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2-tuple Linguistic Bonferroni Mean Operators and Their Application to Multiple Attribute Group Decision Making

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Abstract

Aims: The aim of this paper is to develop the 2-tuple linguistic Bonferroni mean and the weighted 2-tuple linguistic Bonferroni mean.
Study Design: Some desirable properties and special cases of the developed operators are discussed. The geometric Bonferroni mean (GBM) is a generalization of the Bonferroni mean and geometric mean. In this paper, we also investigate the GBM under 2-tuple linguistic environments. We develop the 2-tuple linguistic geometric Bonferroni mean and the weighted 2-tuple linguistic geometric Bonferroni mean. We investigate some fundamental properties and special cases of them.
Place and Duration of Study: The Bonferroni Mean (BM) operator is a traditional mean type aggregation operator, which can capture the expressed interrelationship of the individual arguments and which is only suitable to aggregate crisp data.
Methodology: This paper extends the BM operator to 2-tuple linguistic environments.
Results: Based on these operators, we develop two approaches for multiple attribute group decision making with 2-tuple linguistic information.
Conclusion: Two numerical examples are provided to illustrate the effectiveness and practicality of the proposed approaches.

Keywords: Multiple attribute group decision making; 2-tuple linguistic information; 2-tuple linguistic Bonferroni mean; 2-tuple linguistic geometric Bonferroni mean.

1 Introduction

In many multiple attribute group decision making (MAGDM) problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking [1,2,3]; thus, it may be appropriate and sufficient to assess the decision information in a qualitative form

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rather than a quantitative form. For example, when evaluating a house's cost, linguistic terms such as ‘‘high’’, ‘‘medium’’, and ‘‘low’’ are usually used, and when evaluating a house's design, linguistic terms like ‘‘good’’, ‘‘medium’’, and ‘‘bad’’ can be frequently used. To date, many methods have been developed for dealing with linguistic information [4-13]. Herrera and Martinez [14] introduced a 2-tuple fuzzy linguistic representation model that represents the linguistic information by means of 2-tuples, which are composed by a linguistic term and a number [14,15]. The main advantage of this representation is to allow a continuous representation of the linguistic information on its domain; therefore, it can represent any counting of information obtained in a aggregation process without any loss of information [14,15]. In the past few decades, a variety of 2-tuple linguistic aggregation operators [15-24] have been developed for aggregating 2-tuple linguistic information. However, these 2-tuple linguistic aggregation operators only emphasize the importance of each data or ordered position and they cannot reflect the interrelationships of individual data.

The Bonferroni mean, originally introduced by Bonferroni [25], is a mean-type aggregation operator and it can provide for the aggregation lying between the max and min operators and the logical “or” and “and” operators [26]. The desirable characteristic of the BM is its capability to capture the interrelationship between input arguments [26]. Recently, Yager [27] gave a detailed studied of the BM and proposed some generalizations of the BM. However, the BM [25] and its these generalizations [27] only can accommodate the situations where the input arguments take the form of crisp numbers. In many group decision makings, the attribute values are given in the form of the other types of domains which are not suitable to be aggregated by the BM, such as interval numbers [28], intuitionistic fuzzy numbers [29], interval-valued intuitionistic fuzzy numbers [30], hesitant fuzzy elements [31], uncertain linguistic variables [32], and triangular fuzzy linguistic variables [33]. To address this issue, some authors have suggested some new generalizations of the BM, including the uncertain Bonferroni mean operator [34], the weighted uncertain Bonferroni mean operator [34], the intuitionistic fuzzy BM (IFBM) [26], the weighted intuitionistic fuzzy Bonferroni mean (WIFBM) [26], the Atanassov's intuitionistic fuzzy geometric Bonferroni mean [35], the weighted Atanassov's intuitionistic fuzzy geometric Bonferroni mean [35], the interval-valued intuitionistic fuzzy Bonferroni mean [36], the weighted interval-valued intuitionistic fuzzy Bonferroni mean [36], the hesitant fuzzy geometric Bonferroni mean (HFGBM) [37], the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM) [37], the weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) [37], the weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM) [37], the uncertain linguistic Bonferroni mean (ULBM) operator [38], the uncertain linguistic weighted Bonferroni mean (ULWBM) operator [38], the uncertain linguistic geometric Bonferroni mean (ULGBM) operator [38], the uncertain linguistic weighted geometric Bonferroni mean (ULWGBM) operator [38], the trapezoid fuzzy linguistic Bonferroni mean (TFLBM) operator [39], the trapezoid fuzzy linguistic weighted Bonferroni mean (TFLWBM) operator [39], the trapezoid fuzzy linguistic Bonferroni OWA (TFLBOWA) operator [39], the trapezoid fuzzy linguistic weighted Bonferroni OWA (TFLWBOWA) operator [39], the generalized intuitionistic fuzzy weighted Bonferroni mean (GIFWBM) [40], and the generalized weighted Bonferroni geometric mean (GWBGM) [40]. However, these Bonferroni mean operators cannot accommodate the situations where the input arguments take the form of 2-tuples.

Based on the aforementioned analysis, we can conclude that the existing 2-tuple linguistic aggregation operators do not consider the interrelationship of the individual arguments, while the existing Bonferroni mean operators cannot accommodate the situations in which the input arguments take the form of 2-tuples. To overcome this drawback, it is therefore necessary to develop some new aggregation operators that not only accommodate 2-tuple linguistic information but also consider the interrelationship of the individual arguments. To do this, we extend the BM operator to 2-tuple linguistic environments and then develop two 2-tuple linguistic aggregation operators, including the 2-tuple linguistic Bonferroni mean and the weighted 2-tuple linguistic Bonferroni mean. The desirable characteristic of these two operators is that they not only accommodate the input arguments in the form of 2-tuples but also reflect the interrelationship of the input arguments. Xia et al. [35] introduced a new Bonferroni mean called the geometric Bonferroni mean based on the BM and the geometric mean (GM). We further extend the GBM operator to 2-tuple linguistic environments and develop the 2-tuple linguistic geometric Bonferroni mean and the weighted 2-tuple linguistic geometric Bonferroni mean. Finally, we utilize the proposed operators to develop two approaches for multiple attribute group decision making with 2-tuple linguistic information and then apply both the developed approaches to two practical examples.

The remainder of this paper is organized as follows. In Section 2, we briefly review some basic concepts and operations related to the 2-tuple fuzzy linguistic representation model, the BM, and the GBM. In Section 3, the 2-tuple linguistic Bonferroni mean and the weighted 2-tuple linguistic Bonferroni mean are developed, some desirable properties of these operators are studied, and some special cases are discussed. Section 4 develops the 2-tuple linguistic geometric Bonferroni mean and the weighted 2-tuple linguistic geometric Bonferroni mean. Section 5 introduces two approaches based on these operators for multiple attribute group decision making with 2-tuple linguistic information. In Section 6, we give two practical examples to illustrate the group decision making steps based on the proposed approaches. Section 7 ends this paper with some concluding remarks.

2. Preliminaries

In this section, we will introduce the basic notions of the 2-tuple fuzzy linguistic approach, Bonferroni mean, and geometric Bonferroni mean.

2.1 The 2-tuple Fuzzy Linguistic Representation Model

Let $S=\{s_i \mid i=0,1,2,\cdots,g\}$ be a finite and totally ordered discrete linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [14,41,42].

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;

(2) There is the negation operator: $neg(s_i)=s_j$ such that $j=g-i$;

(3) Max operator: $\max(s_i,s_j)=s_i$ if $s_i \geq s_j$;

(4) Min operator: $\min(s_i,s_j)=s_i$ if $s_i \leq s_j$.

For example, a set of seven terms S could be given as follows [43-47]:

$$S=\{s_0=nothing,\ s_1=very\,low,\ s_2=low,\ s_3=medium,\ s_4=high,\ s_5=very\,high,\ s_6=perfect\}.$$

Based on the concept of symbolic translation, Herrera and Martinez [14,41] introduced a 2-tuple fuzzy linguistic representation model for dealing with linguistic information. This model represents the linguistic assessment information by means of a 2-tuple (s_i, α), where $s_i \in S$ represents a linguistic label from the predefined linguistic term set S and $\alpha \in [-0.5, 0.5)$ is the value of symbolic translation.

Definition 2.1 [14,41]. Let β be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set S, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g+1$ the cardinality of S. Let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$ then α is called a symbolic translation, where $\text{round}(\cdot)$ is the usual round operation.

Definition 2.2 [14,41]. Let $S = \{s_i | i = 0, 1, 2, \cdots, g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation. Then, the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0, g] \to S \times [-0.5, 0.5) \tag{1}$$

$$\Delta(\beta) = (s_i, \alpha), \quad \text{with } \begin{cases} s_i, & i = \text{round}(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} \tag{2}$$

where s_i has the closest index label to β and α is the value of the symbolic translation.

Theorem 2.1 [14,41]. Let $S = \{s_i | i = 0, 1, 2, \cdots, g\}$ be a linguistic term set and (s_i, α) be a 2-tuple. There is always a Δ^{-1} function such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, g] \subset R$, where

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, g] \tag{3}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta. \tag{4}$$

It is obvious that the conversion of a linguistic term into a 2-tuple consists of adding a value zero as symbolic translation

$$s_i \in S \Rightarrow (s_i, 0).$$

Definition 2.3 [14,41]. The comparison of linguistic information represented by 2-tuples is carried out according to an ordinary lexicographic order. Let (s_k, α_k) and (s_l, α_l) be two 2-tuples, with each one representing a counting of information as follows.

(1) If $k < l$ then (s_k, α_k) is smaller than (s_l, α_l).

(2) If $k = l$ then

- if $\alpha_k = \alpha_l$ then (s_k, α_k), (s_l, α_l) represents the same information;
- if $\alpha_k < \alpha_l$ then (s_k, α_k) is smaller than (s_l, α_l);
- if $\alpha_k > \alpha_l$ then (s_k, α_k) is bigger than (s_l, α_l).

Theorem 2.2. Let (s_k, α_k) and (s_l, α_l) be two 2-tuples, $\beta_k = \Delta^{-1}(s_k, \alpha_k)$, and $\beta_l = \Delta^{-1}(s_l, \alpha_l)$. Then, $(s_k, \alpha_k) < (s_l, \alpha_l)$ if and only if $\beta_k < \beta_l$, and $(s_k, \alpha_k) = (s_l, \alpha_l)$ if and only if $\beta_k = \beta_l$.

Proof. (1) We first prove that $(s_k, \alpha_k) < (s_l, \alpha_l)$ if and only if $\beta_k < \beta_l$. Assume that $(s_k, \alpha_k) < (s_l, \alpha_l)$. Then, $k < l$, or $k = l$ and $\alpha_k < \alpha_l$. If $k < l$, then we have $\beta_k = k + \alpha_k < k + 0.5 \le l - 0.5 \le l + \alpha_l = \beta_l$. If $k = l$ and $\alpha_k < \alpha_l$, then we have $\beta_k = k + \alpha_k < l + \alpha_l = \beta_l$.

Assume that $\beta_k < \beta_l$. Then, $k < l$, or $k = l$ and $\alpha_k < \alpha_l$. If $k < l$, then we have $(s_k, \alpha_k) < (s_l, \alpha_l)$. If $k = l$ and $\alpha_k < \alpha_l$, then we have $(s_k, \alpha_k) < (s_l, \alpha_l)$.

(2) We next prove that $(s_k, \alpha_k) = (s_l, \alpha_l)$ if and only if $\beta_k = \beta_l$. If $(s_k, \alpha_k) = (s_l, \alpha_l)$, then $k = l$ and $\alpha_k = \alpha_l$, which implies that $\beta_k = k + \alpha_k = l + \alpha_l = \beta_l$. If $\beta_k = \beta_l$, then $k = l$ and $\alpha_k = \alpha_l$, which implies that $(s_k, \alpha_k) = (s_l, \alpha_l)$.

2.2 Bonferroni Mean and Geometric Bonferroni Mean

Bonferroni [25] originally introduced a mean type aggregation operator, called Bonferroni mean, which can provide for aggregation lying between the max, min operators and the logical "or" and "and" operators.

Definition 2.4 [25]. Let $p, q \ge 0$, and let $a_i\ (i = 1, 2, \cdots, n)$ be a collection of non-negative real numbers. Then, the aggregation function:

$$B^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{5}$$

is called a Bonferroni mean (BM).

Based on the usual geometric mean (GM) and the BM, Xia et al. [35] introduced the geometric Bonferroni mean, which was defined as follows:

Definition 2.5 [35]. Let $p, q \ge 0$, and $a_i\ (i = 1, 2, \cdots, n)$ be a collection of non-negative numbers. If

$$GB^{p,q}(a_1,a_2,\cdots,a_n)=\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(pa_i+qa_j\right)^{\frac{1}{n(n-1)}} \tag{6}$$

then we call $GB^{p,q}$ the geometric Bonferroni mean (GBM).

3. 2-tuple Linguistic Bonferroni Mean and Weighted 2-tuple Linguistic Bonferroni Mean

In this section, we first extend the Bonferroni mean operator (Eq. (5)) to 2-tuple linguistic environment, i.e., develop a 2-tuple linguistic Bonferroni mean operator and its weight form.

Definition 3.1. Let $p \geq 0$, $q \geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n)$ be a collection of 2-tuples. If

$$2\mathrm{TLB}^{p,q}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right), \tag{7}$$

then $2\mathrm{TLB}^{p,q}$ is called the 2-tuple linguistic Bonferroni mean (2TLBM).

In what follows, we investigate some desirable properties of the 2TLBM:

Theorem 3.1. Let $p \geq 0$, $q \geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n)$ be a collection of 2-tuples. Then, the following properties hold.

(1) Commutativity: If $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then

$$2\mathrm{TLB}^{p,q}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))=2\mathrm{TLB}^{p,q}((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')). \tag{8}$$

(2) Idempotency: If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$2\mathrm{TLB}^{p,q}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))=(r,\alpha). \tag{9}$$

(3) Boundedness:

$$\min_{1\leq i\leq n}\{(r_i,\alpha_i)\}\leq 2\mathrm{TLB}^{p,q}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))\leq \max_{1\leq i\leq n}\{(r_i,\alpha_i)\}. \tag{10}$$

(4) Monotonicity: Let $\{(\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\}$ and $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ be two collections of 2-tuples, if $(r_i,\alpha_i)\leq(\bar{r}_i,\bar{\alpha}_i)$, for all i, then

$$2\mathrm{TLB}^{p,q}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))\leq 2\mathrm{TLB}^{p,q}((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)). \tag{11}$$

Proof. (1) Since $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, we have

$$\begin{aligned}
&\mathrm{2TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\
&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i',\alpha_i')\right)^p\cdot\left(\Delta^{-1}(r_j',\alpha_j')\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&=\mathrm{2TLB}^{p,q}\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right)
\end{aligned}$$

(2) If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$\begin{aligned}
&\mathrm{2TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\
&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r,\alpha)\right)^p\cdot\left(\Delta^{-1}(r,\alpha)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r,\alpha)\right)^{p+q}\right)\right)^{\frac{1}{p+q}}\right)=(r,\alpha)
\end{aligned}$$

(3) Because $\min\limits_{1\le i\le n}\{(r_i,\alpha_i)\}\le(r_i,\alpha_i)\le\max\limits_{1\le i\le n}\{(r_i,\alpha_i)\}$, we have

$$\begin{aligned}
\min_{1\le i\le n}\{(r_i,\alpha_i)\}&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}\left(\min_{1\le i\le n}\{(r_i,\alpha_i)\}\right)\right)^p\cdot\left(\Delta^{-1}\left(\min_{1\le i\le n}\{(r_i,\alpha_i)\}\right)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&\le\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&\le\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}\left(\max_{1\le i\le n}\{(r_i,\alpha_i)\}\right)\right)^p\cdot\left(\Delta^{-1}\left(\max_{1\le i\le n}\{(r_i,\alpha_i)\}\right)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&=\max_{1\le i\le n}\{(r_i,\alpha_i)\}.
\end{aligned}\tag{4}$$

$$\begin{aligned}
2\mathrm{TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) &= \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&\leq \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(\bar{r}_i,\bar{\alpha}_i)\right)^p\cdot\left(\Delta^{-1}(\bar{r}_j,\bar{\alpha}_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&= 2\mathrm{TLB}^{p,q}\left((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right)
\end{aligned}$$

The proof of Theorem 3.1 is complete.

Theorem 3.2. Let $p\geq 0$, $q\geq 0$, and p,q do not take the value 0 simultaneously. Let $\left\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. Then, we have

$$2\mathrm{TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=2\mathrm{TLB}^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (12)$$

Proof.

$$\begin{aligned}
2\mathrm{TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) &= \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\\
&= \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{j,i=1\\ j\neq i}}^{n}\left(\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\cdot\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)\right)^{\frac{1}{q+p}}\right) \quad (13)\\
&= 2\mathrm{TLB}^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)
\end{aligned}$$

The proof of Theorem 3.2 is complete.

In the following, let us consider some special cases of the 2TLBM operator by taking different values of the parameters p and q.

Case1. If $q\rightarrow 0$, then, by Eq. (7), we have

$$\begin{aligned}
&\lim_{q\rightarrow 0}2\mathrm{TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\
&=\lim_{q\rightarrow 0}\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right) \quad (14)\\
&=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)^{\frac{1}{p}}\right)=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\left((n-1)\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)\right)^{\frac{1}{p}}\right)\\
&=\Delta\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)^{\frac{1}{p}}\right)=2\mathrm{TLB}^{p,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)
\end{aligned}$$

which we call the generalized 2-tuple linguistic mean [21].

Case 2. If $p=2$ and $q \to 0$, then Eq. (7) is transformed as:

$$\begin{aligned}\text{2TLB}^{2,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) &= \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^2\right)\right)^{\frac{1}{2}}\right) \\ &= \Delta\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^2\right)^{\frac{1}{2}}\right)\end{aligned} \tag{15}$$

which we call the 2-tuple linguistic square mean [21].

Case 3. If $p=1$ and $q \to 0$, then Eq. (7) reduces to the 2-tuple linguistic average [21]:

$$\begin{aligned}\text{2TLB}^{1,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) &= \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^1\right)\right)^{1}\right) \\ &= \Delta\left(\frac{1}{n}\sum_{i=1}^{n}\Delta^{-1}(r_i,\alpha_i)\right)\end{aligned} \tag{16}$$

Case 4. If $p=q=1$, then Eq. (7) reduces to the following:

$$\text{2TLB}^{1,1}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) = \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\cdot\Delta^{-1}(r_j,\alpha_j)\right)\right)^{\frac{1}{2}}\right) \tag{17}$$

which we call the 2-tuple linguistic interrelated square mean.

Case 5. If $p \to +\infty$ and $q \to 0$, then Eq. (7) reduces to the 2-tuple linguistic maximum operator:

$$\begin{aligned}
&\lim_{\substack{p\to+\infty\\ q\to 0}} 2\mathrm{TLB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\
&=\lim_{p\to+\infty}\left(\lim_{q\to 0}\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)\right)\\
&=\lim_{p\to+\infty}\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)^{\frac{1}{p}}\right) \qquad (18)\\
&=\lim_{p\to+\infty}\Delta\left(\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\left((n-1)\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)\right)^{\frac{1}{p}}\right)=\lim_{p\to+\infty}\Delta\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^p\right)^{\frac{1}{p}}\right)\\
&=\lim_{p\to 0}\Delta\left(\max_{1\le i\le n}\left\{\Delta^{-1}(r_i,\alpha_i)\right\}\right)=2\mathrm{TLB}^{+\infty,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)
\end{aligned}$$

In the 2TLBM operator, the importance of the input arguments is not emphasized. Nevertheless, in many practical situations, the weights of the input arguments should be taken into account. If we allow the arguments to have different weights, then the weighted 2-tuple linguistic Bonferroni mean (W2TLBM) operator can be defined as follows:

Definition 3.2. Let $p\ge 0$, $q\ge 0$, and p,q do not take the value 0 simultaneously. Let $\left\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $\left\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right\}$, where w_i indicates the importance degree of (r_i,α_i), satisfying $w_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}w_i=1$. If

$$\mathrm{W2TLB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(w_i\cdot\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(w_j\cdot\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right), \qquad (19)$$

then $\mathrm{W2TLB}_w^{p,q}$ is called the weighted 2-tuple linguistic Bonferroni mean (W2TLBM).

Theorem 3.3. Let $p\ge 0$, $q\ge 0$, and p,q do not take the value 0 simultaneously. Let $\left\{(\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right\}$ and $\left\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right\}$ be two collections of 2-tuples, if $(r_n,\alpha_n)\le(\bar{r}_n,\bar{\alpha}_n)$, for all i, then

$$\mathrm{W2TLB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\le\mathrm{W2TLB}_w^{p,q}\left((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right). \qquad (20)$$

Proof.

$$\mathrm{W2TLB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)$$
$$=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(w_i\cdot\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(w_j\cdot\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)$$
$$\leq\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(w_i\cdot\Delta^{-1}(\bar{r}_i,\bar{\alpha}_i)\right)^p\cdot\left(w_j\cdot\Delta^{-1}(\bar{r}_j,\bar{\alpha}_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)$$
$$=\mathrm{W2TLB}_w^{p,q}\left((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right)$$

The proof of Theorem 3.3 is complete.

The W2TLBM operator is neither idempotent, bounded, nor commutative.

Theorem 3.4. Let $p\geq 0$, $q\geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, satisfying $w_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}w_i=1$. Then, we have

$$\mathrm{W2TLB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\mathrm{W2TLB}_w^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\quad(21)$$

Proof.

$$\mathrm{W2TLB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)$$
$$=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(w_i\cdot\Delta^{-1}(r_i,\alpha_i)\right)^p\cdot\left(w_j\cdot\Delta^{-1}(r_j,\alpha_j)\right)^q\right)\right)^{\frac{1}{p+q}}\right)$$
$$=\Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{j,i=1\\j\neq i}}^{n}\left(\left(w_j\cdot\Delta^{-1}(r_j,\alpha_j)\right)^q\cdot\left(w_i\cdot\Delta^{-1}(r_i,\alpha_i)\right)^p\right)\right)^{\frac{1}{q+p}}\right)$$
$$=\mathrm{W2TLB}_w^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)$$

The proof of Theorem 3.4 is complete.

4. 2-tuple Linguistic Geometric Bonferroni Mean and Weighted 2-tuple Linguistic Geometric Bonferroni Mean

In this section, we shall investigate the geometric Bonferroni mean under 2-tuple linguistic environments, i.e., develop a 2-tuple linguistic geometric Bonferroni mean operator and its weight form.

Definition 4.1. Let $p \geq 0$, $q \geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n\right)$ be a collection of 2-tuples. If

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right), \quad (22)$$

then $\mathrm{2TLGB}^{p,q}$ is called the 2-tuple linguistic geometric Bonferroni mean (2TLGBM).

In what follows, we investigate some desirable properties of the 2TLGBM:

Theorem 4.1. Let $p \geq 0$, $q \geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n\right)$ be a collection of 2-tuples. Then, the following properties hold.

(1) Commutativity: If $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\mathrm{2TLGB}^{p,q}\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right). \quad (23)$$

(2) Idempotency: If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=(r,\alpha). \quad (24)$$

(3) Boundedness:

$$\min_{1\leq i\leq n}\{(r_i,\alpha_i)\}\leq \mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\leq \max_{1\leq i\leq n}\{(r_i,\alpha_i)\}. \quad (25)$$

(4) Monotonicity: Let $\{(\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\}$ and $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ be two collections of 2-tuples, if $(r_i,\alpha_i)\leq(\bar{r}_i,\bar{\alpha}_i)$, for all i, then

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\leq \mathrm{2TLGB}^{p,q}\left((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right). \quad (26)$$

Proof. (1) Since $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then we have

$$\begin{aligned}
&\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\
&=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right)\\
&=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i',\alpha_i')+q\cdot\Delta^{-1}(r_j',\alpha_j')\right)^{\frac{1}{n(n-1)}}\right)\\
&=\mathrm{2TLGB}^{p,q}\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right)
\end{aligned}$$

(2) If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$\begin{aligned}
\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)&=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right)\\
&=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r,\alpha)+q\cdot\Delta^{-1}(r,\alpha)\right)^{\frac{1}{n(n-1)}}\right)\\
&=(r,\alpha)
\end{aligned}$$

(3) Because $\min_{1\le i\le n}\{(r_i,\alpha_i)\}\le \mathrm{2TLPG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\le\max_{1\le i\le n}\{(r_i,\alpha_i)\}$, we have

$$\begin{aligned}
\min_{1\le i\le n}\{(r_i,\alpha_i)\}&=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}\left(\min_{1\le i\le n}\{(r_i,\alpha_i)\}\right)+q\cdot\Delta^{-1}\left(\min_{1\le i\le n}\{(r_i,\alpha_i)\}\right)\right)^{\frac{1}{n(n-1)}}\right)\\
&\le\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right)\\
&\le\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}\left(\max_{1\le i\le n}\{(r_i,\alpha_i)\}\right)+q\cdot\Delta^{-1}\left(\max_{1\le i\le n}\{(r_i,\alpha_i)\}\right)\right)^{\frac{1}{n(n-1)}}\right)\\
&=\max_{1\le i\le n}\{(r_i,\alpha_i)\}.
\end{aligned}$$

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right)$$

(4)

$$\leq\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(\overline{r}_i,\overline{\alpha}_i)+q\cdot\Delta^{-1}(\overline{r}_j,\overline{\alpha}_j)\right)^{\frac{1}{n(n-1)}}\right)$$

$$=\mathrm{2TLGB}^{p,q}\left((\overline{r}_1,\overline{\alpha}_1),(\overline{r}_2,\overline{\alpha}_2),\cdots,(\overline{r}_n,\overline{\alpha}_n)\right)$$

The proof of Theorem 4.1 is complete.

Theorem 4.2. Let $p\geq 0$, $q\geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n)$ be a collection of 2-tuples. Then, we have

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\mathrm{2TLGB}^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (27)$$

Proof.

$$\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right)$$

$$=\Delta\left(\frac{1}{q+p}\prod_{\substack{j,i=1\\ j\neq i}}^{n}\left(q\cdot\Delta^{-1}(r_j,\alpha_j)+p\cdot\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n(n-1)}}\right)$$

$$=\mathrm{2TLGB}^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)$$

The proof of Theorem 4.2 is complete.

In the following, let us consider some special cases of the 2TLGBM operator by taking different values of the parameters p and q.

Case1. If $q\to 0$, then, by Eq. (22), we have

$$\begin{aligned}&\lim_{q\to 0}\mathrm{2TLGB}^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\&=\lim_{q\to 0}\Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)+q\cdot\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right)\\&=\Delta\left(\frac{1}{p}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n(n-1)}}\right)=\Delta\left(\frac{1}{p}\prod_{i=1}^{n}\left(p\cdot\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n}}\right)\\&=\mathrm{2TLGB}^{p,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\end{aligned} \quad (28)$$

which we call the generalized 2-tuple linguistic geometric mean [21].

Case 2. If $p=2$ and $q\to 0$, then Eq. (22) is transformed as:

$$\begin{aligned}\text{2TLGB}^{2,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) &= \Delta\left(\frac{1}{2}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(2\cdot\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n(n-1)}}\right) \\ &= \Delta\left(\frac{1}{2}\prod_{i=1}^{n}\left(2\cdot\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n}}\right)\end{aligned} \tag{29}$$

which we call the 2-tuple linguistic square geometric mean [21].

Case 3. If $p=1$ and $q\to 0$, then Eq. (22) reduces to the 2-tuple linguistic geometric average [25]:

$$\begin{aligned}\text{2TLGB}^{1,0}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) &= \Delta\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n(n-1)}}\right) \\ &= \Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\frac{1}{n}}\right)\end{aligned} \tag{30}$$

Case 4. If $p=q=1$, then Eq. (22) reduces to the following:

$$\text{2TLGB}^{1,1}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) = \Delta\left(\frac{1}{2}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\Delta^{-1}(r_i,\alpha_i)+\Delta^{-1}(r_j,\alpha_j)\right)^{\frac{1}{n(n-1)}}\right) \tag{31}$$

which we call the 2-tuple linguistic interrelated square geometric mean.

It should be noted that the 2TLGBM operator does not consider the importance of the aggregated arguments, but in many practical problem, especially in some group decision makings, the aggregated arguments have different weights, to overcome this drawback, we introduce the following definition:

Definition 4.2. Let $p\geq 0$, $q\geq 0$, and p,q do not take the value 0 simultaneously. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, where w_i indicates the importance degree of (r_i,α_i), satisfying $w_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}w_i=1$. If

$$\begin{aligned}&\text{W2TLGB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \\ &= \Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\cdot\left(\Delta^{-1}(r_i,\alpha_i)\right)^{w_i}+q\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^{w_j}\right)^{\frac{1}{n(n-1)}}\right),\end{aligned} \tag{32}$$

then $\text{W2TLGB}_w^{p,q}$ is called the weighted 2-tuple linguistic geometric Bonferroni mean (W2TLGBM).

Theorem 4.3. Let $p \geq 0$, $q \geq 0$, and p, q do not take the value 0 simultaneously. Let $\{(\bar{r}_1, \bar{\alpha}_1), (\bar{r}_2, \bar{\alpha}_2), \cdots, (\bar{r}_n, \bar{\alpha}_n)\}$ and $\{(r_1, \alpha_1), (r_2, \alpha_2), \cdots, (r_n, \alpha_n)\}$ be two collections of 2-tuples, if $(r_n, \alpha_n) \leq (\bar{r}_n, \bar{\alpha}_n)$, for all i, then

$$\text{W2TLGB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \leq \text{W2TLGB}_w^{p,q}\left((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right). \quad (33)$$

Proof.

$$\begin{aligned}
&\text{W2TLGB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \\
&= \Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(p\cdot\left(\Delta^{-1}(r_i,\alpha_i)\right)^{w_i} + q\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^{w_j}\right)^{\frac{1}{n(n-1)}}\right) \\
&\leq \Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(p\cdot\left(\Delta^{-1}(\bar{r}_i,\bar{\alpha}_i)\right)^{w_i} + q\cdot\left(\Delta^{-1}(\bar{r}_j,\bar{\alpha}_j)\right)^{w_j}\right)^{\frac{1}{n(n-1)}}\right) \\
&= \text{W2TLGB}_w^{p,q}\left((\bar{r}_1,\bar{\alpha}_1),(\bar{r}_2,\bar{\alpha}_2),\cdots,(\bar{r}_n,\bar{\alpha}_n)\right)
\end{aligned}$$

The proof of Theorem 4.3 is complete.

The W2TLGBM operator is neither idempotent, bounded, nor commutative.

Theorem 4.4. Let $p \geq 0$, $q \geq 0$, and p, q do not take the value 0 simultaneously. Let $\{(r_1, \alpha_1), (r_2, \alpha_2), \cdots, (r_n, \alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, \cdots, n)$ be a collection of 2-tuples. $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $\{(r_1, \alpha_1), (r_2, \alpha_2), \cdots, (r_n, \alpha_n)\}$, satisfying $w_i \in [0,1]$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} w_i = 1$. Then, we have

$$\text{W2TLGB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) = \text{W2TLGB}_w^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (34)$$

Proof.

$$\begin{aligned}
&\text{W2TLGB}_w^{p,q}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \\
&= \Delta\left(\frac{1}{p+q}\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(p\cdot\left(\Delta^{-1}(r_i,\alpha_i)\right)^{w_i} + q\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^{w_j}\right)^{\frac{1}{n(n-1)}}\right) \\
&= \Delta\left(\frac{1}{q+p}\prod_{\substack{j,i=1 \\ j\neq i}}^{n}\left(q\cdot\left(\Delta^{-1}(r_j,\alpha_j)\right)^{w_j} + p\cdot\left(\Delta^{-1}(r_i,\alpha_i)\right)^{w_i}\right)^{\frac{1}{n(n-1)}}\right) \\
&= \text{W2TLGB}_w^{q,p}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)
\end{aligned}$$

The proof of Theorem 4.4 is complete.

5. Approaches for Multiple Attribute Group Decision making with 2-tuple Linguistic Information

In this section, we utilize the proposed aggregation operators to develop some approaches for multiple attribute group decision making with 2-tuple linguistic information.

The multiple attribute group decision making (MAGDM) with 2-tuple linguistic information can be formulated as follows: Let $X=\{x_1,x_2,\cdots,x_m\}$ be a set of m alternatives, and let $C=\{c_1,c_2,\ldots,c_n\}$ be a collection of n attributes, whose weight vector is $w=(w_1,w_2,\cdots,w_n)^T$, with $w_j\in[0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n} w_j=1$, where w_j denotes the importance degree of the attribute c_j, and let $D=\{d_1,d_2,\cdots,d_l\}$ be a set of l decision makers, whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_l)^T$ with $\omega_k\in[0,1]$, $k=1,2,\cdots,l$, and $\sum_{k=1}^{l}\omega_k=1$, where ω_k denotes the importance degree of the decision maker d_k. Each decision maker provides his/her own linguistic decision matrix $A^{(k)}=\left(a_{ij}^{(k)}\right)_{m\times n}$ $(k=1,2,\cdots,l)$, where $a_{ij}^{(k)}\in S$ is a performance value, which takes the form of linguistic variable, given by the decision maker $d_k\in D$, for the alternative $x_i\in X$ with respect to the attribute $c_j\in C$.

If all the attributes c_j ($j=1,2,\cdots,n$) are of the same type, then the performance values do not need normalization. Whereas there are, generally, benefit attributes (the bigger the performance values the better) and cost attributes (the smaller the performance values the better) in multiple attribute group decision making, in such cases, we may transform the performance values of the cost type into the performance values of the benefit type. Then, we can transform $A^{(k)}=\left(a_{ij}^{(k)}\right)_{m\times n}$ into the matrix $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$, where

$$r_{ij}^{(k)}=\begin{cases} a_{ij}^{(k)}, & \text{for benefit attribute } c_j \\ neg\left(a_{ij}^{(k)}\right), & \text{for cost attribute } c_j \end{cases}, \quad i=1,2,\cdots,m,\ j=1,2,\cdots,n,\ k=1,2,\cdots,l \tag{35}$$

where $neg\left(a_{ij}^{(k)}\right)$ is the complement of $a_{ij}^{(k)}$.

In the following, we utilize the W2TLBM (or W2TLGBM) operator to develop an approach for multiple attribute group decision making under 2-tuple linguistic environments. The algorithm involves the following steps.

Approach I

Step 1. Transform the linguistic decision matrix $A^{(k)}=\left(a_{ij}^{(k)}\right)_{m\times n}$ into a normalized linguistic decision matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m\times n}$ using Eq. (35).

Step 2. Transform the linguistic decision matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m\times n}$ $\left(k = 1, 2, \cdots, l\right)$ into 2-tuple linguistic decision matrix $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{m\times n}$ $\left(k = 1, 2, \cdots, l\right)$.

Step 3. Utilize the W2TLBM operator (Eq. (19)),

$$\begin{aligned}\overline{r}_{ij} &= \left(r_{ij}, \alpha_{ij}\right) = \mathrm{W2TLB}_{\omega}^{p,q}\left(\left(r_{ij}^{(1)}, 0\right), \left(r_{ij}^{(2)}, 0\right), \ldots, \left(r_{ij}^{(l)}, 0\right)\right) \\ &= \Delta\left(\left(\frac{1}{l\left(l-1\right)}\sum_{\substack{s,t=1\\ s\neq t}}^{l}\left(\left(\omega_s \cdot \Delta^{-1}\left(r_{ij}^{(s)}, 0\right)\right)^{p} \cdot \left(\omega_t \cdot \Delta^{-1}\left(r_{ij}^{(t)}, 0\right)\right)^{q}\right)\right)^{\frac{1}{p+q}}\right)\end{aligned} \tag{36}$$

or the W2TLGBM operator (Eq. (32)),

$$\begin{aligned}\overline{r}_{ij} &= \left(r_{ij}, \alpha_{ij}\right) = \mathrm{W2TLGB}_{\omega}^{p,q}\left(\left(r_{ij}^{(1)}, 0\right), \left(r_{ij}^{(2)}, 0\right), \ldots, \left(r_{ij}^{(l)}, 0\right)\right) \\ &= \Delta\left(\frac{1}{p+q}\prod_{\substack{s,t=1\\ s\neq t}}^{l}\left(p \cdot \left(\Delta^{-1}\left(r_{ij}^{(s)}, 0\right)\right)^{\omega_s} + q \cdot \left(\Delta^{-1}\left(r_{ij}^{(t)}, 0\right)\right)^{\omega_t}\right)^{\frac{1}{l(l-1)}}\right)\end{aligned} \tag{37}$$

to aggregate all of the individual 2-tuple linguistic decision matrices $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{m\times n}$ ($k = 1, 2, \cdots, l$) into the collective 2-tuple linguistic decision matrix $\overline{R} = \left(\overline{r}_{ij}\right)_{m\times n} = \left(\left(r_{ij}, \alpha_{ij}\right)\right)_{m\times n}$.

Step 4. Utilize the W2TLBM operator (Eq. (19)),

$$\begin{aligned}\overline{r}_{i} &= \left(r_{i}, \alpha_{i}\right) = \mathrm{W2TLB}_{w}^{p,q}\left(\left(r_{i1}, \alpha_{i1}\right), \left(r_{i2}, \alpha_{i2}\right), \ldots, \left(r_{in}, \alpha_{in}\right)\right) \\ &= \Delta\left(\left(\frac{1}{n\left(n-1\right)}\sum_{\substack{x,y=1\\ x\neq y}}^{n}\left(\left(w_x \cdot \Delta^{-1}\left(\left(r_{ix}, \alpha_{ix}\right)\right)\right)^{p} \cdot \left(w_y \cdot \Delta^{-1}\left(\left(r_{iy}, \alpha_{iy}\right)\right)\right)^{q}\right)\right)^{\frac{1}{p+q}}\right)\end{aligned} \tag{38}$$

or the W2TLGBM operator (Eq. (32)),

$$\begin{aligned}\overline{r}_{i} &= \left(r_{i}, \alpha_{i}\right) = \mathrm{W2TLGB}_{w}^{p,q}\left(\left(r_{i1}, \alpha_{i1}\right), \left(r_{i2}, \alpha_{i2}\right), \ldots, \left(r_{in}, \alpha_{in}\right)\right) \\ &= \Delta\left(\frac{1}{p+q}\prod_{\substack{x,y=1\\ x\neq y}}^{n}\left(p \cdot \left(\Delta^{-1}\left(r_{ix}, \alpha_{ix}\right)\right)^{w_x} + q \cdot \left(\Delta^{-1}\left(r_{iy}, \alpha_{iy}\right)\right)^{w_y}\right)^{\frac{1}{n(n-1)}}\right)\end{aligned} \tag{39}$$

to aggregate all of the preference values $\overline{r}_{ij}$ ($j=1,2,\cdots,n$) in the i th line of $\overline{R}$, and then derive the collective overall preference value $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,\cdots,m)$ of alternative x_i $(i=1,2,\cdots,m)$.

Step 5. Rank the $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,\cdots,m)$ in descending order using Definition 2.3.

Step 6. Rank all of the alternatives x_i $(i=1,2,\cdots,m)$, and then select the best one(s) in accordance with the collective overall preference values $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,\cdots,m)$.

Step 7. End.

If the information regarding the weights of the decision makers and attributes is unknown, then we utilize the 2TLBM (or 2TLGBM) operator to develop an alternative approach for the MAGDM problem with 2-tuple linguistic information, which is described below.

Approach II

Step 1. Transform the linguistic decision matrix $A^{(k)}=\left(a_{ij}^{(k)}\right)_{m\times n}$ into a normalized linguistic decision matrix $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ using Eq. (35).

Step 2. Transform the linguistic decision matrix $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ $(k=1,2,\cdots,l)$ into 2-tuple linguistic decision matrix $\overline{R}^{(k)}=\left(\left(r_{ij}^{(k)},0\right)\right)_{m\times n}$ $(k=1,2,\cdots,l)$.

Step 3. Utilize the 2TLBM operator (Eq. (7)),

$$\overline{r}_{ij}=\left(r_{ij},\alpha_{ij}\right)=2\mathrm{TLB}_{\omega}^{p,q}\left(\left(r_{ij}^{(1)},0\right),\left(r_{ij}^{(2)},0\right),\ldots,\left(r_{ij}^{(l)},0\right)\right)$$
$$=\Delta\left(\left(\frac{1}{l(l-1)}\sum_{\substack{s,t=1\\s\neq t}}^{l}\left(\left(\Delta^{-1}\left(r_{ij}^{(s)},0\right)\right)^{p}\cdot\left(\Delta^{-1}\left(r_{ij}^{(t)},0\right)\right)^{q}\right)\right)^{\frac{1}{p+q}}\right) \tag{40}$$

or the 2TLGBM operator (Eq. (22)),

$$\overline{r}_{ij}=\left(r_{ij},\alpha_{ij}\right)=2\mathrm{TLGB}_{\omega}^{p,q}\left(\left(r_{ij}^{(1)},0\right),\left(r_{ij}^{(2)},0\right),\ldots,\left(r_{ij}^{(l)},0\right)\right)$$
$$=\Delta\left(\frac{1}{p+q}\prod_{\substack{s,t=1\\s\neq t}}^{l}\left(p\cdot\Delta^{-1}\left(r_{ij}^{(s)},0\right)+q\cdot\Delta^{-1}\left(r_{ij}^{(t)},0\right)\right)^{\frac{1}{l(l-1)}}\right) \tag{41}$$

to aggregate all of the individual 2-tuple linguistic decision matrices $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{m\times n}$ ($k = 1, 2, \cdots, l$) into the collective 2-tuple linguistic decision matrix $\overline{R} = \left(\overline{r}_{ij}\right)_{m\times n} = \left(\left(r_{ij}, \alpha_{ij}\right)\right)_{m\times n}$.

Step 4. Utilize the 2TLBM operator (Eq. (7)),

$$\overline{r}_i = (r_i, \alpha_i) = \text{2TLB}_w^{p,q}\left((r_{i1}, \alpha_{i1}), (r_{i2}, \alpha_{i2}), \ldots, (r_{in}, \alpha_{in})\right)$$
$$= \Delta\left(\left(\frac{1}{n(n-1)}\sum_{\substack{x,y=1\\x\neq y}}^{n}\left(\left(\Delta^{-1}\left((r_{ix}, \alpha_{ix})\right)\right)^p \cdot \left(\Delta^{-1}\left((r_{iy}, \alpha_{iy})\right)\right)^q\right)\right)^{\frac{1}{p+q}}\right) \tag{42}$$

or the 2TLGBM operator (Eq. (22)),

$$\overline{r}_i = (r_i, \alpha_i) = \text{2TLGB}_w^{p,q}\left((r_{i1}, \alpha_{i1}), (r_{i2}, \alpha_{i2}), \ldots, (r_{in}, \alpha_{in})\right)$$
$$= \Delta\left(\frac{1}{p+q}\prod_{\substack{x,y=1\\x\neq y}}^{n}\left(p\cdot\Delta^{-1}(r_{ix}, \alpha_{ix}) + q\cdot\Delta^{-1}(r_{iy}, \alpha_{iy})\right)^{\frac{1}{n(n-1)}}\right) \tag{43}$$

to aggregate all of the preference values $\overline{r}_{ij}$ ($j = 1, 2, \cdots, n$) in the ith line of $\overline{R}$, and then derive the collective overall preference value $\overline{r}_i = (r_i, \alpha_i)$ $(i = 1, 2, \cdots, m)$ of alternative x_i $(i = 1, 2, \cdots, m)$.

Step 5. Rank the $\overline{r}_i = (r_i, \alpha_i)$ $(i = 1, 2, \cdots, m)$ in descending order using Definition 2.3.

Step 6. Rank all of the alternatives x_i $(i = 1, 2, \cdots, m)$, and then select the best one(s) in accordance with the collective overall preference values $\overline{r}_i = (r_i, \alpha_i)$ $(i = 1, 2, \cdots, m)$.

Step 7. End.

Remark 5.1. Approach I is designed for situations where the weights of the decision makers and attributes can be predefined and it utilizes the W2TLBM (or W2TLGBM) operator to aggregate all of the individual 2-tuple linguistic decision matrices into the collective 2-tuple linguistic decision matrix. Approach II is designed for situations where the information regarding the weights of the decision makers and attributes is unknown and it utilizes the 2TLBM (or 2TLGBM) operator to aggregate all of the individual 2-tuple linguistic decision matrices into the collective 2-tuple linguistic decision matrix.

6. Illustrative examples

In this subsection, let us consider a numerical example adapted from Herrera et al. [48] and Herrera and Martínez [41].

Example 6.1 [41,48]. Suppose that an investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money: (1) x_1 is a car industry; (2) x_2 is a food company; (3) x_3 is a computer company; and (4) x_4 is an arms industry. The investment company must make a decision according to the following four attributes: (1) c_1 is the risk analysis; (2) c_2 is the growth analysis; (3) c_3 is the social-political impact analysis; and (4) c_4 is the environmental impact analysis. Among the considered attributes, c_2 is the benefit attribute, and c_j $(j=1,3,4)$ are the cost attributes. The weight vector of attributes c_j $(j=1,2,3,4)$ is $w=(0.3,0.25,0.25,0.2)^T$. The four possible alternatives x_i $(i=1,2,3,4)$ are to be evaluated using the linguistic term set

$$S=\left\{\begin{array}{l} s_0=\text{extremely poor},\ s_1=\text{very poor},\ s_2=\text{poor},\ s_3=\text{slightly poor},\ s_4=\text{fair},\\ s_5=\text{slightly good},\ s_6=\text{good},\ s_7=\text{very good},\ s_8=\text{extremely good}\end{array}\right\}$$

by three decision makers d_k $(k=1,2,3)$ (suppose that the weight vector of three decision makers is $\omega=(0.2,0.5,0.3)^T$) under the above four attributes, and construct, respectively, the linguistic decision matrices $A^{(k)}=\left(a_{ij}^{(k)}\right)_{4\times 4}$ $(k=1,2,3)$ as shown in Tables 1-3.

Table 1. Linguistic decision matrix $A^{(1)}$ provided by d_1.

1	c_1	c_2	c_3	c_4
x_1	s_4	s_3	s_7	s_3
x_2	s_5	s_6	s_2	s_0
x_3	s_5	s_2	s_1	s_3
x_4	s_0	s_1	s_5	s_2

Table 2. Linguistic decision matrix $A^{(2)}$ provided by d_2.

2	c_1	c_2	c_3	c_4
x_1	s_3	s_2	s_1	s_5
x_2	s_1	s_4	s_0	s_2
x_3	s_1	s_8	s_3	s_2
x_4	s_0	s_6	s_3	s_5

Table 3. Linguistic decision matrix $A^{(3)}$ provided by d_3.

3	c_1	c_2	c_3	c_4
x_1	s_3	s_1	s_6	s_0
x_2	s_1	s_8	s_2	s_3
x_3	s_3	s_6	s_5	s_4
x_4	s_2	s_8	s_3	s_1

Assume that the weights of the decision makers and attributes are known. We use Approach I to find the decision result.

Step 1. Using Eq. (35), we transform the linguistic decision matrix $A^{(k)} = \left(a_{ij}^{(k)}\right)_{4\times 4}$ into a normalized linguistic matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{4\times 4}$ (see Table 4, 5, and 6).

Table 4. Linguistic decision matrix $R^{(1)}$ provided by d_1.

4	c_1	c_2	c_3	c_4
x_1	s_4	s_3	s_1	s_5
x_2	s_3	s_6	s_5	s_8
x_3	s_3	s_2	s_7	s_5
x_4	s_8	s_1	s_3	s_6

Table 5. Linguistic decision matrix $R^{(2)}$ provided by d_2.

5	c_1	c_2	c_3	c_4
x_1	s_5	s_2	s_7	s_3
x_2	s_7	s_4	s_8	s_6
x_3	s_7	s_8	s_5	s_6
x_4	s_8	s_6	s_5	s_3

Table 6. Linguistic decision matrix $R^{(3)}$ provided by d_3.

6	c_1	c_2	c_3	c_4
x_1	s_5	s_1	s_2	s_8
x_2	s_7	s_8	s_6	s_5
x_3	s_5	s_6	s_3	s_4
x_4	s_6	s_8	s_5	s_7

Step 2. Transform the linguistic decision matrices $R^{(k)} = \left(r_{ij}^{(k)}\right)_{4\times 4}$ $\left(k = 1,2,3\right)$ given in Tables 4-

6 into 2-tuple linguistic decision matrices $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{4\times 4}$ $(k = 1, 2, 3)$ which are given in Tables 7-9.

Table 7. 2-tuple linguistic decision matrix $\overline{R}^{(1)}$

7	c_1	c_2	c_3	c_4
x_1	$(s_4, 0)$	$(s_3, 0)$	$(s_1, 0)$	$(s_5, 0)$
x_2	$(s_3, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_8, 0)$
x_3	$(s_3, 0)$	$(s_2, 0)$	$(s_7, 0)$	$(s_5, 0)$
x_4	$(s_8, 0)$	$(s_1, 0)$	$(s_3, 0)$	$(s_6, 0)$

Table 8. 2-tuple linguistic decision matrix $\overline{R}^{(2)}$**.**

8	c_1	c_2	c_3	c_4
x_1	$(s_5, 0)$	$(s_2, 0)$	$(s_7, 0)$	$(s_3, 0)$
x_2	$(s_7, 0)$	$(s_4, 0)$	$(s_8, 0)$	$(s_6, 0)$
x_3	$(s_7, 0)$	$(s_8, 0)$	$(s_5, 0)$	$(s_6, 0)$
x_4	$(s_8, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_3, 0)$

Table 9. 2-tuple linguistic decision matrix $\overline{R}^{(3)}$**.**

9	c_1	c_2	c_3	c_4
x_1	$(s_5, 0)$	$(s_1, 0)$	$(s_2, 0)$	$(s_8, 0)$
x_2	$(s_7, 0)$	$(s_8, 0)$	$(s_6, 0)$	$(s_5, 0)$
x_3	$(s_5, 0)$	$(s_6, 0)$	$(s_3, 0)$	$(s_4, 0)$
x_4	$(s_6, 0)$	$(s_8, 0)$	$(s_5, 0)$	$(s_7, 0)$

Step 3. Use the W2TLBM operator (Eq. (36)) (here, we take $p = q = 1$) to aggregate all of the individual 2-tuple linguistic decision matrices $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{4\times 4}$ ($k = 1, 2, 3$) into the collective 2-tuple linguistic decision matrix $\overline{R} = \left(\overline{r}_{ij}\right)_{4\times 4} = \left(\left(r_{ij}, \alpha_{ij}\right)\right)_{4\times 4}$ (see Table 10).

Table 10. The collective 2-tuple linguistic decision matrix $\overline{R}$

10	c_1	c_2	c_3	c_4
x_1	$(s_2,-0.4779)$	$(s_1,-0.4000)$	$(s_1,-0.0134)$	$(s_2,-0.4189)$
x_2	$(s_2,-0.1106)$	$(s_2,-0.1670)$	$(s_2,0.0817)$	$(s_2,-0.0252)$
x_3	$(s_2,-0.3417)$	$(s_2,-0.2186)$	$(s_2,-0.4714)$	$(s_2,-0.3875)$
x_4	$(s_2,0.3438)$	$(s_2,-0.3387)$	$(s_1,0.4318)$	$(s_2,-0.4220)$

Step 4. Use the W2TLBM operator (Eq. (38)) to aggregate all of the preference values (r_{ij},α_{ij}) ($j=1,2,3,4$) in the ith line of $\overline{R}$ and then derive the collective overall preference value $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,3,4)$ of the alternative x_i $(i=1,2,3,4)$.

$\overline{r}_1=(s_0,0.2852)$, $\overline{r}_2=(s_0,0.4837)$, $\overline{r}_3=(s_0,0.4101)$, $\overline{r}_4=(s_0,0.4394)$.

Using Definition 2.3, we then rank the $\overline{r}_i$ $(i=1,2,3,4)$ in descending order:

$$\overline{r}_2>\overline{r}_4>\overline{r}_3>\overline{r}_1.$$

Step 5. Rank all of the alternatives x_i $(i=1,2,3,4)$ as follows:

$$x_2\succ x_4\succ x_3\succ x_1.$$

The best alternative is x_2.

We can find that as the values of the parameters p and q change according to the decision makers' subjective preferences, we may obtain different rankings of the alternatives, which can reflect the decision makers' risk preferences. As the values of the parameters p and q change, the collective overall preference value $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,3,4)$ and the ranking of alternatives can be obtained and shown in Table 11.

Table 11. The collective overall preference values obtained with the W2TLBM operator and rankings of the alternatives.

11	$p=0,\ q=20$	$p=0.5,\ q=15$	$p=q=5$	$p=15,\ q=0.1$	$p=20,$ $q=0.05$
x_1	$(s_1,-0.2255)$	$(s_1,-0.3110)$	$(s_0,0.3779)$	$(s_1,-0.2668)$	$(s_1,-0.2308)$
x_2	$(s_1,-0.0577)$	$(s_1,-0.1233)$	$(s_1,-0.4132)$	$(s_1,-0.0948)$	$(s_1,-0.0610)$
x_3	$(s_1,-0.0577)$	$(s_1,-0.1302)$	$(s_1,-0.4712)$	$(s_1,-0.0963)$	$(s_1,-0.0615)$
x_4	$(s_1,0.0598)$	$(s_1,-0.0280)$	$(s_1,-0.4210)$	$(s_1,0.0074)$	$(s_1,0.0560)$
Ranking	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$

Furthermore, it is possible to analyze how the different attitudinal characters p and q play a role in the aggregation results. As the values of the parameters p and q change between 0 and 20, different results of a symbolic aggregation operation $\beta_i = \Delta^{-1}(\bar{r}_i)$ $(i=1,2,3,4)$ of the collective overall preference values $\bar{r}_i$ $(i=1,2,3,4)$ of the four alternatives x_i $(i=1,2,3,4)$ can be obtained. Figs. 1-4 illustrate the values $\beta_i = \Delta^{-1}(\bar{r}_i)$ $(i=1,2,3,4)$ of the four alternatives x_i $(i=1,2,3,4)$ obtained by the W2TLBM operator in detail.

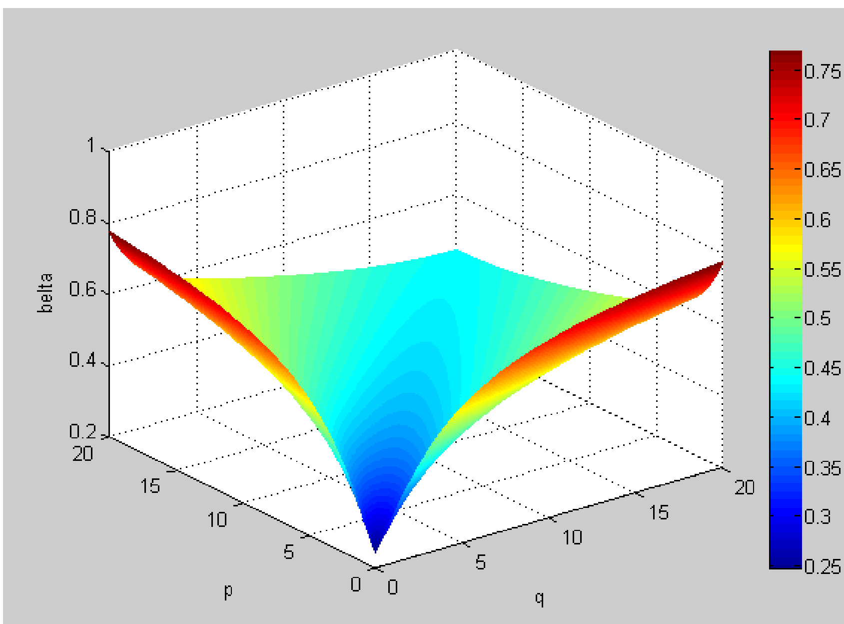

Fig. 1. The values β for alternative x_1 obtained by the W2TLBM operator ($p \in (0,20]$, $q \in (0,20]$).

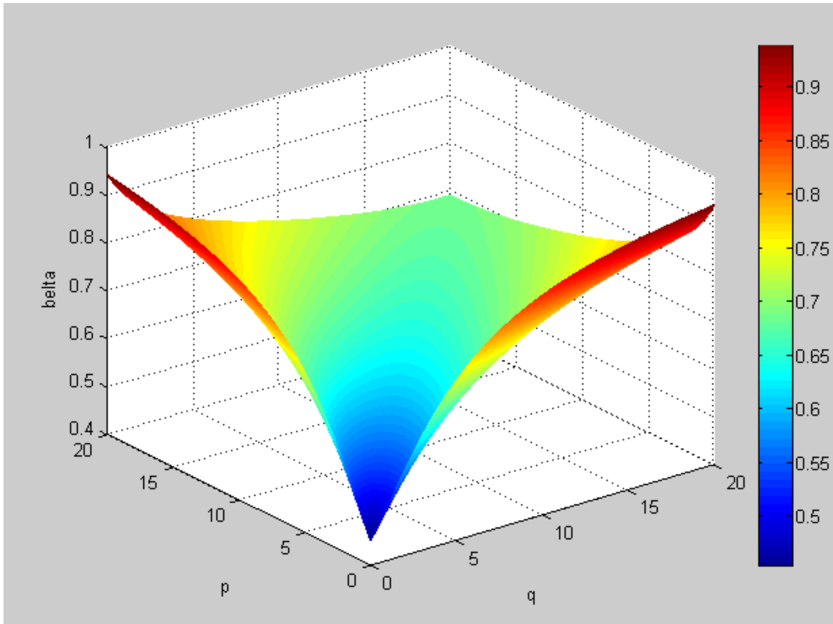

Fig. 2. The values β for alternative x_2 obtained by the W2TLBM operator ($p \in (0,20]$, $q \in (0,20]$).

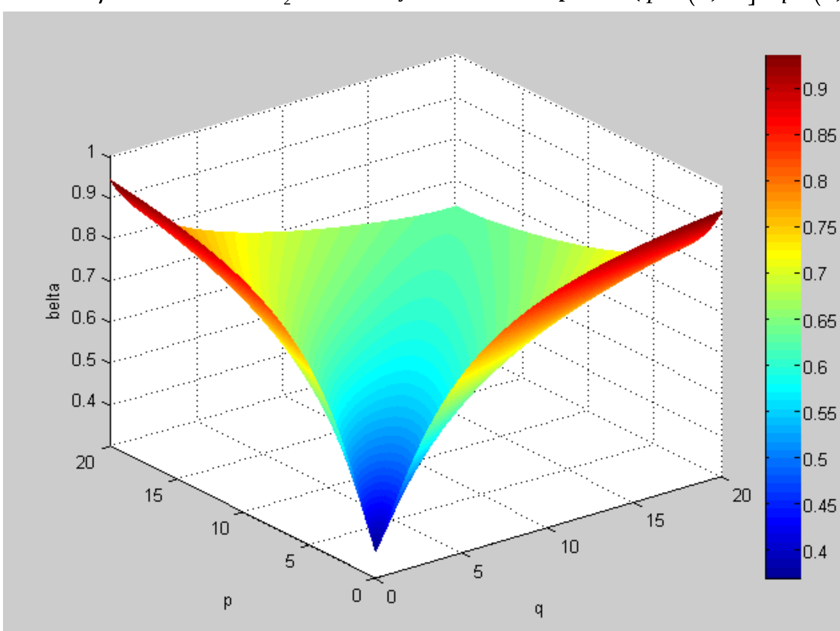

Fig. 3. The values β for alternative x_3 obtained by the W2TLBM operator ($p \in (0,20]$, $q \in (0,20]$).

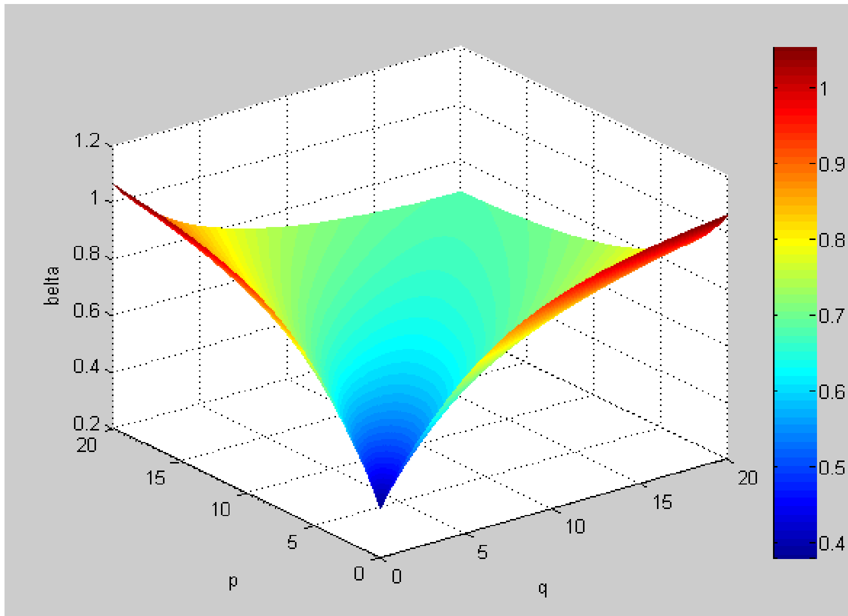

Fig. 4. The values β for alternative x_4 obtained by the W2TLBM operator ($p \in (0,20]$, $q \in (0,20]$).

If we let the parameter p fixed, different values β_i and rankings of the alternatives can be obtained as the parameter q changed which was shown in Fig. 5.

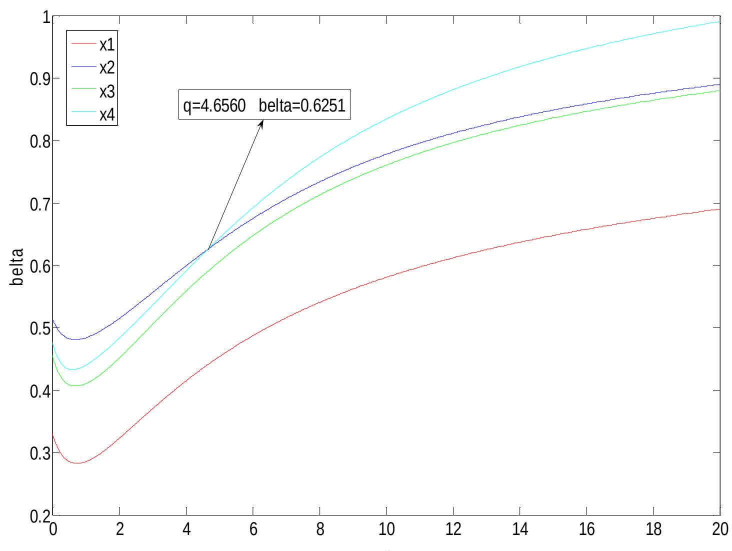

Fig. 5. Variation of β obtained with the W2TLBM operator ($p=1$, $q \in (0,20]$).

From Fig. 5, we can find that,

1) when $q \in (0, 4.6560]$, the ranking of the four alternatives is $x_2 \succ x_4 \succ x_3 \succ x_1$ and the best choice is x_2.

2) when $q \in (4.6560, 20]$, the ranking of the four alternatives is $x_4 \succ x_2 \succ x_3 \succ x_1$ and the best choice is x_4.

From Fig. 5, we also find that the collective overall preference values obtained by the W2TLBM aggregation operators become bigger as parameter p and q increase for the same aggregation arguments. Therefore, it plays a crucial part in decision making. For example, in the real group decision making problems, the decision makers who takes a gloomy view of the prospects could choose the smaller values of the parameters p and q while the decision makers who are optimistic could choose the bigger values of the parameters p and q.

If the W2TLGBM operator is used in place of the W2TLBM operator to aggregate the values of the alternatives in steps 3 and 4, then the collective overall preference values and the rankings of the alternatives are listed in Table 12.

Table 12. The collective overall preference values obtained with the W2TLGBM operator and rankings of the alternatives

12	$p=0,\ q=20$	$p=0.5,\ q=15$	$p=q=5$	$p=15,\ q=0.1$	$p=20,\ q=0.05$
x_1	$(s_1, 0.1057)$	$(s_1, 0.1066)$	$(s_1, 0.1130)$	$(s_1, 0.1059)$	$(s_1, 0.1057)$
x_2	$(s_1, 0.1604)$	$(s_1, 0.1614)$	$(s_1, 0.1677)$	$(s_1, 0.1606)$	$(s_1, 0.1605)$
x_3	$(s_1, 0.1464)$	$(s_1, 0.1476)$	$(s_1, 0.1558)$	$(s_1, 0.1466)$	$(s_1, 0.1465)$
x_4	$(s_1, 0.1477)$	$(s_1, 0.1488)$	$(s_1, 0.1560)$	$(s_1, 0.1479)$	$(s_1, 0.1478)$
Ranking	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$

Furthermore, it is possible to analyze how the different attitudinal characters p and q play a role in the aggregation results. As the values of the parameters p and q change between 0 and 20, different results of a symbolic aggregation operation $\beta_i = \Delta^{-1}(\overline{r}_i)$ $(i = 1,2,3,4)$ of the collective overall preference values $\overline{r}_i$ $(i = 1,2,3,4)$ of the four alternatives x_i $(i = 1,2,3,4)$ can be obtained. Figs. 6-9 illustrate the values $\beta_i = \Delta^{-1}(\overline{r}_i)$ $(i = 1,2,3,4)$ of the four alternatives x_i $(i = 1,2,3,4)$ obtained by the W2TLGBM operator in detail.

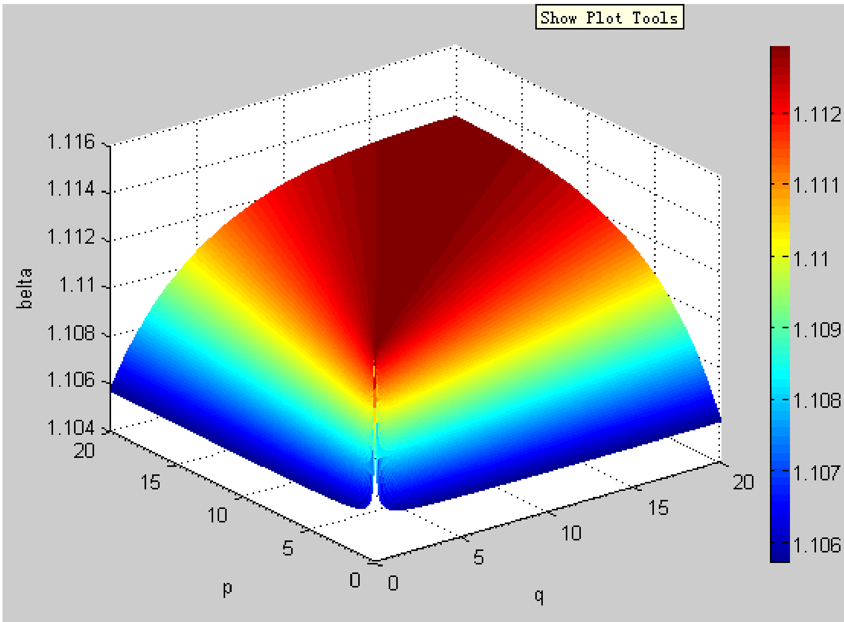

Fig. 6. The values β for alternative x_1 obtained by the W2TLGBM operator ($p \in (0,20]$, $q \in (0,20]$).

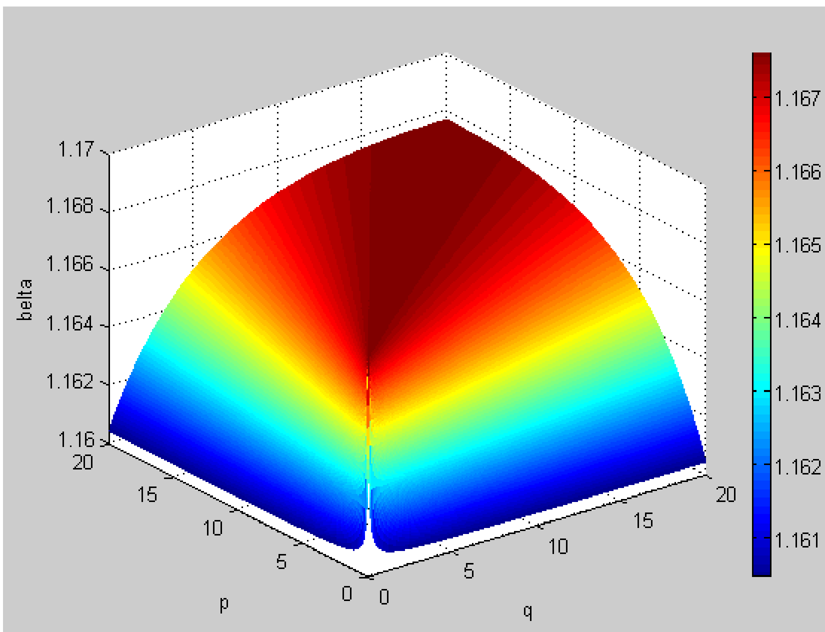

Fig. 7. The values β for alternative x_2 obtained by the W2TLGBM operator ($p \in (0,20]$, $q \in (0,20]$).

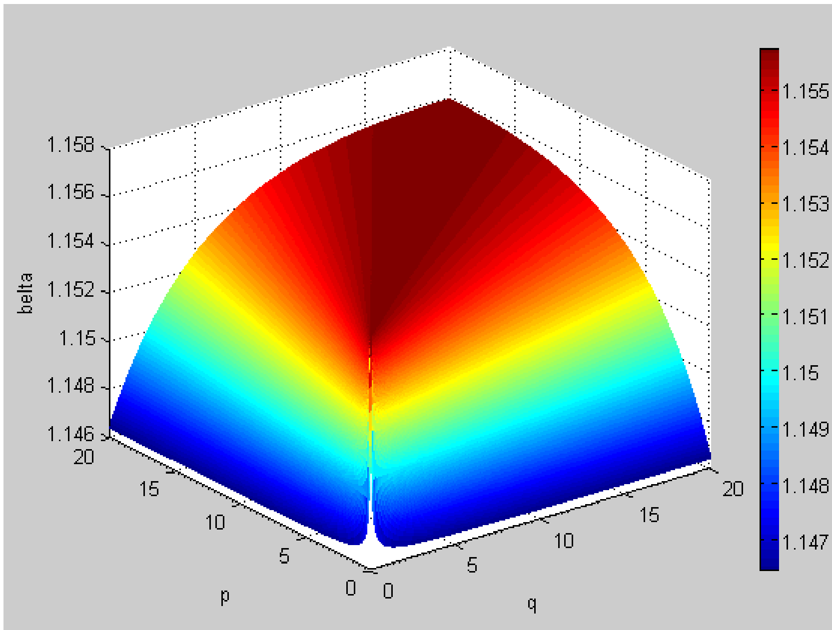

Fig. 8. The values β for alternative x_3 obtained by the W2TLGBM operator ($p \in (0,20]$, $q \in (0,20]$).

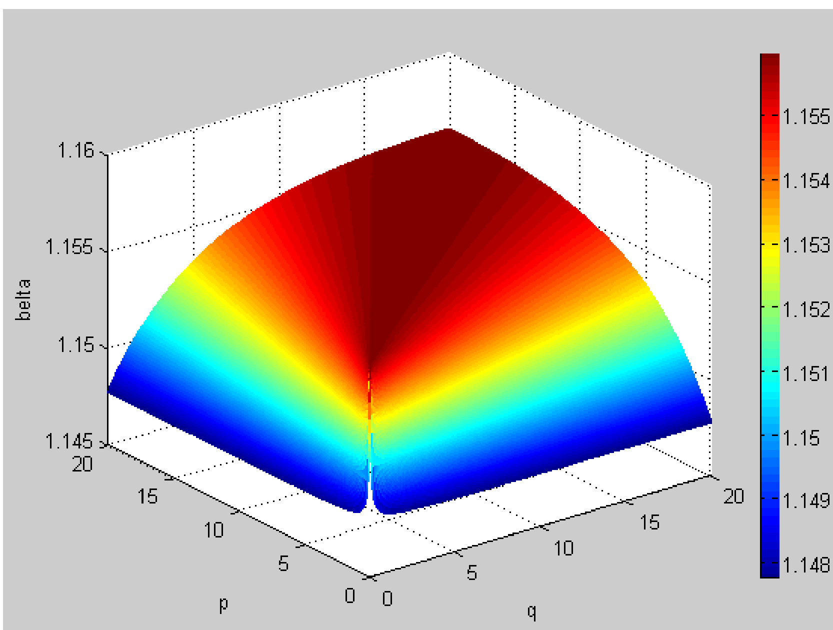

Fig. 9. The values β for alternative x_4 obtained by the W2TLGBM operator ($p \in (0,20]$, $q \in (0,20]$).

If we let the parameter p fixed, different values β_i and rankings of the alternatives can be obtained as the parameter q changed which was shown in Fig. 10.

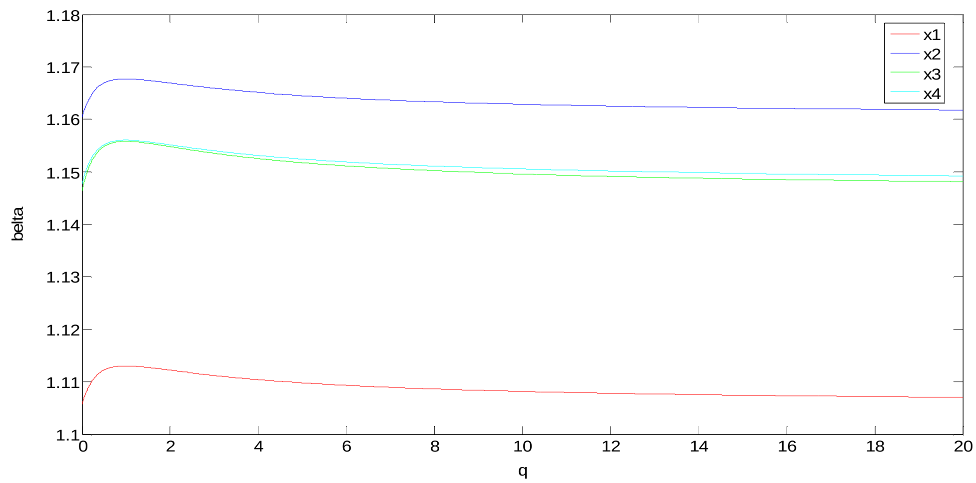

Fig. 10. Variation of β obtained with the W2TLGBM operator ($p=1$, $q\in(0,20]$).

From Fig. 10, we can find that when $q\in(0,20]$, the ranking of the four alternatives is $x_2 \succ x_4 \succ x_3 \succ x_1$ and the best choice is x_2.

From Fig. 10, we find that the collective overall preference values obtained by the W2TLGBM aggregation operators become smaller as parameter p and q increase for the same aggregation arguments. Therefore, it plays a crucial part in decision making. For example, in the real group decision making problems, the decision makers who take a gloomy view of the prospects could choose the bigger values of the parameters p and q while the decision makers who are optimistic could choose the smaller values of the parameters p and q.

It should be noted that the belta obtained by the W2TLBM operator are smaller than the belta obtained by the W2TLGBM operator, which indicates that the W2TLBM operator can obtain more unfavorable (or pessimistic) expectations, while the W2TLGBM operator has more favorable (or optimistic) expectations. Therefore, we can conclude that the W2TLBM operator can be considered as the optimistic operator, while the W2TLGBM operator can be considered as the pessimistic operator and the values of the parameters can be considered as the optimistic or pessimistic levels. By Figs. 1-10, we can conclude that the decision makers who take a gloomy view of the prospects could use the W2TLBM operator and choose the smaller values of the parameters p and q, while the decision makers who are optimistic could use the W2TLGBM operator and choose the smaller values of the parameters p and q.

In order to obtain the more neutral results, we can use the arithmetic averages of the optimistic and pessimistic results, which can be found in Figs. 11-14.

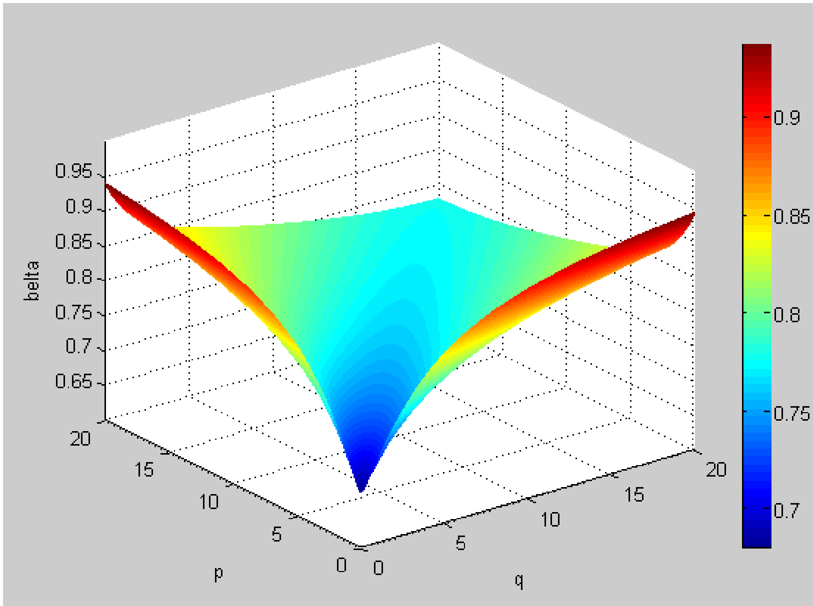

Fig. 11. The values β for alternative x_1 obtained by the W2TLBM and W2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

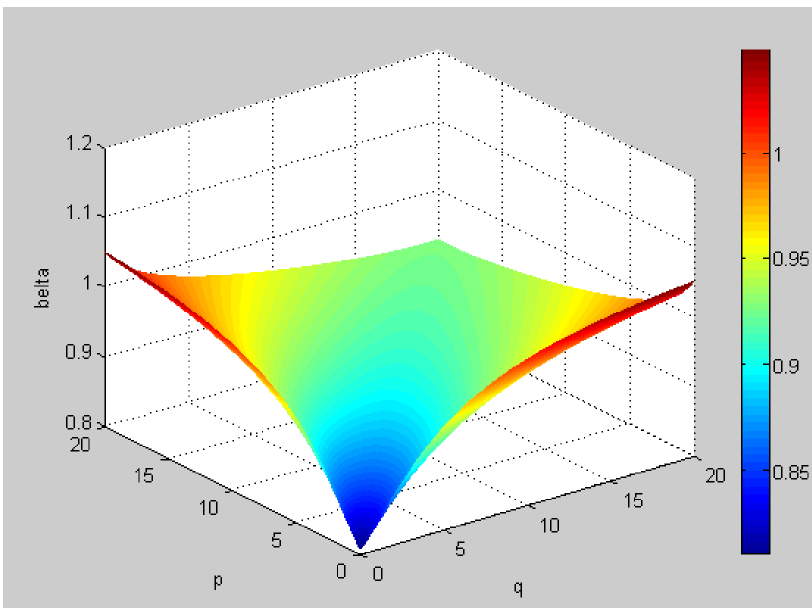

Fig. 12. The values β for alternative x_2 obtained by the W2TLBM and W2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

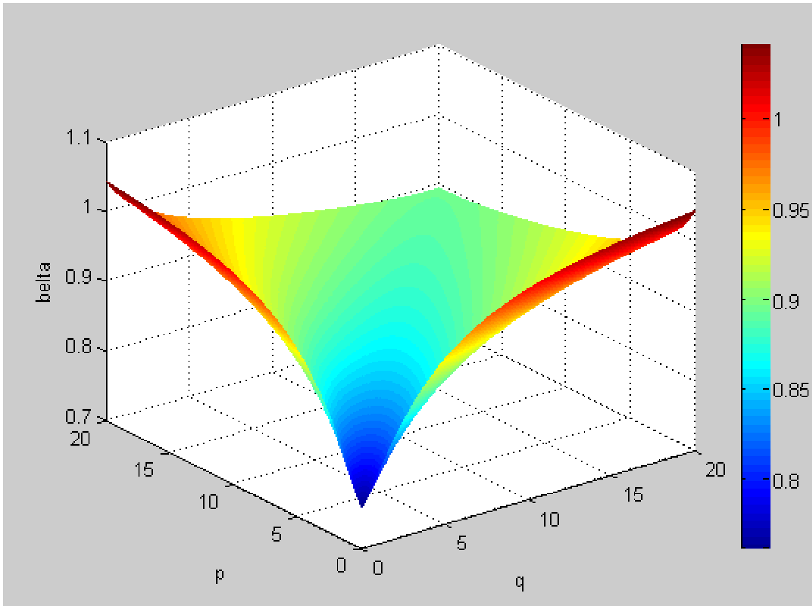

Fig. 13. The values β for alternative x_3 obtained by the W2TLBM and W2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

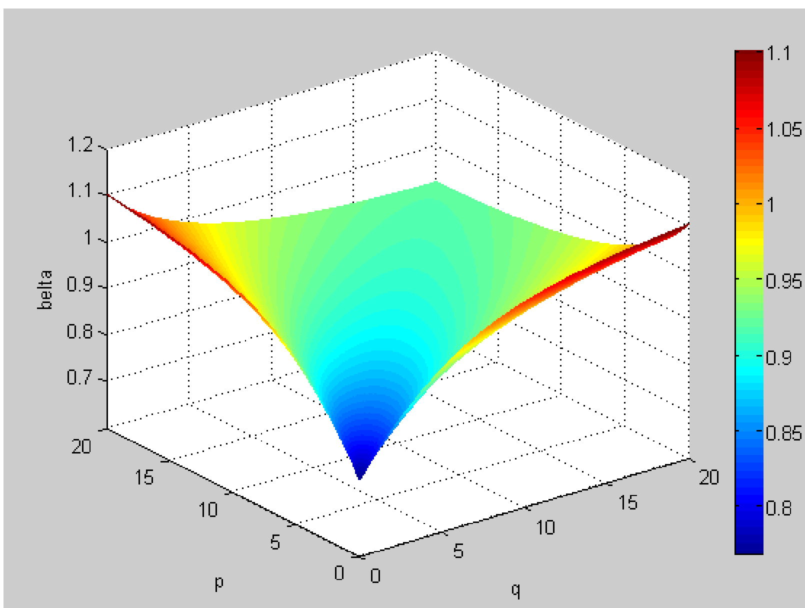

Fig. 14. The values β for alternative x_4 obtained by the W2TLBM and W2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

In the following, we compare our operators and approaches with the existing 2-tuple linguistic aggregation operators and approaches so as to demonstrate the advantages of the operators and approaches proposed here. In Example 5.1, we utilize the 2-tuple linguistic weighted average (2TLWA) operator [14,41]:

$$\text{2TLWA}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\Delta\left(\sum_{i=1}^{n} w_i\cdot\Delta^{-1}\left(r_i,\alpha_i\right)\right) \tag{44}$$

to replace the W2TLBM operator; then the collective overall preference value $\overline{r}_i=\left(r_i,\alpha_i\right)$ $\left(i=1,2,3,4\right)$ of alternative x_i $\left(i=1,2,3,4\right)$ are shown as follows:

$\overline{r}_1=\left(s_4,-0.0300\right)$, $\overline{r}_2=\left(s_6,0.1800\right)$, $\overline{r}_3=\left(s_5,0.4700\right)$, $\overline{r}_4=\left(s_6,-0.2700\right)$.

Using Definition 2.3, we then rank the $\overline{r}_i$ $\left(i=1,2,3,4\right)$ in descending order:

$\overline{r}_2>\overline{r}_4>\overline{r}_3>\overline{r}_1$.

According to the ranking of the $\overline{r}_i$ $\left(i=1,2,3,4\right)$, rank all of the alternatives x_i $\left(i=1,2,3,4\right)$ as follows:

$x_2\succ x_4\succ x_3\succ x_1$.

Thus, the best alternative is x_2.

Based on the aforementioned analysis, we can see that the 2TLWA operator is simpler than the W2TLBM and W2TLGBM operators from the computational point of view, but the W2TLBM and W2TLGBM operators can capture the interrelationship of the aggregated arguments [35] and can provide the decision makers more choices by changing the values of the parameters determined by the preferences of the decision makers [35].

Example 6.2. Let us reconsider Example 6.1. Assume that $A^{(k)}=\left(a_{ij}^{(k)}\right)_{4\times4}$ $\left(k=1,2,3\right)$ are three linguistic decision matrices shown in Tables 1-3. $R^{(k)}=\left(r_{ij}^{(k)}\right)_{4\times4}$ $\left(k=1,2,3\right)$ are three normalized linguistic decision matrices given in Tables 4-6. $\overline{R}^{(k)}=\left(\left(r_{ij}^{(k)},0\right)\right)_{4\times4}$ $\left(k=1,2,3\right)$ are three corresponding 2-tuple linguistic decision matrices given in Tables 7-9. Suppose that the weights of the decision makers and the attributes are unknown; then, we use Approach II to determine the decision. The collective 2-tuple linguistic decision matrix $\overline{R}'=\left(\overline{r}'_{ij}\right)_{4\times4}=\left(\left(r'_{ij},\alpha'_{ij}\right)\right)_{4\times4}$ is given in Table 13. The best alternative is x_2.

Table 13. The collective 2-tuple linguistic decision matrix $\overline{R}'$

13	c_1	c_2	c_3	c_4
x_1	$\left(s_5,-0.3453\right)$	$\left(s_2,-0.0851\right)$	$\left(s_3,-0.2311\right)$	$\left(s_5,0.1316\right)$
x_2	$\left(s_6,-0.4924\right)$	$\left(s_6,-0.1122\right)$	$\left(s_6,0.2716\right)$	$\left(s_6,0.2716\right)$
x_3	$\left(s_5,-0.1352\right)$	$\left(s_5,0.0332\right)$	$\left(s_5,-0.1352\right)$	$\left(s_5,-0.0334\right)$
x_4	$\left(s_7,0.3030\right)$	$\left(s_5,-0.4539\right)$	$\left(s_4,0.2817\right)$	$\left(s_5,0.1962\right)$

We can find that as the values of the parameters p and q change according to the decision makers' subjective preferences, we may obtain different rankings of the alternatives, which can reflect the decision makers' risk preferences. As the values of the parameters p and q change, the collective overall preference value $\bar{r}_i'=(r_i',\alpha_i')$ $(i=1,2,3,4)$ and the ranking of alternatives can be obtained and shown in Table 14.

Table 14. The collective overall preference values obtained with the 2TLBM operator and rankings of the alternatives.

14	$p=0,\ q=20$	$p=0.5,\ q=15$	$p=q=5$	$p=15,\ q=0.1$	$p=20,\ q=0.05$
x_1	$(s_7,0.0891)$	$(s_7,-0.4043)$	$(s_4,0.4483)$	$(s_7,-0.2153)$	$(s_7,0.0684)$
x_2	$(s_7,0.4823)$	$(s_7,0.2594)$	$(s_6,0.3807)$	$(s_7,0.3246)$	$(s_7,0.4752)$
x_3	$(s_7,0.1133)$	$(s_7,-0.2625)$	$(s_6,-0.4656)$	$(s_7,-0.1379)$	$(s_7,0.0981)$
x_4	$(s_7,0.4740)$	$(s_7,0.2032)$	$(s_6,0.2012)$	$(s_7,0.2934)$	$(s_7,0.4625)$
Ranking	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$

As the values of the parameters p and q change between 0 and 20, different results of a symbolic aggregation operation $\beta_i=\Delta^{-1}(\bar{r}_i')$ $(i=1,2,3,4)$ of the collective overall preference values $\bar{r}_i$ $(i=1,2,3,4)$ of the four alternatives x_i $(i=1,2,3,4)$ can be obtained. Figs. 15-18 illustrate the values $\beta_i=\Delta^{-1}\left(\bar{r}_i'\right)$ $(i=1,2,3,4)$ of the four alternatives x_i $(i=1,2,3,4)$ obtained by the 2TLBM operator in detail.

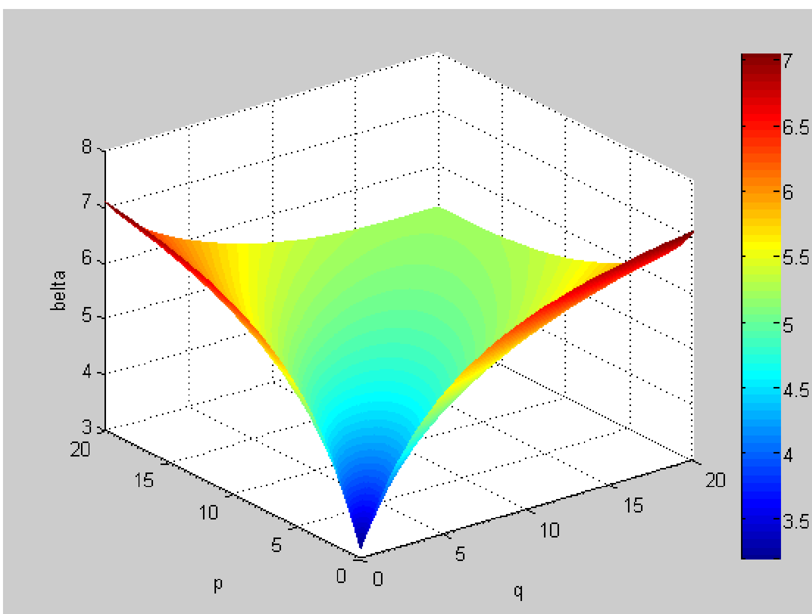

Fig. 15. The values β for alternative x_1 obtained by the 2TLBM operator ($p\in(0,20]$, $q\in(0,20]$).

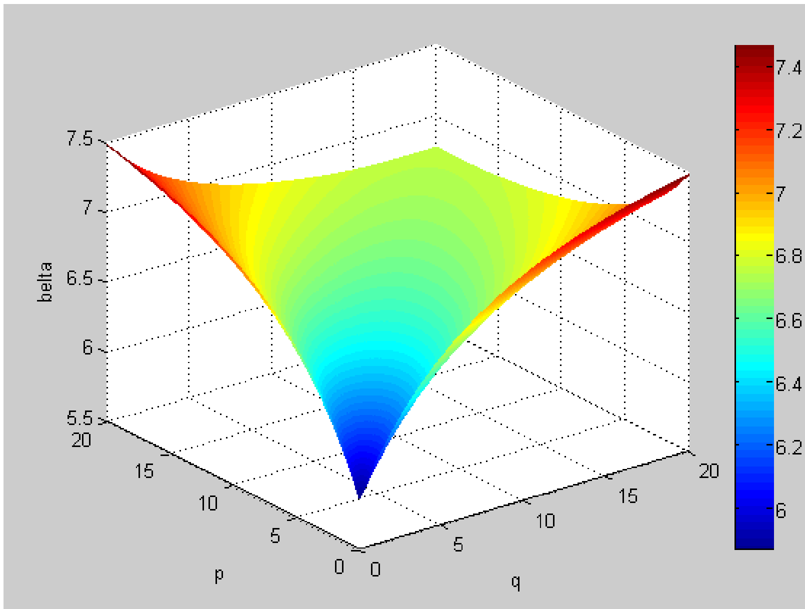

Fig. 16. The values β **for alternative** x_2 **obtained by the 2TLBM operator (** $p \in (0,20]$, $q \in (0,20]$ **).**

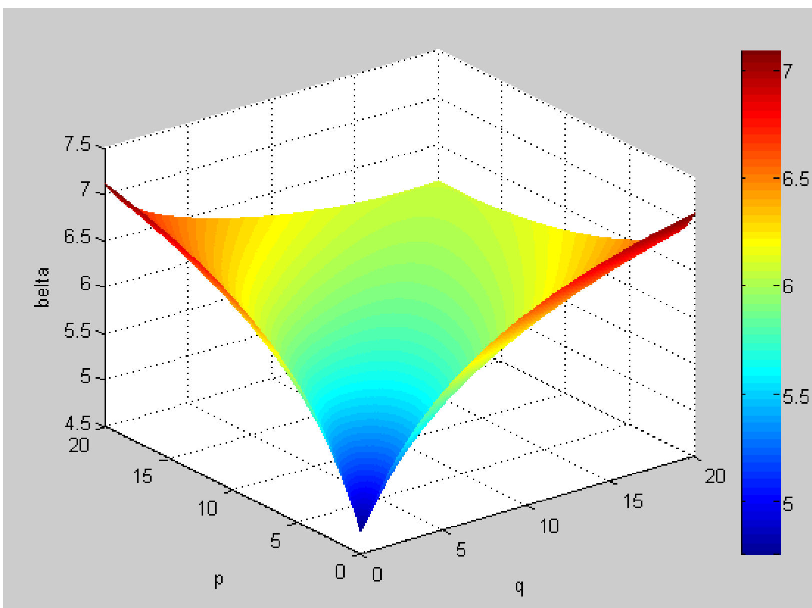

Fig. 17. The values β **for alternative** x_3 **obtained by the 2TLBM operator (** $p \in (0,20]$, $q \in (0,20]$ **).**

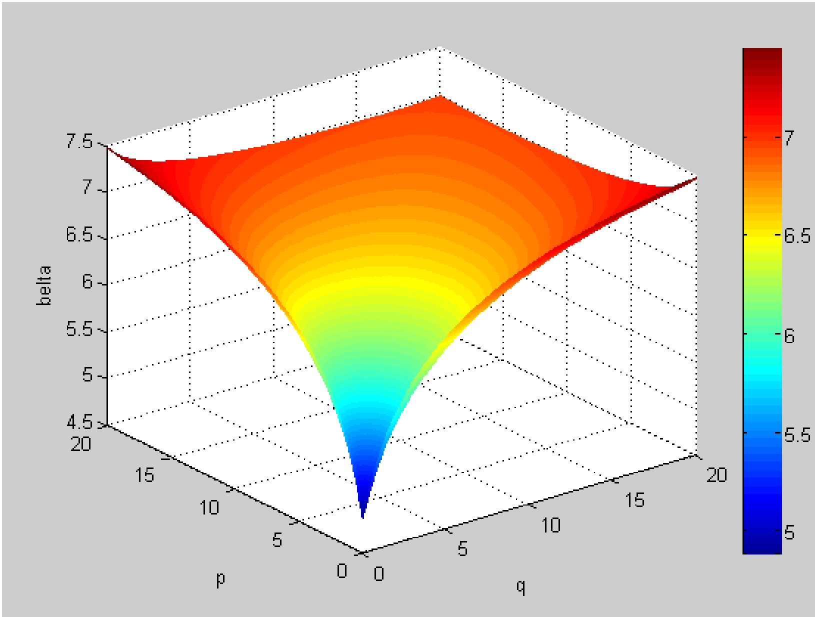

Fig. 18. The values β for alternative x_4 obtained by the 2TLBM operator ($p \in (0,20]$, $q \in (0,20]$).

If we let the parameter p fixed, different values β_i and rankings of the alternatives can be obtained as the parameter q changed which was shown in Fig. 19.

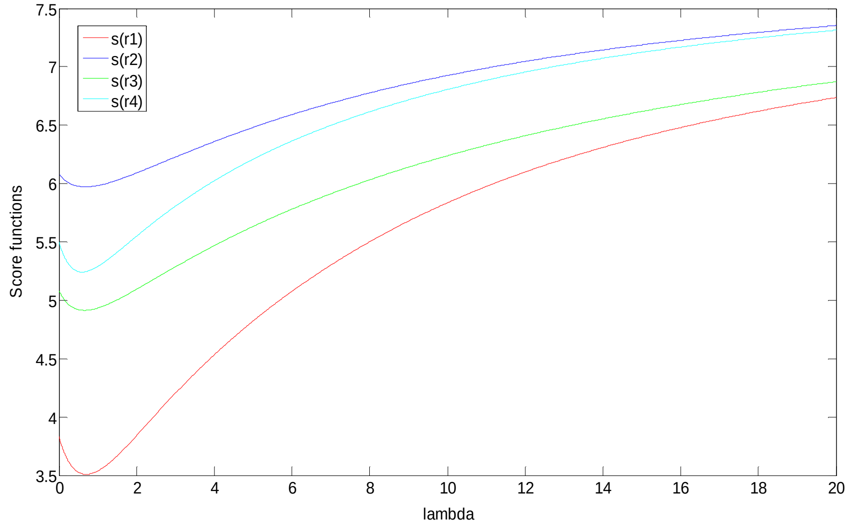

Fig. 19. Variation of β obtained with the 2TLBM operator ($p=1$, $q \in (0,20]$).

From Fig. 19, we can find that when $q \in (0, 20]$, the ranking of the four alternatives is $x_2 \succ x_4 \succ x_3 \succ x_1$ and the best choice is x_2.

From Fig. 19, we find that the collective overall preference values obtained by the 2TLBM aggregation operators become bigger as parameter p and q increase for the same aggregation arguments. Therefore, it plays a crucial part in decision making. For example, in the real group decision making problems, the decision makers who take a gloomy view of the prospects could choose the smaller values of the parameters p and q while the decision makers who are optimistic could choose the bigger values of the parameters p and q.

If the 2TLGBM operator is used in place of the 2TLBM operator to aggregate the values of the alternatives in steps 1 and 2, then the collective overall preference values and the rankings of the alternatives are listed in Table 15.

Table 15. The collective overall preference values obtained with the 2TLGBM operator and rankings of the alternatives.

15	$p=0,\ q=20$	$p=0.5,\ q=15$	$p=q=5$	$p=15,\ q=0.1$	$p=20,\ q=0.05$
x_1	$(s_3, 0.1644)$	$(s_3, 0.2330)$	$(s_4, -0.3675)$	$(s_3, 0.1792)$	$(s_3, 0.1700)$
x_2	$(s_6, -0.1442)$	$(s_6, -0.1208)$	$(s_6, 0.0328)$	$(s_6, -0.1392)$	$(s_6, -0.1423)$
x_3	$(s_5, -0.2650)$	$(s_5, -0.2273)$	$(s_5, 0.0098)$	$(s_5, -0.2570)$	$(s_5, -0.2620)$
x_4	$(s_5, -0.1387)$	$(s_5, -0.0524)$	$(s_5, 0.3822)$	$(s_5, -0.1196)$	$(s_5, -0.1314)$
Ranking	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_2 \succ x_4 \succ x_3 \succ x_1$

As the values of the parameters p and q change between 0 and 20, different results of a symbolic aggregation operation $\beta_i = \Delta^{-1}(\overline{r}_i)\ (i = 1, 2, 3, 4)$ of the collective overall preference values $\overline{r}_i\ (i = 1, 2, 3, 4)$ of the four alternatives $x_i\ (i = 1, 2, 3, 4)$ can be obtained. Figs. 20-23 illustrate the values $\beta_i = \Delta^{-1}(\overline{r}_i)\ (i = 1, 2, 3, 4)$ of the four alternatives $x_i\ (i = 1, 2, 3, 4)$ obtained by the 2TLGBM operator in detail.

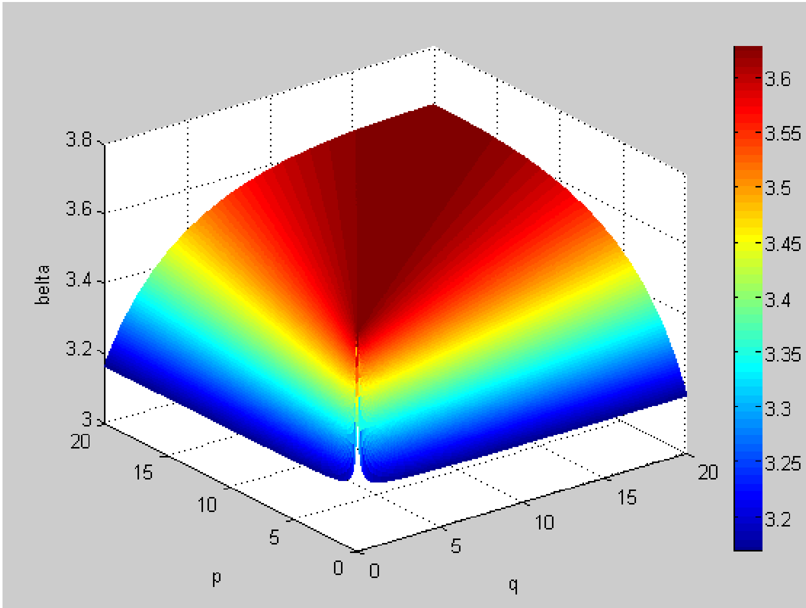

Fig. 20. The values β **for alternative** x_1 **obtained by the 2TLGBM operator (** $p \in (0,20]$, $q \in (0,20]$ **).**

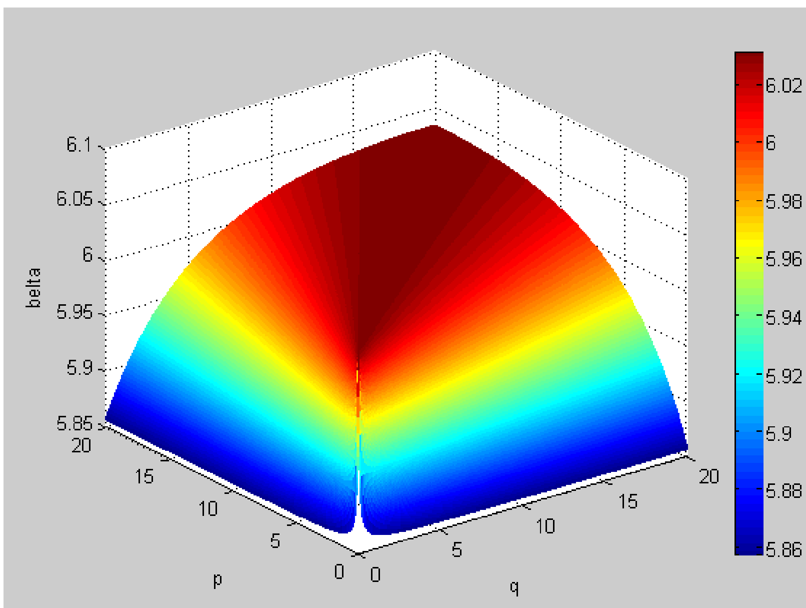

Fig. 21. The values β **for alternative** x_2 **obtained by the 2TLGBM operator (** $p \in (0,20]$, $q \in (0,20]$ **).**

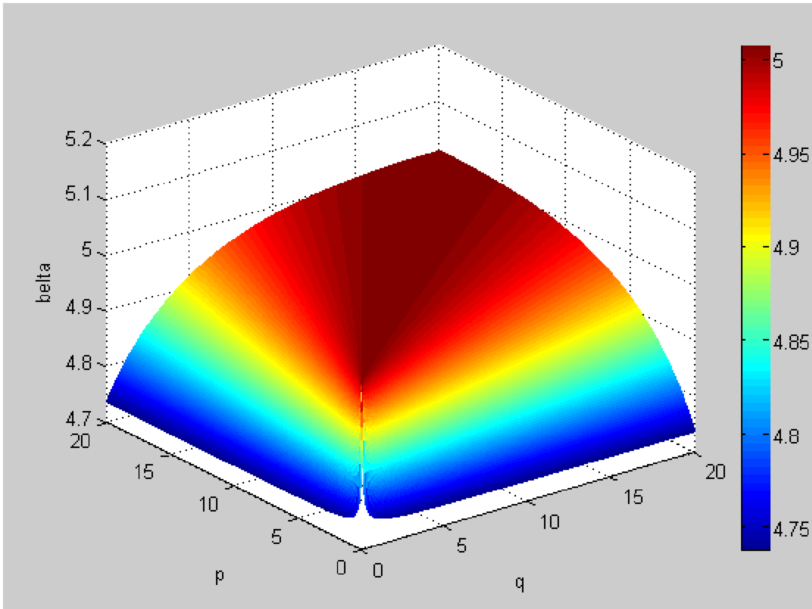

Fig. 22. The values β for alternative x_3 obtained by the 2TLGBM operator ($p \in (0,20]$, $q \in (0,20]$).

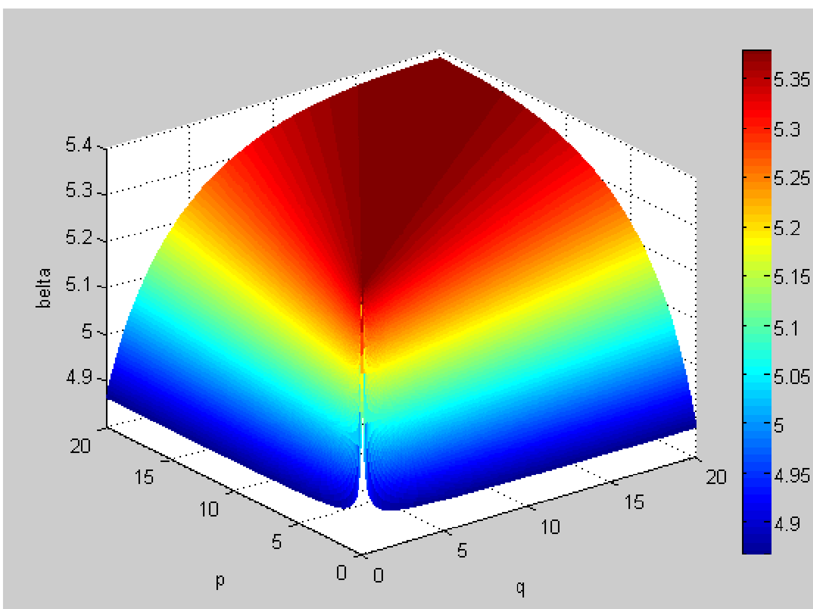

Fig. 23. The values β for alternative x_4 obtained by the 2TLGBM operator ($p \in (0,20]$, $q \in (0,20]$).

If we let the parameter p fixed, then different values β_i and rankings of the alternatives can be obtained as the parameter q changed, which was shown in Fig. 24.

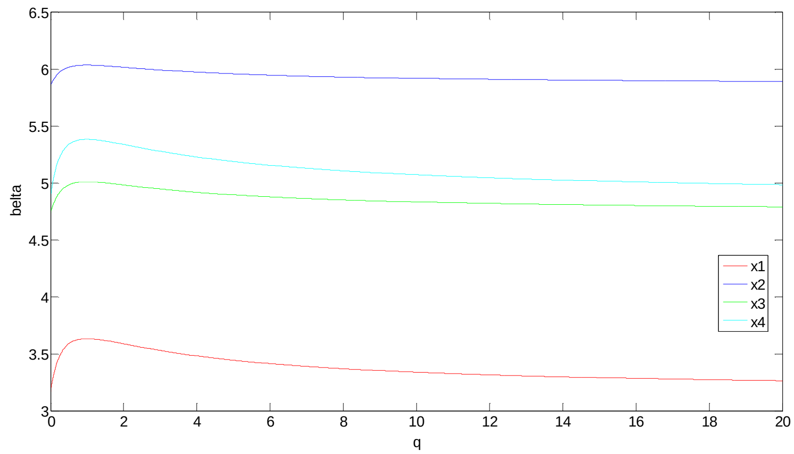

Fig. 24. Variation of β obtained with the 2TLGBM operator ($p=1$, $q\in(0,20]$).

From Fig. 24, we can find that when $q\in(0,20]$, the ranking of the four alternatives is $x_2 \succ x_4 \succ x_3 \succ x_1$ and the best choice is x_2.

From Fig. 24, we find that the collective overall preference values obtained by the 2TLGBM aggregation operators become smaller as parameter p and q increase for the same aggregation arguments. Therefore, it plays a crucial part in decision making. For example, in the real group decision making problems, the decision makers who take a gloomy view of the prospects could choose the bigger values of the parameters p and q while the decision makers who are optimistic could choose the smaller values of the parameters p and q.

It is worth noting that most of the belta obtained by the 2TLBM operator are bigger than most of the belta obtained by the 2TLGBM operator, which indicates that the 2TLBM operator can obtain more favorable (or optimistic) expectations, while the 2TLGBM operator has more unfavorable (or pessimistic) expectations. Therefore, we can conclude that the 2TLBM operator can be considered as the optimistic operator, while the 2TLGBM operator can be considered as the pessimistic operator and the values of the parameters can be considered as the optimistic or pessimistic levels. By Figs. 15-24, we can conclude that the decision makers who take a gloomy view of the prospects could use the 2TLGBM operator and choose the bigger values of the parameters p and q, while the decision makers who are optimistic could use the 2TLBM operator and choose the bigger values of the parameters p and q.

In order to obtain the more neutral results, we can use the arithmetic averages of the optimistic and pessimistic results, which can be found in Figs. 25-28.

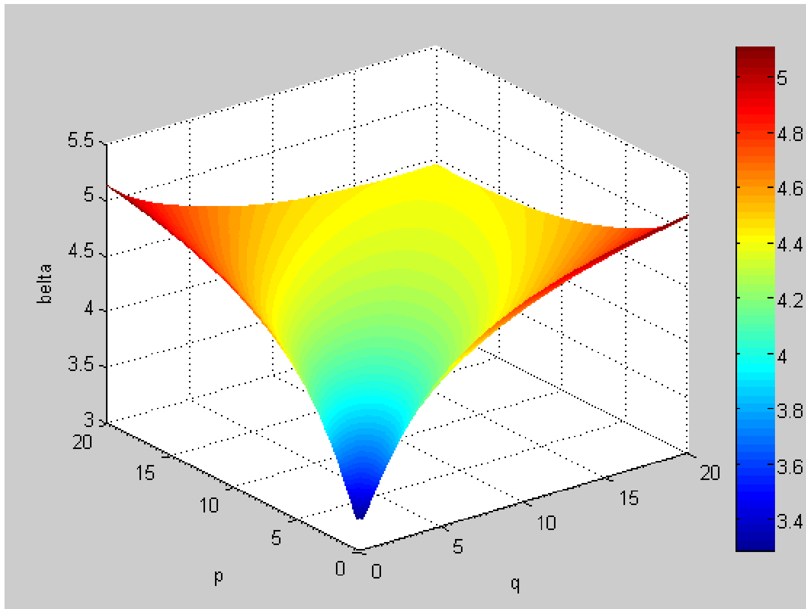

Fig. 25. The values β for alternative x_1 obtained by the 2TLBM and 2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

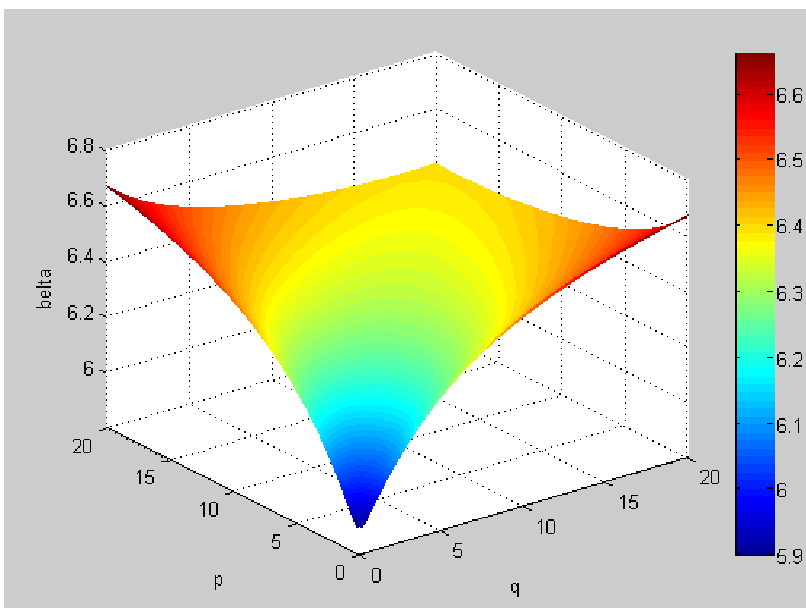

Fig. 26. The values β for alternative x_2 obtained by the 2TLBM and 2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

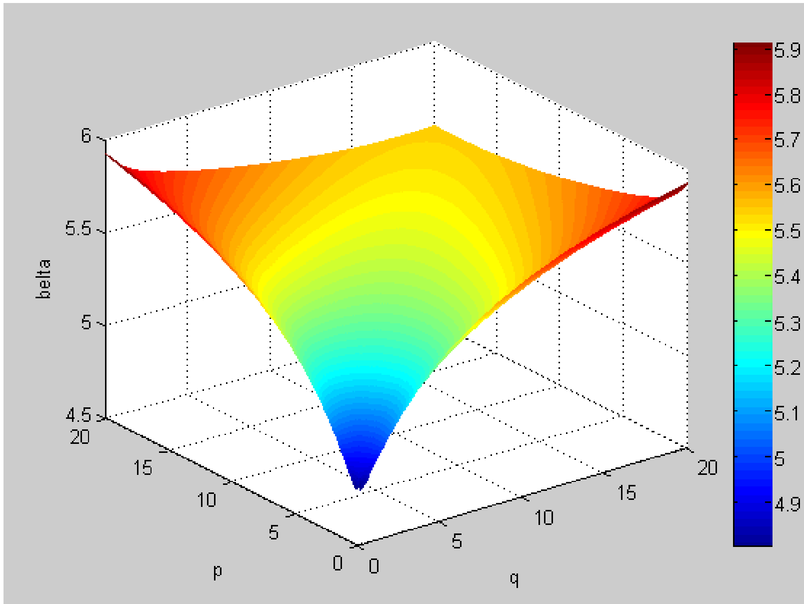

Fig. 27. The values β for alternative x_3 obtained by the 2TLBM and 2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

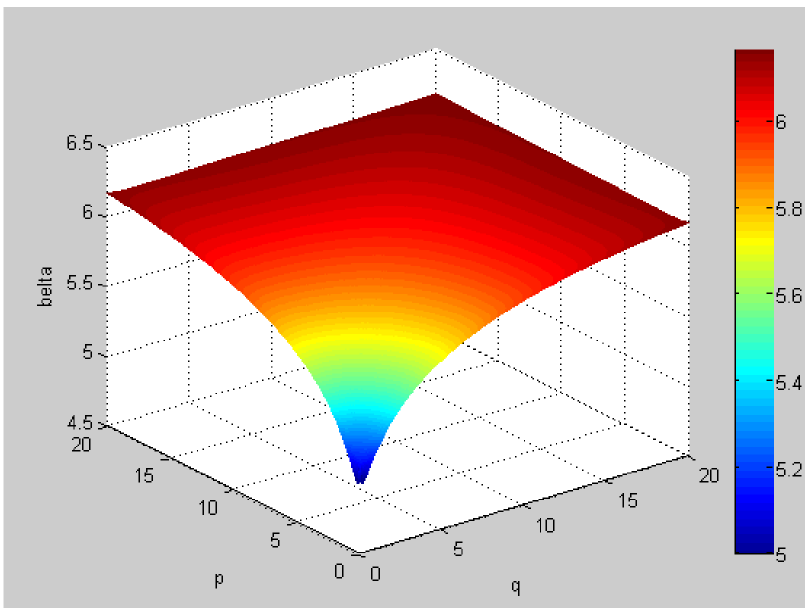

Fig. 28. The values β for alternative x_4 obtained by the 2TLBM and 2TLGBM operators ($p \in (0,20]$, $q \in (0,20]$).

By examples 1 and 2, we can see that the decision results obtained with Approach I may be different from the decision results obtained with Approach II. Approach I utilizes the W2TLBM (or W2TLGBM) operator to aggregate all of the individual 2-tuple linguistic decision matrices into a collective 2-tuple linguistic decision matrix and then utilize the W2TLBM (or W2TLGBM) operator to derive the collective overall preference values of each alternative. Approach II utilizes the 2TLBM (or 2TLGBM) operator to aggregate all of the individual 2-tuple linguistic decision matrices into a collective 2-tuple linguistic decision matrix and then utilize the 2TLBM (or 2TLGBM) operator to derive the collective overall preference values of each alternative. The 2TLBM and 2TLGBM operators only involve the input data and their interrelationships, but the importance of each datum is not emphasized. The W2TLBM and W2TLGBM operators not only involve the input data and their interrelationships but also take the importance of each datum into account.

7. Conclusions

In this paper, we have developed several new 2-tuple linguistic aggregation operators, including the 2TLBM, W2TLBM, 2TLGBM, and W2TLGBM operators. We have studied some fundamental properties of the developed operators, such as commutativity, idempotency, boundedness, and monotonicity. We also discuss some special cases of the proposed operators. Compared with the existing 2-tuple linguistic aggregation operators, the primary advantage of these operators is that they capture the interrelationship of the input arguments and thus consider the decision information as much as possible. Furthermore, we have used the proposed operators to develop two approaches for multiple attribute group decision making with 2-tuple linguistic information. Finally, two numerical examples are provided to illustrate the developed approaches and to compare the developed approaches with the existing ones. In the future, we will study the Bonferroni mean and the geometric Bonferroni mean under interval-valued 2-tuple linguistic environments [49,50].

Acknowledgements

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Authors' Contributions

'Zhiming Zhang' designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. 'Chong Wu' managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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