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A Casson Fluid Model for the Steady Flow through a Stenosed Blood Vessel

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Abstract

Laminar flow of blood considering the blood as a Casson fluid has been studied. It is observed that the axial velocity, volumetric flow rate and pressure gradient increase with the increase in slip velocity and decrease with growth in yield stress. The results derived have been presented both analytically and graphically for a better understanding by choosing the appropriate parameters.

Keywords: Blood flow, casson fluid, steady flow, stenosis, yield stress.

1 Introduction

Many researchers have now established this fact that stenosis is causing a number of deaths in several countries and this problem needs to be dealt seriously. In an stenosed artery, the wall thickens because of an abnormal development along the lumen of the wall which in turn; affects the hemodynamic behaviour of the blood flow. According to medical experts, the blood vessel narrows when the macrophage white blood cells gather near the arterial wall; and the fat and cholesterol from the macrophages are not sufficiently removed by the high density lipoproteins. It has been observed that the blood behaves like a Newtonian fluid at high shear rate and it behaves like a non – Newtonian fluid at low shear rate due to which it requires a certain yield stress for smooth flow. So far various mathematical models have been proposed by many researchers to study the different features of the blood.

S. Rodbard [1] studied the dynamics of blood flow in stenotic lesions. D.F. Young [2] analyzed the effect of an axially symmetric time-dependent growth into the lumen of a tube of constant cross-section on the steady flow of a Newtonian fluid. P. Chaturani and D. Biswas [3] made a theoretical study of blood flow through stenosed artery with slip velocity at wall. S. Chakravarty [4] studied the effects of stenosis on the flow behaviour of blood in an artery. N.P. Smith et al. [5] presented an anatomically based model of transient coronary blood flow through an artery with mild stenosis viz. two – layered model for different shapes of stenosis and slip velocity at the wall. B. K. Mishra et al. [6] studied the effect of resistance parameter on uniform and non – uniform

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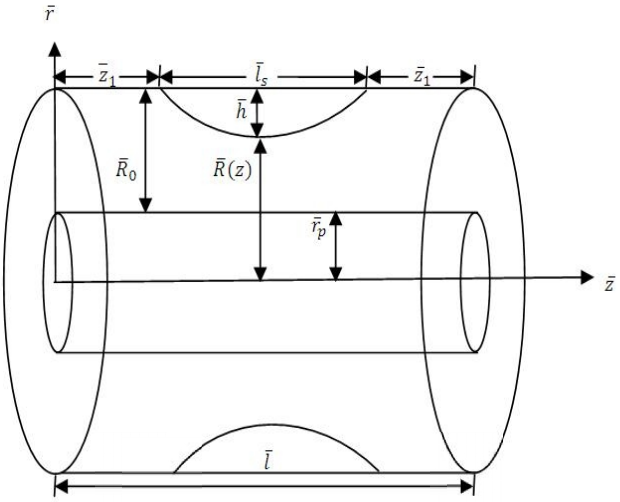
portion of artery for non – Newtonian fluid model of blood flow through an arterial stenosis. B. Singh et al. [7] explored blood flow through an artery having radially non–symmetric mild stenosis by taking blood as a power law fluid and D. Biswas et al. [8] represented a non–Newtonian model for the steady flow of blood through a stenosed artery by assuming blood as a Herschel – Bulkley fluid and a slip velocity near the arterial wall. S. S. yadav et al. [9] suggested a Bingham plastic model to discuss the blood flow characteristics through a generalized atherosclerotic artery with multiple stenosis.

2 Mathematical Formulation

Consider a laminar, steady and incompressible blood flow through a cylindrical artery; which is stenosed in an axially symmetric manner. The geometry of the artery is given by figure below:



Let the radius of the artery is $\bar{R}(\bar{z})$ in the stenotic region and $\bar{R}_0$ in the non – stenotic region which can be given as (Young [2]):

$$\bar{R}(\bar{z}) = \begin{cases} \bar{R}_0 - \frac{\bar{h}}{2}\left[1 + \cos\frac{2\pi}{\bar{l}_s}(\bar{z}_1 + \bar{l}_s - \bar{z})\right]; & \bar{z}_1 \leq \bar{z} \leq \bar{z}_1 + \bar{l}_s \\ \bar{R}_0 & ; \text{ otherwise} \end{cases} \tag{2.1}$$

where $\bar{h}, \bar{l}_s$ and $\bar{z}_1$ represent the maximum height, length and the position of the stenosis in the artery whose whole length is $\bar{l}$. Also, let $\bar{r}$ and $\bar{z}$ are the radial and axial coordinates.

In this study the blood is considered to behave like a Casson fluid.

Under the above assumptions, the equations of motion for the blood can be written as

$$-\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{\tau}_c) = 0 \tag{2.2}$$

$$\frac{\partial \bar{p}}{\partial \bar{r}} = 0 \tag{2.3}$$

where $\bar{p}$ represents the pressure at any point and $\bar{\tau}_c$ denotes the shear stress of the blood. The constitutive equation of a Casson fluid can be simplified as

$$F(\bar{\tau}_c) = -\frac{\partial \bar{v}_c}{\partial \bar{r}} = \frac{1}{\bar{k}_c}\left(\bar{\tau}_c^{1/2} - \bar{\tau}_0^{1/2}\right)^2 \quad \text{for } \bar{\tau}_c \geq \bar{\tau}_0 \tag{2.4}$$

$$\frac{\partial \bar{v}_c}{\partial \bar{r}} = 0 \qquad \text{for } \bar{\tau}_c \leq \bar{\tau}_0 \tag{2.5}$$

where $\bar{v}_c$ represents the axial velocity of blood, $\bar{\tau}_0$ stands for the yield stress and $\bar{k}_c$ is the viscosity of the fluid. The equations (2.2) to (2.5) are applied to the following boundary conditions:

$$\left.\begin{array}{ll} \bar{v}_c = \bar{v}_s & \text{at } \bar{r} = \bar{R}(\bar{z}) \\ \bar{\tau}_c = \text{Finite value} & \text{at } \bar{r} = 0 \end{array}\right\} \tag{2.6}$$

where $\bar{v}_s$ denotes the slip velocity in the axial direction

Introducing following non – dimensional quantities:

$$R(z) = \frac{\bar{R}(\bar{z})}{\bar{R}_0},\ z = \frac{\bar{z}_1+\bar{l}_s-\bar{z}}{\bar{l}_s},\ r = \frac{\bar{r}}{\bar{R}_0},\ \frac{\partial p}{\partial z} = \frac{\partial \bar{p}/\partial \bar{z}}{\bar{p}_0},\ \tau_c = \frac{\bar{\tau}_c}{\bar{p}_0\bar{R}_0/2},\ \tau_0 = \frac{\bar{\tau}_0}{\bar{p}_0\bar{R}_0/2},\ H = \frac{\bar{h}}{\bar{R}_0}$$

$$v_c = \frac{\bar{v}_c}{\bar{p}_0\bar{R}_0^2/2\bar{k}_c},\ v_s = \frac{\bar{v}_s}{\bar{p}_0\bar{R}_0^2/2\bar{k}_c}, \tag{2.7}$$

where $\bar{p}_0$ represents absolute typical pressure gradient.

Using the above non – dimensional scheme, the radius of the stenotic region of the artery becomes

$$R(z) = \begin{cases} 1 - H\cos^2 \pi z\,; & 0 \leq z \leq 1 \\ 1 & ; \text{otherwise} \end{cases} \tag{2.8}$$

The equations of the motion in the non – dimensional form are

$$-2\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_c) = 0 \tag{2.9}$$

$$\frac{\partial p}{\partial r} = 0 \tag{2.10}$$

Constitutive equations of Casson fluid in the non – dimensional form are

$$-\frac{\partial v_c}{\partial r} = (\tau_c^{1/2} - \tau_0^{1/2})^2 \qquad \text{for } \tau_c \geq \tau_0 \tag{2.11}$$

$$\frac{\partial v_c}{\partial r} = 0 \qquad \text{for } \tau_c \leq \tau_0 \tag{2.12}$$

The dimensionless boundary conditions (2.6) are

$$\left.\begin{array}{ll} v_c = v_s & \text{at } r = R(z) \\ \tau_c = \text{Finite value} & \text{at } r = 0 \end{array}\right\} \tag{2.13}$$

Applying the condition (2.13) to the equation (2.9), we can write the expressions for the shear stress τ_c and wall shear stress τ_R given as

$$\tau_c = -r\frac{\partial p}{\partial z} \tag{2.14}$$

$$\tau_R = -R(z)\frac{\partial p}{\partial z} \tag{2.15}$$

From equations (14) and (15),

$$\frac{\tau_c}{\tau_R} = \frac{r}{R} \tag{2.16}$$

where $R = R(z)$

3 Method of Solution

Integrating equation (2.11) using equations (2.13) and (2.15), we get the velocity function for $r_p \le r \le R(z)$ where $r_p = \frac{\bar{r}_p}{\bar{R}_0}$ is the dimensionless radius of the plug flow region, given as

$$v_c = v_s + \frac{R}{2\tau_R}\left[(\tau_R^2 - \tau_c^2) - \frac{8}{3}\tau_0^{1/2}\left(\tau_R^{3/2} - \tau_c^{3/2}\right) + 2\tau_0(\tau_R - \tau_c)\right] \tag{3.1}$$

within plug flow region i.e. $0 \le r \le r_p$, $\tau_c = \tau_0$ at $r = r_p$

Then equation (3.1) gives the plug flow velocity as

$$v_p = v_s + \frac{R}{2\tau_R}\left(\tau_R^2 - \frac{1}{3}\tau_0^2 - \frac{8}{3}\tau_0^{1/2}\tau_R^{3/2} + 2\tau_0\tau_R\right) \tag{3.2}$$

The volumetric flow rate in the dimensionless form for the region $0 \le r \le R(z)$ can be obtained as

$$Q = 4\int_0^R rv(r)dr$$

$$= 4\int_0^{r_p} rv_p dr + 4\int_{r_p}^R rv_c dr$$

Hence

$$Q = 2R^2 v_s + \frac{2R^3}{\tau_R^3}\left(\frac{1}{4}\tau_R^4 - \frac{4}{7}\tau_0^{1/2}\tau_R^{7/2} + \frac{1}{3}\tau_0\tau_R^3 - \frac{1}{84}\tau_0^4\right) \tag{3.3}$$

If $\tau_0 \ll \tau_R$ i.e. $\frac{\tau_0}{\tau_R} \ll 1$, then equation (3.3) takes the form

$$Q = 2R^2 v_s + \frac{R^3}{2}\left(\tau_R - \frac{16}{7}\tau_0^{1/2}\tau_R^{1/2} + \frac{4}{3}\tau_0\right) \quad (3.4)$$

which gives us the wall shear stress for the artery with stenosis as

$$\tau_R = \left[\frac{8}{7}\tau_0^{1/2} + \left\{\frac{2}{R^3}(Q - 2R^2 v_s) - \frac{4}{147}\tau_0\right\}^{1/2}\right]^2 \quad (3.5)$$

When there is no stenosis i.e. $R(z) = R_0$ then the wall shear stress for the non – stenotic artery is given as

$$\tau_N = \left[\frac{8}{7}\tau_0^{1/2} + \left\{\frac{2}{R_0^3}(Q - 2R_0^2 v_s) - \frac{4}{147}\tau_0\right\}^{1/2}\right]^2 \quad (3.6)$$

Using equation (3.5) in equation (2.15), we get the pressure gradient as

$$\frac{\partial p}{\partial z} = -\frac{1}{R}\left[\frac{8}{7}\tau_0^{1/2} + \left\{\frac{2}{R^3}(Q - 2R^2 v_s) - \frac{4}{147}\tau_0\right\}^{1/2}\right]^2 \quad (3.7)$$

4 Results and Discussion

The velocity profile for the axial velocity in the non – plug flow region has been obtained in equation (3.1) and results are analyzed graphically in Figures 1(a), 1(b), 2(a) and 2(b).

Figures 1(a) shows the variations of the axial velocity along the axial distance z for the different values of the shear stress τ_c and wall yield stress τ_o with some fixed values like $\tau_R = 0.070$ and $v_s = 0.0$ i.e. the no slip condition. It is clear that the axial velocity first increases and after a certain point, it starts decreasing and again increases along the axial distance z. The axial velocity slows down when there is a decrease in the yield stress.

In Figure 1(b), the variations of the axial velocity along the radial distance $R(z)$ have been shown for the different values of the shear stress τ_c and wall yield stress τ_o taking $\tau_R = 0.070$ and $v_s = 0.0$. It is clear that the axial velocity decreases with the increase in shear stress but it decreases rapidly with the increase in yield stress. The axial velocity is greater for the greater radius of the stenosis region of the artery.

Figure 2(a) shows the variations of the axial velocity along the axial distance z for different values of shear stress τ_c and slip velocity v_s with $\tau_o = 0.010$ and $\tau_R = 0.070$. It shows that the axial velocity fluctuates i.e. increases and after certain point decreases and again starts increasing along the axial distance z. The axial velocity increases with the increase in slip velocity.

Figure 2(b) shows the variations of the axial velocity along radial distance $R(z)$ for the different values of the shear stress τ_c and slip velocity v_s fixing some values as $\tau_o = 0.010$ and $\tau_R = 0.070$. It is observed that there is an increase in the axial velocity with a decrease in shear stress.

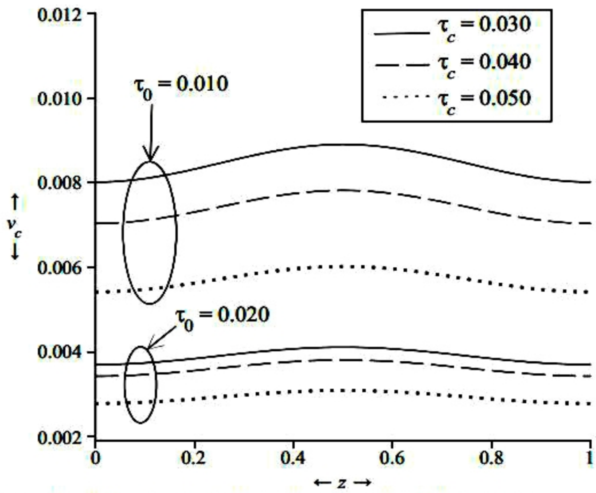


Figure 1 (a): Variation of Axial velocity Along Axial Distance for Different Values of the Shear Stress τ_c and Yeild Stress τ_0 with some fixed values $\tau_R = 0.070, v_s = 0.0, H = 0.1$

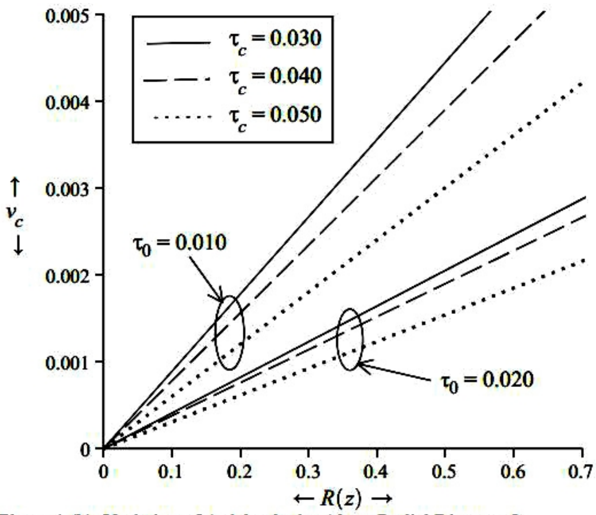


Figure 1 (b): Variation of Axial velocity Along Radial Distance for Different Values of the Shear Stress τ_c and Yeild Stress τ_0 with some fixed values $\tau_R = 0.070$ and $v_s = 0.0$

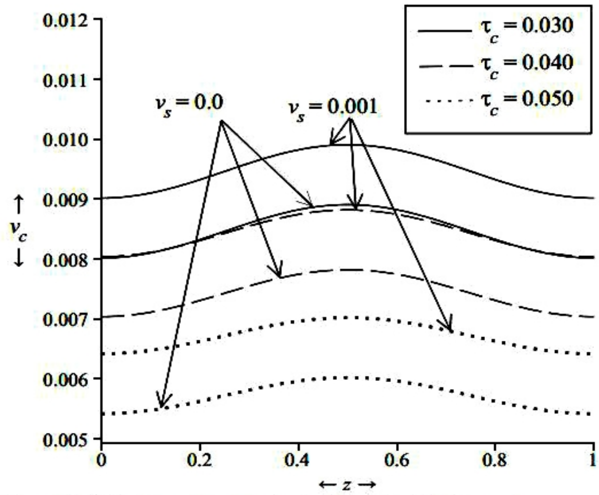


Figure 2 (a): Variation of Axial velocity Along Axial Distance for Different Values of the Shear Stress τ_c and Slip Velocity v_s with some fixed values $\tau_0 = 0.010$, $\tau_R = 0.070$, $H = 0.1$

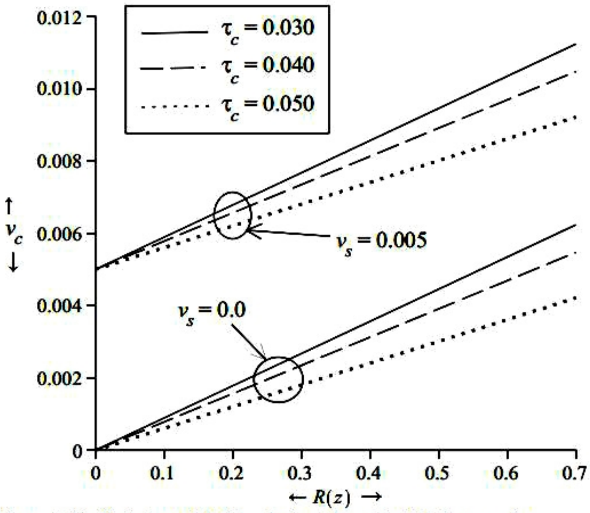


Figure 2 (b): Variation of Axial velocity Along Radial Distance for Different Values of the Shear Stress τ_c and Slip Velocity v_s with Some Fixed Values $\tau_0 = 0.010$ and $\tau_R = 0.070$

The axial velocity for the plug flow region obtained through equation (3.2) has been graphically presented in Figure 3(a). It shows the variations of the plug flow velocity along the axial distance z for the various values of the yield stress τ_o and slip velocity v_s with a fixed value $\tau_R = 0.070$. The plug flow velocity has wave – like variations along the axial distance z. Also the plug flow velocity increases when the slip velocity increases and it decreases for an increase in the yield stress.

Figure 3(b) shows the variations of the plug flow velocity with the change in radial distance $R(z)$ for the different values of the yield stress τ_0 and slip velocity v_s with $\tau_R = 0.070$. The graph also clarifies the fact that the axial velocity in the plug flow region increases due to an increase in slip velocity but the increase in yield stress decreases the plug flow velocity rapidly.

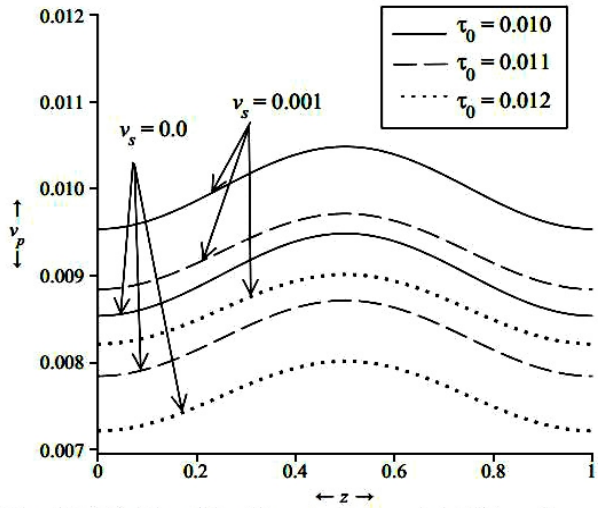


Figure 3 (a): Variation of Plug Flow velocity Along Axial Distance for Different Values of the Yeild Stress τ_0 and Slip Velocity v_s with some fixed values $\tau_R = 0.070, H = 0.1$

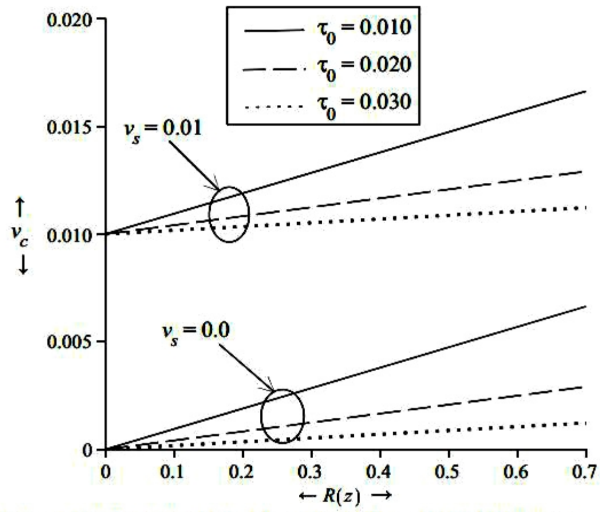


Figure 3 (b): Variation of Plug Flow velocity Along Radial Distance for Different Values of the Yeild Stress τ_0 and Slip Velocity v_s with some fixed values $\tau_R = 0.070$.

Figure 4(a) shows the variations of the volumetric flow rate derived in equation (3.4) along the radial distance $R(z)$ for the various values of the yield stress τ_0 and slip velocity v_s with $\tau_R =$

0.070. The volumetric flow rate increases as the slip velocity increases but it decreases with increase in yield stress.

Figure 4(b) shows the changes in the volumetric flow rate along the height H of the stenosis for the different values of the yield stress τ_0 and wall slip velocity v_s with $\tau_R = 0.070$. It is observed that the volumetric flow rate decreases as the height of the stenosis increases but the volumetric flow rate increases with increase in slip velocity.

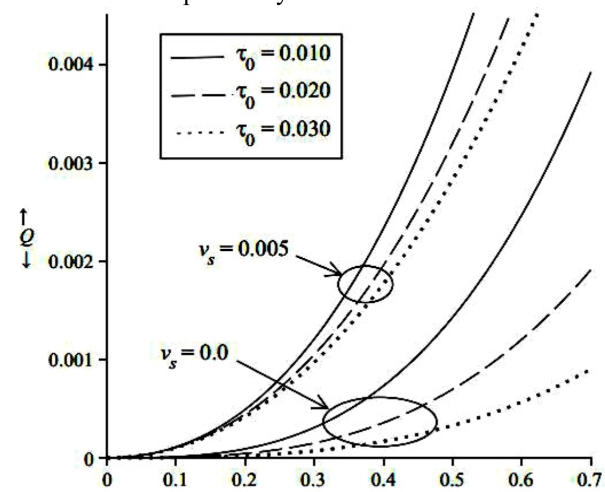


Figure 4 (a): Variation of Volumetric Flow Rate Along Radial Distance for Different Values of the Yeild Stress τ_0 and Slip Velocity v_s with Some Fixed Values $\tau_R = 0.070$.

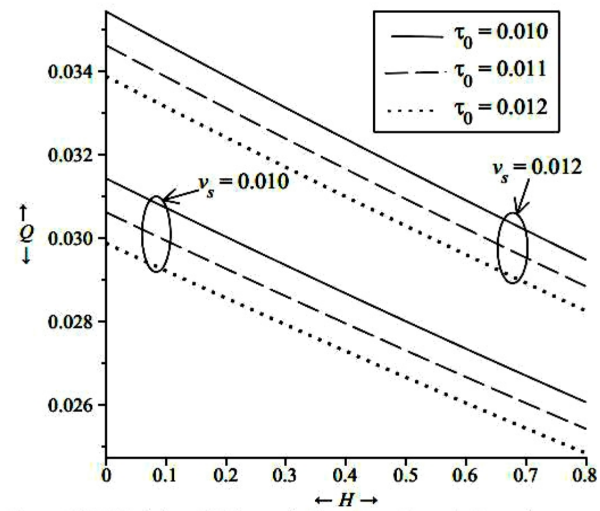


Figure 4 (b): Variation of Volumetric Flow Ratr Along the Stenosis Height for Different Values of the Yeild Stress τ_0 and Slip Velocity v_s with Some Fixed Value $\tau_R = 0.070$.

Figure 5(a) shows variations of the wall shear stress obtained in equation (3.5) along the radial distance $R(z)$ for the different values of the yield stress τ_0 and slip velocity v_s with $Q = 1$. It shows that the wall stress decreases as the slip velocity increases and increases with the increase in yield stress.

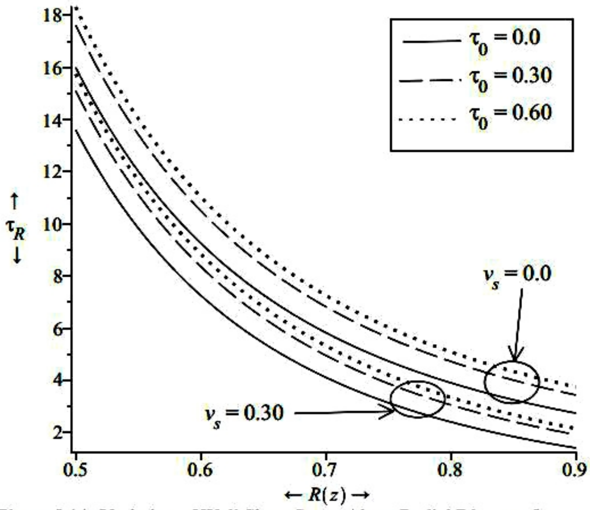


Figure 5 (a): Variation of Wall Shear Stress Along Radial Distance for Different Values of the Yeild Stress τ_0 and Slip Velocity v_s with some fixed value $Q = 1$.

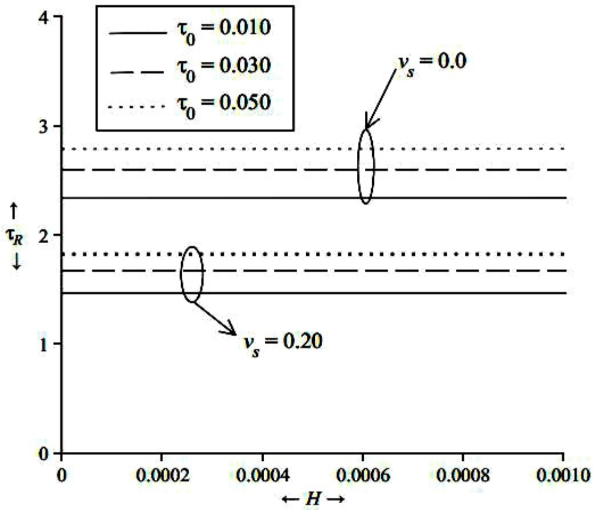


Figure 5 (b): Variation of Wall Shear Stress Along the Stenosis Height for Different Values of the Yeild Stress τ_0 and Slip Velocity v_s with some fixed value $Q = 1$

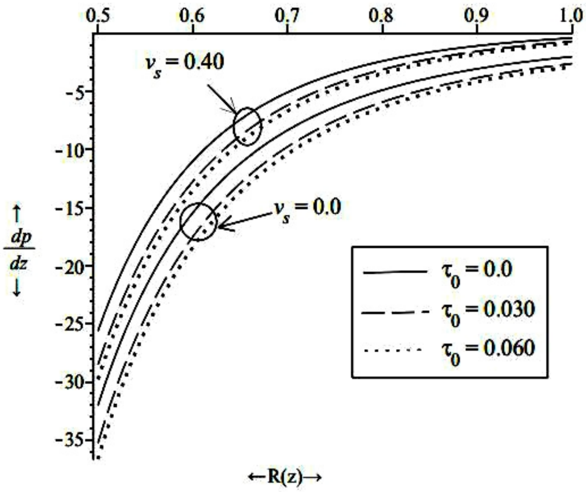


Figure 6 (a): Variation of Pressure Gradient Along Radial Distance $R(z)$ for Differentt Values of Yeild Stress τ_0 and Slip Velocity v_s with Some fixed Value Q = 1.

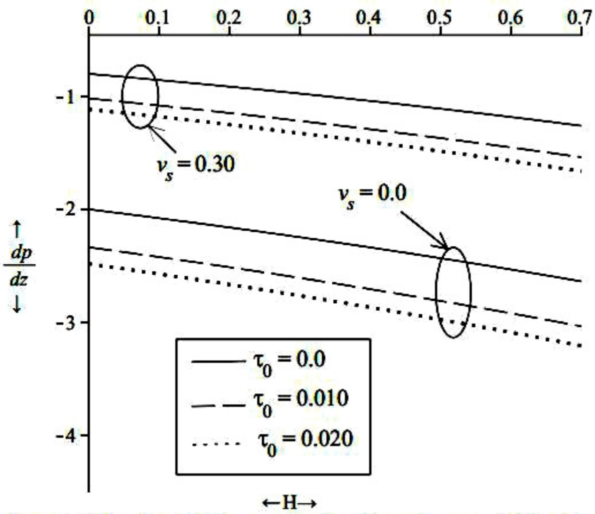


Figure 6 (b): Variation of Pressure Gradient Along the Stenosis Height for Different Values of the Yeild Sress τ_0 and Slip Velocity v_s with Some Fixed Value Q = 1.

In figure 5(b), the changes in the wall shear stress are plotted against the height H of the stenosis for the different values of the yield stress τ_0 and wall slip velocity v_s with a fixed value $Q = 1$. It shows that the wall shear stress increases continuously as the height of the stenosis increases but goes on decreasing when the yield stress increases. The wall shear stress decreases as the slip velocity increases.

Figure 6(a) shows the variations of the pressure gradient obtained in equation (3.7) along the radial distance $R(z)$ for various values of the yield stress τ_0 and slip velocity v_s assuming a fixed value $Q = 1$. Figure also shows that the pressure gradient increases ~~as~~ with the increase in slip velocity with a slower rate for higher values of the yield stress.

Figure 6(b) shows the variations of the pressure gradient along the height H of the stenosis. These variations are plotted for the various values of the yield stress τ_0 and wall slip velocity v_s with a fixed value $Q = 1$. It is observed that the pressure gradient decreases as the height of the stenosis increases but it increases when the slip velocity increases.

5 Conclusion

In this work authors tried to make a theoretical study regarding the various blood flow properties through a stenosed artery assuming blood as a Casson fluid. The study shows that the axial velocity increases due to an increase in slip velocity but it decreases as the shear stress and yield stress increase along the radial distance $R(z)$. The yield stress slows down the axial velocity in both plug flow as well as non – plug flow regions. The volumetric flow rate increases when the slip velocity increases but it begins to decrease when the yield stress increases. The wall stress decreases with increase in slip velocity and increases with increase in yield stress. The pressure gradient grows with a growth in slip velocity and yield stress. The effects of the stenosis on other flow properties like volumetric flow rate, wall shear stress and pressure gradient have also been studied and the analysis shows that these flow properties decrease as the height of the stenosis increases.

Competing Interests

Authors have declared that no competing interests exist.

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