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On Hyers-Ulam-Rassias Stability for Bernoulli's and First Order Linear and Nonlinear Differential Equations

Maher Nazmi Qarawani[1*]

[1]*Department of Mathematics, Alquds Open University, Salfit, Palestine.*

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Abstract

This paper considers the stability of linear and nonlinear differential equations of first order in the sense of Hyers-Ulam-Rassias. It also considers the Hyers-Ulam-Rassias stability for Bernoulli's differential equation. Some illustrative examples are given.

Keywords: Approximate solution, Bernoulli's differential equation, Hyers-Ulam-Rassias stability.

1 Introduction

The study of stability problems for various functional equations originated from a famous talk given by Ulam in 1940. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [1] gave a partial solution to Ulam's problem. During the last two decades many mathematicians have extensively investigated the stability problems of functional equations (see [2,3,4,5,6,7-9,10,11,12,13,14,15]).

Alsina and Ger [16] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$. In 1998, they proved that if a differentiable function $y : I \to R$ satisfies $|y' - y| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \le 3\varepsilon$, for all $t \in I$. This result of Alsina and Ger has been generalized by Takahasi et al. [17] to the case of the complex Banach space valued differential equation $y' = \lambda y$.

Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura et al. [18], Jung [19,20] and Wang et al. [21]. Li [22] established the stability of linear differential equation of second order in the sense of the Hyers and Ulam $y'' = \lambda y$. Li and Shen [23] proved the stability of nonhomogeneous linear differential equation of second order in the sense of the Hyers and Ulam $y'' + p(x)y' + q(x)y + r(x) = 0$, while Gavruta

*Corresponding author: mkerawani@qou.edu;

et al. [24] proved the Hyers-Ulam stability of the equation $y'' + \beta(x)y = 0$ with boundary and initial conditions. The author in his studies [25,26] established the Hyers-Ulam stability of the differential equations of second order with initial conditions.

In this paper we investigate the Hyers-Ulam-Rassias stability of the following linear differential equation of order one

$$y' + P(t)y = Q(t) \tag{1}$$

with the initial condition

$$y(t_0) = y_0 \tag{2}$$

where $y \in C^1(I)$, $I = [t_0, \ t_1]$, $-\infty < t_0 < t_1 \leq \infty$.

We also consider a nonlinear differential equation of order one

$$y' + p(t)y = G(t, y) \tag{3}$$

with initial condition

$$y(t_0) = y_0 \tag{4}$$

where $G(t, y) : [t_0, t_1] \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous function that satisfies Lipschitz condition $\left|G(t, y) - G(t, z)\right| \leq L\left|y - z\right|$, for any $t \in [t_0, t_1]$ and $y, z \in \boldsymbol{R}$, $G(t, 0) = 0, -\infty < t_0 < t_1 \leq \infty.$.

Moreover we establish the Hyers-Ulam-Rassias stability for Bernoulli's equation

$$y' + p(t)y = q(t)y^n \ , \quad n \neq 1 \tag{5}$$

with the initial condition

$$y(t_0) = y_0 \tag{6}$$

Definition 1.1 We say that the equation (1) has the Hyers –Ulam-Rassias (HUR) stability if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0$, $y \in C^1(I)$, if

$$| y' + P(t)y - Q(t) | \leq \varepsilon\varphi(t) \tag{7}$$

with the initial condition (2), then there exists a solution $x(t) \in C^1(I)$ of the equation (1), such that $| y(t) - x(t) | \leq K\varepsilon\varphi(t)$, where K is a constant that does not depend on ε nor on $y(t)$.

Definition 1.2 We say that the equation (1) has the Hyers –Ulam-Rassias-Gavruta (HURG) stability with respect to $\varphi(t) > 0$, if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0$, $y \in C^1(I)$, if

$$| y' + P(t)y - Q(t) | \le \varepsilon\varphi(t) \tag{8}$$

with the initial condition (2), then there exists a solution $x(t) \in C^1(I)$ of the equation (1), such that $| y(t) - x(t) | \le K\varepsilon\phi(t)$, where K is a constant that does not depend on ε nor on $y(t)$.

Definition 1.3 We say that the equation (3) has the Hyers-Ulam-Rassias stability if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0$, $y \in C^1(I)$, if

$$| y' + p(t)y - G(t, y) | \le \varepsilon\varphi(t) \tag{9}$$

with the initial condition (4), then there exists a solution $x(t) \in C^1(I)$ of the equation (1),such that $| y(t) - x(t) | \le K\varepsilon\varphi(t)$, where K is a constant that does not depend on ε nor on $y(t)$.

Definition 1.4 We say that the equation (3) has the Hyers-Ulam-Rassias-Gavruta stability if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0$, $y \in C^1(I)$, if

$$| y' + p(t)y - q(t)y^n | \le \varepsilon\varphi(t) \tag{10}$$

with the initial condition (4), then there exists a solution $x(t) \in C^1(I)$ of the equation (1),such that $| y(t) - x(t) | \le K\varepsilon\phi(t)$, where K is a constant that does not depend on ε nor on $y(t)$.

2. Hyers-Ulam-Rassias Stability of First Order Linear Equation

Theorem 2.1 If $y \in C^1(I)$ and $P(t)$ and $Q(t)$ are continuous functions on I then the initial value problem (1), (2) is stable in the sense of HUR.

Proof. Let $\varepsilon > 0$ and $y(t)$ be an approximate solution of the initial value problem (1),(2)· We will show that there exists a function $x(t) \in C^1(I)$ satisfying (1) and (2) such that

$$\left| y(t) - x(t) \right| \le K\varepsilon\varphi(t)$$

Setting $\varphi(t) = e^{-\int_{t_0}^{t} P(s)ds}$ in (7), we have

$$-\varepsilon e^{-\int_{t_0}^{t} P(s)ds} \le y' + P(t)y - Q(t) \le \varepsilon e^{-\int_{t_0}^{t} P(s)ds} \tag{11}$$

Multiplying (11) by $e^{\int_{t_0}^{t} P(s)ds}$, we obtain

$$-\varepsilon \le \left(y e^{\int_{t_0}^{t} P(s)ds} \right)' - e^{\int_{t_0}^{t} P(s)ds} Q(t) \le \varepsilon \tag{12}$$

Integrating (12) with respect to t, we have

$$-\varepsilon(t-t_0) \le y e^{\int_{t_0}^{t} P(s)ds} - y_0 - \int_{t_0}^{t} e^{\int_{t_0}^{t} P(s)ds} Q(t) \le \varepsilon(t-t_0) \tag{13}$$

that is,

$$-\varepsilon(t-t_0) e^{-\int_{t_0}^{t} P(s)ds} \le y - e^{-\int_{t_0}^{t} P(s)ds} \left(y_0 + \int_{t_0}^{t} e^{\int_{t_0}^{t} P(s)ds} Q(s)ds \right) \le \varepsilon(t-t_0) e^{-\int_{t_0}^{t} P(s)ds} \tag{14}$$

From which we get

$$\left| y - e^{-\int_{t_0}^{t} P(s)ds} \left(y_0 + \int_{t_0}^{t} Q(s) e^{\int_{t_0}^{s} P(\tau)d\tau} ds \right) \right| \le \varepsilon(t_1 - t_0) e^{-\int_{t_0}^{t} P(s)ds}$$

One can easily show that

$$x(t) = e^{-\int_{t_0}^{t} P(s)ds} \left(y_0 + \int_{t_0}^{t} Q(s) e^{\int_{t_0}^{s} P(\tau)d\tau} ds \right)$$

is an exact solution of initial value problem (1),(2).
Whence

$$\max_{t_0 \le t \le t_1} |y(t) - x(t)| \le K\varepsilon\varphi(t) \text{ , where } \varphi(t) = e^{-\int_{t_0}^{t} P(s)ds}$$

Hence the problem (1),(2) is stable in the sense of HUR.

In the following theorem we establish the Hyers-Ulam-Rassias stability for (1),(2) in the interval $-\infty < t_0 < t < t_1 \le \infty$.

Theorem 2.2 If $y \in C^1(\boldsymbol{R})$, $P(t)$, $Q(t)$ are continuous function in $\boldsymbol{R}$, and $P(t) \ge c > 0$, then the initial value problem (1),(2) is stable in the sense of HURG.
Proof. Given the following inequality

$$-\varepsilon e^{-\int_{t_0}^{t} P(s)ds} \le y' + P(t)y - Q(t) \le \varepsilon e^{-\int_{t_0}^{t} P(s)ds}$$

By applying an argument quite similar to the one used in proof of Theorem 1.1, we arrived at the same inequality

$$\left| y - e^{-\int_{t_0}^{t} P(s)ds} \left(y_0 + \int_{t_0}^{t} Q(s) e^{\int_{t_0}^{s} P(\tau)d\tau} ds \right) \right| \le \varepsilon (t - t_0) e^{-\int_{t_0}^{t} P(s)ds} \tag{15}$$

and

$$x(t) = e^{-\int_{t_0}^{t} P(s)ds} \left(y_0 + \int_{t_0}^{t} Q(s) e^{\int_{t_0}^{s} P(\tau)d\tau} ds \right)$$

is an exact solution of problem (1),(2).

Now sending $t \to \infty$ in (15) and taking into account that $\int_{t_0}^{\infty} P(t)dt = \infty$, we find

$\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} (t - t_0) e^{-\int_{t_0}^{t} P(s)ds} = 0$, whence

$$\lim_{t\to\infty} \left(y(t) - x(t) \right) = 0$$

Thus, the problem (1),(2) is stable in the sense of HURG.

Remark 2.1 If $P(t) = -\beta(t)$, $\beta(t) \ge c > 0$, then replacing $P(t)$ by $-\beta(t)$ in the proof of Theorems 2.1 and 2.2, then problem (1), (2) has HUR stability and HURG stability, respectively.

Example 2.1 Let $$y' + 2y\sin^2 t = \cos t, \quad y(0) = 1, \quad t \ge 0 \tag{16}$$

Suppose that $y(t)$ is a solution of the inequality

$$| y' + 2y\sin^2 t - \cos t \;| \le \varepsilon e^{-2\int_0^t \sin^2 s ds} \tag{17}$$

Using the same argument used above, we obtain

$$\left| y - e^{-t + \frac{1}{2}\sin 2t} \left(1 + \int_0^t \cos s e^{s - \frac{1}{2}\sin 2s} ds \right) \right| \le \varepsilon t e^{-t + \frac{1}{2}\sin 2t} \tag{18}$$

One shows that $x(t) = e^{-t + \frac{1}{2}\sin 2t} \left(1 + \int_0^t \cos s e^{s - \frac{1}{2}\sin 2s} ds \right)$ is a solution of Eq. (16), such that

$| y - x | \leq K\varepsilon t e^{-t+\frac{1}{2}\sin 2t}, \forall t \geq 0.$

Now, $\lim_{t\to\infty} t e^{-t+\frac{1}{2}\sin 2t} = 0,$ then from (18) we obtain HURG stability of Eq. (16).

3. Hyers-Ulam-Rassias Stability for Bernoulli's Differential Equation

In this section we investigate stability and asymptotic stability in the sense of Hyers-Ulam-Rassias for Bernoulli's differential equation.

First let's consider a differential equation

$$y' + p(t)y = G(t, y) \tag{19}$$

with initial condition

$$y(t_0) = y_0 \tag{20}$$

where $G(t, y) : [t_0, t_1] \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous function which satisfies a Lipschitz condition $\left|G(t, y) - G(t, z)\right| \leq L\left|y - z\right|$ for any $t \in [t_0, t_1]$ and $y, z \in \boldsymbol{R}$, $G(t, 0) = 0$.

Theorem 3.1 If $y \in C^1(I)$ and $p(t)$ are continuous functions on I , then the initial value problem(19), (20) is stable in the sense of HUR.

Proof. Let $\varepsilon > 0$ and $y(t)$ be an approximate solution of the initial value problem (19),(20). We will show that there exists a function $x(t) \in C^1(I)$ satisfying (19) and (20) such that

$$\left|y(t) - x(t)\right| \leq K\varepsilon\varphi(t)$$

Consider the inequality

$$-\varepsilon\varphi(t) \leq y' + p(t)y - G(t, y) \leq \varepsilon\varphi(t) \tag{21}$$

Setting $\varphi(t) = e^{-\int_{t_0}^{t} p(s)ds}$ in (21), we have

$$-\varepsilon e^{-\int_{t_0}^{t} p(s)ds} \leq y' + p(t)y - G(t, y) \leq \varepsilon e^{-\int_{t_0}^{t} p(s)ds} \tag{22}$$

Multiplying (22) by $e^{\int_{t_0}^{t} p(s)ds}$, we obtain

$$-\varepsilon \le \left(y e^{\int_{t_0}^{t} p(s)ds} \right)' - e^{\int_{t_0}^{t} p(s)ds} G(t,y) \le \varepsilon \tag{23}$$

Integrating (23) with respect to t, we have

$$-\varepsilon(t-t_0) \le y e^{\int_{t_0}^{t} p(s)ds} - y_0 - \int_{t_0}^{t} e^{\int_{t_0}^{s} p(\tau)d\tau} G(s,y)ds \le \varepsilon(t-t_0) \tag{24}$$

that is,

$$-\varepsilon(t-t_0) e^{-\int_{t_0}^{t} p(s)ds} \le y - e^{-\int_{t_0}^{t} p(s)ds} \left(y_0 + \int_{t_0}^{t} e^{\int_{t_0}^{s} p(\tau)d\tau} G(s,y)ds \right) \le \varepsilon(t-t_0) e^{-\int_{t_0}^{t} p(s)ds} \tag{25}$$

From which we get

$$\left| y - e^{-\int_{t_0}^{t} p(s)ds} \left(y_0 + \int_{t_0}^{t} G(s,y) e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right) \right| \le \varepsilon(t_1-t_0) e^{-\int_{t_0}^{t} p(s)ds}$$

One can easily show that

$$x(t) = e^{-\int_{t_0}^{t} p(s)ds} \left(y_0 + \int_{t_0}^{t} G(s,x) e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right)$$

satisfies the initial value problem (1),(2).
Now consider the difference

$$\begin{aligned}
|y(t)-x(t)| &\le \left| y(t) - e^{-\int_{t_0}^{t} p(\tau)d\tau} \left(y_0 + \int_{t_0}^{t} G(s,y) e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right) \right| + \\
&\quad \left| e^{-\int_{t_0}^{t} p(\tau)d\tau} \left(y_0 + \int_{t_0}^{t} G(s,y) e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right) - e^{-\int_{t_0}^{t} p(\tau)d\tau} \left(y_0 + \int_{t_0}^{t} G(s,x) e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right) \right| \\
&\le \varepsilon(t_1-t_0) e^{-\int_{t_0}^{t} p(s)ds} + e^{-\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} |G(s,y)-G(s,x)| e^{\int_{t_0}^{s} p(\tau)d\tau} ds \\
&\le \varepsilon(t_1-t_0) e^{-\int_{t_0}^{t} p(s)ds} + L e^{-\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} |y-x| e^{\int_{t_0}^{s} p(\tau)d\tau} ds \\
&\le \varepsilon(t_1-t_0) e^{-\int_{t_0}^{t} p(s)ds} + L \int_{t_0}^{t} |y-x| e^{\int_{t}^{s} p(\tau)d\tau} ds
\end{aligned}$$

From Gronwall's inequality it follows that

$$\left|y(t)-x(t)\right| \le \varepsilon(t_1-t_0)e^{-\int_{t_0}^{t} p(s)ds} e^{L[1-e^{-p(t^*)(t-t_0)}]} \le \varepsilon(t_1-t_0)e^{L}e^{-\int_{t_0}^{t} p(s)ds}$$

Whence

$$\max_{t_0\le t\le t_1}\left|y(t)-x(t)\right| \le K\varepsilon\varphi(t)\ , \text{ where } \varphi(t)=e^{-\int_{t_0}^{t} P(s)ds}$$

Hence the problem (19),(20) is stable in the sense of HUR.

In the following theorem we establish the Hyers-Ulam-Rassias stability for (19),(20) in the interval $-\infty < t_0 < t < t_1 \le \infty$.

Theorem 3.2 If $y \in C^1(\boldsymbol{R})$, $p(t)$ are continuous function in $\boldsymbol{R}$, and $p(t) \ge c > 0$, then the initial value problem (19),(20) is stable in the sense of HURG.

Proof. Using a similar argument to above we get

$$\left|y(t)-x(t)\right| \le \varepsilon(t_1-t_0)e^{-\int_{t_0}^{t} p(s)ds} + L\int_{t_0}^{t}\left|y-x\right| e^{\int_{t}^{s} p(\tau)d\tau} ds$$

Using Gronwall's inequality we infer that

$$\begin{aligned}\left|y(t)-x(t)\right| &\le \varepsilon(t-t_0)e^{-\int_{t_0}^{t} cds} e^{L\int_{t_0}^{t} e^{-\int_{s}^{t} cd\tau} ds} \le \varepsilon(t-t_0)e^{-c(t-t_0)}e^{L\int_{t_0}^{t} e^{-c(t-s)}ds} \\ &\le \varepsilon(t-t_0)e^{-c(t-t_0)}e^{\frac{L}{c}(1-e^{-c(t-t_0)})} \le \varepsilon(t-t_0)e^{-c(t-t_0)}e^{\frac{L}{c}} \qquad (26)\end{aligned}$$

Now, sending $t \to \infty$ in (26), we find $\lim_{t\to\infty}(t-t_0)e^{-c(t-t_0)} = 0$, whence

$$\lim_{t\to\infty}\left(y(t)-x(t)\right)=0$$

Thus, the problem (19),(20) is stable in the sense of HURG.

Now we consider stability in the sense of Hyers-Ulam-Rassias for Bernoulli's differential equation.

Theorem 3.3 Assume that in Eq. (3) $n>1$. If $y \in C^1(I)$, $p(t)$ and $q(t)$ are continuous functions on I, then the initial value problem (3), (4) is stable in the sense of HUR.

Proof. Let $\varepsilon > 0$ and $y(t)$ be an approximate solution of the initial value problem (3),(4). We will show that there exists a function $x(t) \in C^1(I)$ satisfying (3) and (4) such that

$$\left| y(t) - x(t) \right| \le K\varepsilon\varphi(t)$$

Consider the inequality

$$-\varepsilon\varphi(t) \le y' + p(t)y - q(t)y^n \le \varepsilon\varphi(t) \tag{27}$$

Substituting $\varphi(t) = e^{-\int_{t_0}^{t} p(s)ds}$ in (27), we get

$$-\varepsilon e^{-\int_{t_0}^{t} p(s)ds} \le y' + p(t)y - q(t)y^n \le \varepsilon e^{-\int_{t_0}^{t} p(s)ds} \tag{28}$$

Multiplying (28) by $e^{\int_{t_0}^{t} p(s)ds}$, we obtain

$$-\varepsilon \le \left(y e^{\int_{t_0}^{t} p(s)ds} \right)' - e^{\int_{t_0}^{t} p(s)ds} q(t)y^n \le \varepsilon \tag{29}$$

Integrating (29) with respect to t, we have

$$-\varepsilon(t - t_0) \le y e^{\int_{t_0}^{t} p(s)ds} - y_0 - \int_{t_0}^{t_1} e^{\int_{t_0}^{s} p(\tau)d\tau} q(s)y^n dt \le \varepsilon(t - t_0) \tag{30}$$

Now, multiplying (30) by $e^{-\int_{t_0}^{t} p(s)ds}$, we obtain

$$-\varepsilon(t - t_0)e^{-\int_{t_0}^{t} p(s)ds} \le y - e^{-\int_{t_0}^{t} p(s)ds} \left(y_0 + \int_{t_0}^{t_1} e^{\int_{t_0}^{s} p(\tau)d\tau} q(s)y^n dt \right) \le \varepsilon(t - t_0)e^{-\int_{t_0}^{t} p(s)ds} \tag{31}$$

From which it follows that

$$\left| y - e^{-\int_{t_0}^{t} p(s)ds} \left(y_0 + \int_{t_0}^{t} q(s)y^n e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right) \right| \le \varepsilon(t_1 - t_0)e^{-\int_{t_0}^{t} p(s)ds}$$

It can be easily shown that

$$x(t) = e^{-\int_{t_0}^{t} p(s)ds} \left(y_0 + \int_{t_0}^{t} q(s)x^n e^{\int_{t_0}^{s} p(\tau)d\tau} ds \right)$$

satisfies the initial value problem (3),(4).
Consider the difference

$$\left|y(t)-x(t)\right| \le \left|y(t)-e^{-\int_{t_0}^{t} p(\tau)d\tau}\left(y_0+\int_{t_0}^{t} q(s)y^n e^{\int_{t_0}^{s} p(\tau)d\tau} ds\right)\right| +$$

$$\left|e^{-\int_{t_0}^{t} p(\tau)d\tau}\left(y_0+\int_{t_0}^{t} q(s)y^n e^{\int_{t_0}^{s} p(\tau)d\tau} ds\right)-e^{-\int_{t_0}^{t} p(\tau)d\tau}\left(y_0+\int_{t_0}^{t} q(s)x^n e^{\int_{t_0}^{s} p(\tau)d\tau} ds\right)\right|$$

$$\le \varepsilon(t_1-t_0)e^{-\int_{t_0}^{t} p(s)ds}+e^{-\int_{t_0}^{t} p(\tau)d\tau}\int_{t_0}^{t}\left|q(s)\right|\left|y^n-x^n\right|e^{\int_{t_0}^{s} p(\tau)d\tau} ds$$

Now, $\left|\dfrac{\partial\left(q(t)y^n\right)}{\partial y}\right|=\left|nq(t)y^{n-1}\right|$ is bounded for all $(t,y)\in[t_0,t_1]\times \boldsymbol{R},$ then $q(t)y^n$ satisfies Lipschitz condition for $n>1.$

Hence

$$\left|y(t)-x(t)\right| \le \varepsilon(t_1-t_0)e^{-\int_{t_0}^{t} p(s)ds}+L\int_{t_0}^{t}\left|y-x\right|e^{\int_{t}^{s} p(\tau)d\tau} ds$$

Using Gronwall's inequality we infer

$$\left|y(t)-x(t)\right| \le \varepsilon(t_1-t_0)e^{-\int_{t_0}^{t} p(s)ds}e^{Lq_0[1-e^{-p(t^*)(t-t_0)}]} \le \varepsilon(t_1-t_0)e^{Lq_0}e^{-\int_{t_0}^{t} p(s)ds}$$

Whence

$$\max_{t_0\le t\le t_1}\left|y(t)-x(t)\right| \le K\varepsilon\varphi(t)\ , \text{ where } \varphi(t)=e^{-\int_{t_0}^{t} P(s)ds}$$

Hence, the proof is completed.

Example 3.1 Let

$$y'+4ty=ty^2, \quad y(0)=1, \quad t\in[0,t_1] \tag{32}$$

Consider the inequality

$$-\varepsilon\varphi(t) \le y'+4ty-ty^2 \le \varepsilon\varphi(t) \tag{33}$$

Substituting $\varphi(t)=e^{-2t^2}$ in (33), we get

$$-\varepsilon e^{-2t^2} \le y'+4ty-ty^2 \le \varepsilon e^{-2t^2} \tag{34}$$

Multiplying (34) by e^{2t^2}, we obtain

$$-\varepsilon \le \left(ye^{2t^2}\right)' - e^{2t^2}ty^2 \le \varepsilon \tag{35}$$

Integrating (35) with respect to t, we have

$$-\varepsilon t \le ye^{2t^2} - 1 - \int_0^{t_1} e^{2s^2} sy^2 ds \le \varepsilon t \tag{36}$$

Now, multiplying (36) by e^{-2t^2}, we obtain

$$-\varepsilon te^{-2t^2} \le y - e^{-2t^2}\left(1 + \int_0^t e^{2s^2} sy^2 ds\right) \le \varepsilon te^{-2t^2} \tag{37}$$

From which it follows that

$$\left| y - e^{-2t^2}\left(1 + \int_0^t e^{2s^2} sy^2 ds\right)\right| \le \varepsilon t_1 e^{-2t^2}$$

It can be easily shown that

$$x(t) = e^{-2t^2}\left(1 + \int_0^t se^{2s^2} x^2 ds\right)$$

satisfies the initial value problem (3),(4).

Consider the difference

$$\left|y(t) - x(t)\right| \le \left| y(t) - e^{-2t^2}\left(1 + \int_0^t se^{2s^2} y^2 ds\right)\right| + e^{-2t^2}\int_0^t se^{2s^2}\left|x^2 - y^2\right| ds$$

Using Gronwall's inequality we infer

$$\left|y(t) - x(t)\right| \le \varepsilon t_1 e^{-2t^2} e^{e^{-2t^2} L \int_0^t se^{2s^2} ds} \le \varepsilon t_1 e^{-2t^2} e^{L}$$

Whence

$$\max_{t_0 \le t \le t_1} \left|y(t) - x(t)\right| \le K\varepsilon\varphi(t)$$

Hence, the proof is completed.

4. Conclusion

In this paper we have obtained criteria for stability of linear and nonlinear differential equations of first order in the sense of Hyers-Ulam-Rassias. It also considers the Hyers-Ulam-Rassias stability for Bernoulli's differential equation.

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The author has declared that no competing interests exist.

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