frac{p}{q}} \left(\int_a^b \varphi^q(t)\chi^{\frac{-q}{p}}(t) \int_t^b (x-a)^{q-1}\Delta t\, \Delta x\right)^{\frac{p}{q}}. \quad (2.5)$$

By chain rule, we have that

$$(x-a)^{q-1} \leq \frac{1}{q}\, [(x-a)^q]^{\Delta}. \quad (2.6)$$

Then

$$\int_a^b (H(x))^{\frac{p}{q}} \Delta x \leq (b-a)^{1-\frac{p}{q}} \left(\frac{1}{q}\int_a^b \varphi^q(t)\chi^{\frac{-q}{p}}(t) \int_t^b [(x-a)^q]^{\Delta}\, \Delta t\, \Delta x\right)^{\frac{p}{q}}.$$

Therefore,

$$\int_a^b \frac{G_{\mu}{}^{p}(x)}{\chi(x)}\, \Delta x \;\leq\; \mu(a)^p (b-a)^{1-\frac{p}{q}} \left(\frac{1}{q}\int_a^b \varphi^q(t)\chi^{\frac{-q}{p}}(t)[(b-a)^q-(t-a)^q]\,\Delta t\right)^{\frac{p}{q}}.$$

$$= \left(\frac{\mu(a)}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}} \left\{(b-a)^q \int_a^b \frac{\varphi^q(t)}{\chi^{\frac{p}{q}}(t)}\, \Delta t \;-\; \int_a^b (t-a)^q\, \frac{\varphi^q(t)}{\chi^{\frac{p}{q}}(t)}\, \Delta t\right\}^{\frac{p}{q}}$$

which is the desired inequality (2.1).

(ii) By using reverse Hölder's inequality (1.13), we see that

$$\int_a^b \frac{G_{\mu}{}^{p}(x)}{\chi(x)}\, \Delta x = \int_a^b \chi^{-1}(x) \left(\int_a^x \varphi(t)\; \mu(t)\Delta t\right)^p \Delta x$$

$$\geq \int_a^b \chi^{-1}(x)\left\{\left(\int_a^x \varphi^p(t)\; \mu(t)\Delta t\right)^{\frac{1}{p}} \left(\int_a^x \mu(t)\,\Delta t\right)^{\frac{1}{q}}\right\}^p \Delta x$$

$$= \int_a^b \chi^{-1}(x)\left\{\left(\int_a^x \varphi^p(t)\; \mu(t)\Delta t\right) \left(\int_a^x \mu(t)\,\Delta t\right)^{p-1}\right\} \Delta x\,. \quad (2.7)$$

Applying reverse inequality (1.10), then from (2.7), we get that

$$\int_a^b \frac{G_{\mu}{}^{p}(x)}{\chi(x)}\, \Delta x \;\geq\; \int_a^b \chi^{-1}(x) \left(\int_a^x \mu(t)\,\Delta t\right)^{p-\frac{p}{q}} \left(\int_a^x \varphi^q(t)\; \mu(t)\Delta t\right)^{\frac{p}{q}} \Delta x$$

$$\geq \quad \int_a^b \chi^{-1}(x)\mu^p(a)(x-a)^{\frac{p}{q}(q-1)} \left(\int_a^x \varphi^q(t)\; \Delta t\right)^{\frac{p}{q}} \Delta x$$

$$= \quad \mu^p(b)\int_a^b (H(x))^{\frac{p}{q}} \quad \Delta x, \quad (2.8)$$

where

$$H(x) = \int_a^x \chi^{\frac{-q}{p}}(x)(x-a)^{q-1}\; \varphi^q(t)\Delta t.$$

Let $\phi(x) = x^{\frac{p}{q}}$, be a convex function and χ be nondecreasing function, by Jensen's inequality, we get that

$$\int_a^b \left(H(x)\right)^{\frac{p}{q}} \, \Delta x = \int_a^b \phi\left(H(x)\right) \Delta x$$

$$\geq (b-a)\, \phi\left(\frac{1}{b-a} \int_a^b H(x)\, \Delta x\right)$$

$$= (b-a)^{1-\frac{p}{q}} \left(\int_a^b \int_a^x \chi^{-1}(x)(x-a)^{q-1} \varphi^q(t)\, \Delta t \Delta x\right)^{\frac{p}{q}}$$

$$= (b-a)^{1-\frac{p}{q}} \left(\int_a^b \varphi^q(t) \int_t^b \chi^{\frac{-q}{p}}(x)(x-a)^{q-1} \Delta t\, \Delta x\right)^{\frac{p}{q}}$$

$$\geq \frac{(b-a)^{1-\frac{p}{q}}}{\chi(b)} \left(\int_a^b \varphi^q(t) \int_t^b (x-a)^{q-1} \Delta t\, \Delta x\right)^{\frac{p}{q}}. \qquad (2.9)$$

By chain rule, we have that

$$(x-a)^{q-1} \geq \frac{1}{q} \left[(x-a)^q\right]^{\Delta}. \qquad (2.10)$$

Then

$$\int_a^b (H(x))^{\frac{p}{q}} \Delta x \geq \frac{(b-a)^{1-\frac{p}{q}}}{\chi(b)} \left(\frac{1}{q} \int_a^b \varphi^q(t) \int_t^b \left[(x-a)^q\right]^{\Delta} \Delta t\, \Delta x\right)^{\frac{p}{q}}.$$

Therefore,

$$\int_a^b \frac{G_{\mu}{}^p(x)}{\chi(x)} \Delta x \geq \frac{\mu(b)(b-a)^{1-\frac{p}{q}}}{\chi(b)} \left(\frac{1}{q} \int_a^b \varphi^q(t)\left[(b-a)^q - (t-a)^q\right] \Delta t\right)^{\frac{p}{q}}$$

$$= \frac{\mu(b)(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}} \chi(b)} \left\{(b-a)^q \int_a^b \varphi^q(t)\ \Delta t - \int_a^b (t-a)^q \varphi^q(t)\ \Delta t\right\}^{\frac{p}{q}}.$$

which is the desired inequality (2.2).

Remark 2.2. In Theorem 2.1, if we take $\mathrm{T} = \mathbb{R}$, then we get Theorem 1.5.

Corollary 2.3. In Theorem 2.1, if we take $\mathrm{T} = \mathbb{Z}$, and $a = 1$ then we get the following inequalities

(i) for $1 \leq p \leq q$,

$$\sum_{s=1}^{b-1} \frac{G_{\mu}{}^p(s)}{\chi(s)} \leq \left(\frac{\mu(1)}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}} \left\{(b-a)^q \sum_{m=1}^{b-1} \frac{\varphi^q(m)}{\chi^{\frac{p}{q}}(m)} - \sum_{m=1}^{b-1} (m-a)^q \frac{\varphi^q(m)}{\chi^{\frac{p}{q}}(m)}\right\}^{\frac{p}{q}}.$$

(ii) For $0 \leq q \leq p \leq 1$,

$$\sum_{s=1}^{b-1} \frac{G_{\mu}{}^{p}(s)}{\chi\ (s)} \ \geq \ \frac{(b-a)^{1-\frac{p}{q}}\mu(b)}{q^{\frac{p}{q}}\,\chi\ ^{p}(b)} \left\{ \sum_{m=1}^{b-1} \varphi^{q}(m) - \sum_{m=1}^{b-1} \ (m-a)^{q}\ \ \varphi^{q}(m) \right\}^{\frac{p}{q}},$$

where $G_{\mu}(s) = \sum_{m=1}^{s} \varphi(m)\, \mu(m)$.

In the same data on the functions χ, φ and μ with $G_{\mu}(x) = \int_{a}^{x} \varphi(t)\mu(t)\Delta t$ and by reasoning analogously to the proof of Theorem 2.1, we obtain the following remarks.

Remark 2.4. If μ and χ are nondecreasing functions, then
(i) for $1 \leq p \leq q$,

$$\int_{a}^{b} \frac{\left(G_{\mu}\right)^{p}(x)}{\chi(x)} \Delta x \leq \left(\frac{\mu(b)}{q^{\frac{1}{q}}}\right)^{p} (b-a)^{1-\frac{p}{q}} \left\{ (b-a)^{q} \int_{a}^{b} \frac{\varphi^{q}(x)}{\chi^{\frac{q}{p}}(x)} - \int_{a}^{b} (x-a)^{q} \frac{\varphi^{q}(x)}{\chi^{\frac{q}{p}}(x)} \Delta x \right\}^{\frac{p}{q}}.$$

(ii) For $0 \leq q \leq p < 1$,

$$\int_{a}^{b} \frac{\left(G_{\mu}\right)^{p}(x)}{\chi(x)} \Delta x \geq \frac{\mu^{p}(a)(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}}\,\chi(a)} \left\{ (b-a)^{q} \int_{a}^{b} \varphi^{q}(x)\Delta x - \int_{a}^{b} (x-a)^{q}\, \varphi^{q}(x)\Delta x \right\}^{\frac{p}{q}}.$$

Remark 2.5. If χ is nonincreasing and μ is nondecresing, then
(i) for $1 \leq p \leq q$,

$$\int_{a}^{b} \frac{\left(G_{\mu}\right)^{p}(x)}{\chi(x)} \Delta x \leq \frac{\mu^{p}(b)(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}}\,\chi(b)} \left\{ (b-a)^{q} \int_{a}^{b} \varphi^{q}(x)\Delta x - \int_{a}^{b} (x-a)^{q}\, \varphi^{q}(x)\Delta x \right\}^{\frac{p}{q}}.$$

(ii) For $0 \leq q \leq p < 1$,

$$\int_{a}^{b} \frac{\left(G_{\mu}\right)^{p}(x)}{\chi(x)} \Delta x \geq \left(\frac{\mu(a)}{q^{\frac{1}{q}}}\right)^{p} (b-a)^{1-\frac{p}{q}} \left\{ (b-a)^{q} \int_{a}^{b} \frac{\varphi^{q}(x)}{\chi^{\frac{q}{p}}(x)} \Delta x - \int_{a}^{b} (x-a)^{q} \frac{\varphi^{q}(x)}{\chi^{\frac{q}{p}}(x)} \Delta x \right\}^{\frac{p}{q}}.$$

Remark 2.6. If χ and μ are nonincresing functions, then
(i) for $1 \leq p \leq q$,

$$\int_{a}^{b} \frac{\left(G_{\mu}\right)^{p}(x)}{\chi(x)} \Delta x \leq \frac{\mu^{p}(a)(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}}\,\chi(b)} \left\{ (b-a)^{q} \int_{a}^{b} \varphi^{q}(x)\Delta x - \int_{a}^{b} (x-a)^{q}\, \varphi^{q}(x)\Delta x \right\}^{\frac{p}{q}}.$$

(ii) For $1 \leq q \leq p < 1$,

$$\int_{a}^{b} \frac{\left(G_{\mu}\right)^{p}(x)}{\chi(x)} \Delta x \geq \left(\frac{\mu(b)}{q^{\frac{1}{q}}}\right)^{p} (b-a)^{1-\frac{p}{q}} \left\{ (b-a)^{q} \int_{a}^{b} \frac{\varphi^{q}(x)}{\chi^{\frac{q}{p}}(x)} \Delta x - \int_{a}^{b} (x-a)^{q} \frac{\varphi^{q}(x)}{\chi^{\frac{q}{p}}(x)} \Delta x \right\}^{\frac{p}{q}}.$$

3. Applications

Now, we give some new consequences of the above results.

3.1. The reverses weighted Hardy's type inequalities on time scales

If we put $p = q$ in Theorem 2.1, we get the following corollary.

Corollary 3.1. Let $\mathbb{T}$ be time scale with $a, b \in \mathbb{T}$, and assume that φ, χ are nonnegative, rd-continuous and $\Delta-$integrable functions on $[a, b]_{\mathbb{T}}$. Let μ be weight function on $[a, b]_{\mathbb{T}}$ and

$$G_\mu(x) = \int_a^x \varphi(t)\mu(t)\Delta t.$$

If χ in nondecreasing and μ is nonincresing, then

(i) for $1 \leq p$,

$$\int_a^b \frac{\left(G_\mu\right)^p(x)}{\chi(x)}\Delta x \leq \frac{\mu^p(a)}{p}\left\{(b-a)^p\int_a^b \frac{\varphi^p(x)}{\chi(x)}\Delta x - \int_a^b (x-a)^p\frac{\varphi^p(x)}{\chi(x)}\Delta x\right\}.$$

(ii) For $0 < p < 1$,

$$\int_a^b \frac{(G_\mu(x))^p(x)}{\chi(x)}\Delta x \geq \frac{\mu^p(b)}{p\chi(b)}\left\{(b-a)^p\int_a^b \varphi^p(x)\Delta x - \int_a^b (x-a)^p\varphi^p(x)\Delta x\right\}.$$

3.2. The reverses Hardy's type inequalities on time scales

If we put $\mu \equiv 1$ in Theorem 2.1 we get the following corollaries.

Corollary 3.2. Let $\mathbb{T}$ be time scale with $a, b \in \mathbb{T}$, and assume that φ, χ are nonnegative, rd-continuous and $\Delta-$integrable functions on $[a, b]_{\mathbb{T}}$. Define

$$G(x) = \int_a^x \varphi(t)\Delta t.$$

If χ is nondecreasing, then

(i) for $1 \leq p \leq q$,

$$\int_a^b \frac{G^p(x)}{\chi(x)}\Delta x \leq \left(\frac{1}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}}\left\{(b-a)^q\int_a^b \frac{\varphi^q(x)}{\chi^{\left(\frac{q}{p}\right)}(x)}\Delta x - \int_a^b (x-a)^q\frac{\varphi^p(x)}{\chi^{\frac{q}{p}}(x)}\Delta x\right\}^{\frac{p}{q}}.$$

(ii) For $0 \leq q \leq p < 1$,

$$\int_a^b \frac{G^p(x)}{\chi(x)}\Delta x \geq \frac{(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}}\chi(b)}\left\{(b-a)^q\int_a^b \varphi^q(x)\Delta x - \int_a^b (x-a)^q\varphi^p(x)\Delta x\right\}^{\frac{p}{q}}.$$

Remark 3.3. In Corollary 3.2, if we take $\mathbb{T} = \mathbb{R}$ and $q = p$ we get Theorem 1.4.

Corollary 3.4. Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and assume that φ, χ are nonnegative, rd-continuous and $\Delta-$integrable functions on $[a, b]_{\mathbb{T}}$. Define

$$G(x) = \int_a^x \varphi(t)\Delta t.$$

If χ is nonincreasing, then

(i) for $1 \leq p \leq q$,

$$\int_a^b \frac{G^p(x)}{\chi(x)}\Delta x \leq \frac{(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}}\chi(b)}\left\{(b-a)^q\int_a^b \varphi^q(x)\Delta x - \int_a^b (x-a)^q\varphi^q(x)\Delta x\right\}^{\frac{p}{q}}.$$

(ii) For $0 \leq q \leq p < 1$,

$$\int_a^b \frac{(G)^p\,(x)}{\chi(x)}\Delta x \geq \left(\frac{1}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}} \left\{(b-a)^q \int_a^b \frac{\varphi^q(x)}{\chi^{\frac{p}{q}}(x)}\Delta x - \int_a^b (x-a)^q \frac{\varphi^p(x)}{\chi^{\frac{p}{q}}(x)}\Delta x\right\}^{\frac{p}{q}}.$$

If we put $p = q$, in Corollary 3.4, then we obtain the following result.

Corollary 3.5. Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and assume that φ, χ are nonnegative, rd-continuous and Δ −integrable functions on $[a,b]_{\mathbb{T}}$. Define

$$G(x) = \int_a^x \varphi(t)\Delta t.$$

If χ is nonincreasing, then

(i) for $1 \leq p$,

$$\int_a^b \frac{G^p(x)}{\chi(x)}\Delta x \leq \frac{1}{p\chi(b)} \left\{(b-a)^p \int_a^b \varphi^p(x)\Delta x - \int_a^b (x-a)^p\,\varphi^p(x)\Delta x\right\}.$$

(ii) For $0 < p < 1$,

$$\int_a^b \frac{G^p(x)}{\chi(x)}\Delta x \geq \left(\frac{1}{p}\right) \left\{(b-a)^p \int_a^b \frac{\varphi^p(x)}{\chi(x)}\Delta x - \int_a^b (x-a)^p \frac{\varphi^p(x)}{\chi(x)}\Delta x\right\}.$$

4. Conclusions

In this paper, by applying Hölder's inequality and its inverse, Jensen's integral inequality and its inverse on time scale, we generalized some integral inequalities relating to the inverse-weighted Hardy inequalities to a general time scale. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete and continuous calculus. In the future, we can generalize these inequalities in a different way by using other mathematical tools.

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Conflict of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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بعض التعميمات العكسية لمتباينات هاردي بتكامل جينسن على مقاييس الزمن

سمير مخارش، مصطفى العويضي، احمد الديب*

قسم الرياضيات، كلية العلوم (بنين)، جامعة الازهر، مدينة نصر، القاهرة ، مصر

الملخص:

تعتبر متباينات هاردي من اهم المتباينات في التحليل الرياضي. حيث جذبت مؤخراً اهتمام الكثير من الباحثين والدارسين ومن اهم هذهِ الدراسات هي تعميم متباينات هاردي على مقاييس الزمن. في هذا البحث, تم بناء و تعميم بعض متباينات هاردي ومعكوساتها على مقاييس الزمن, اثبات هذهِ التعميمات تمت بواسطة بعض المتباينات الجبرية مثل متباينة هولدر ومعكوسها و متباينة جنسين ومعكوسها . تم استنتاج بعض الحالات الخاصة من النتائج الاساسية كما حصلنا على متباينات هاردي و هاردي العكسية في حالتي $\mathbb{R}$ و $\mathbb{Z}$.