British Journal of Mathematics & Computer Science
4(11): 1542-1550, 2014

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Some New Results and Generalizations in G-cone Metric Fixed Point Theory

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Original Research Article

Received: 12 September 2013
Accepted: 13 March 2014
Published: 29 March 2014

Abstract

In this paper, we present sufficient generalized contractive conditions for the existence of fixed points in what so-called G-cone metric spaces. Importantly, we have obtained our results using contractive conditions stated in terms of variable coefficients and with no use of the normality property of cone.

Keywords: Generalized metric space, cone metric space, fixed point
2000 Mathematics Subject Classification: 47H10,54H25,54E35,46B20

1 Introduction

Some attempts to generalize the usual metric spaces had been made since sixties of 20^{th} century, see for example [1, 2]. Separate work conducted by other researchers see for example [3], refuted these generalizations. A different generalization was introduced by Dhage [4], but unfortunately, this one has also many fundamental flaws that demonstrated by other workers, see for example [5, 6, 7].

Later on, generalized metric spaces or more specifically, G-metric spaces, in an appropriate new structure were introduced by Mustafa and Sims [8]. This new structure was a great alternative to amend the flaws in the concept of D-metric spaces [4]. In [8], it is proved that in this new structure every G-metric space is a topologically equivalent to a metric space, which allows transforming directly many concepts and results from metric spaces into the G-metric space setting.

Separately, Huang and Zhang generalized in [9] the notation of metric spaces by replacing the set of real numbers by ordered Banach space, and define the concept of cone metric spaces.

Beg *et. al.* [10] introduced a generalization of the G-metric spaces and cone metric spaces in what is called G-cone metric spaces, and proved some convergence properties as well as some fixed point theorems.

Several fixed point theorems were obtained in the G-metric spaces and the cone metric spaces for mappings satisfying certain contractive conditions, see for example [9, 11, 12, 13] and references

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therein. Whereas, only few results were obtained in the G-cone metric spaces, see for example [10, 14].

In this paper, we prove in these new metric spaces, the G-cone metric spaces, new fixed point results that are generalizations of previous results from literature, e.g. [10, 14], for mappings satisfying contractions with variable coefficients. We should also note that we prove our fixed point results with no use of the normality property of cone that is used to prove some similar results in the literature, e.g., in [14].

2 Preliminaries

We give in this section, preliminaries and basic definitions which will be used throughout the paper.

Throughout the paper, let E be a real Banach space.

Definition 2.1 (See [10]). A subset P of E is called a cone if and only if:

(P1) P is closed, nonempty and $P \neq \{0\}$,

(P2) If $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ then $ax + by \in P$. More generally, if $a, b, c \in \mathbb{R}$, $a, b, c \geq 0$, and $x, y, z \in P$ then $ax + by + cz \in P$,

(P3) $P \cap (-P) = \{0\}$.

A partial ordering $\preceq$ with respect to a given cone $P \subset E$ is defined by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \prec\prec y$ stands for $y - x \in IntP$, i.e., $y - x$ in interior of P. A cone P is called normal if there exists a number $K > 0$ such that for all $x, y \in E$, we have $0 \preceq x \preceq y \Rightarrow \|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P, and it is proved in [15] that there are no normal cones with normal constant $K < 1$.

Definition 2.2 (See [10]). Let X be a nonempty set. Suppose a mapping $G : X \times X \times X \to E$ satisfies

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 \prec G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$,

(G3) $G(x, x, y) \preceq G(x, y, z)$ whenever $y \neq z$, for all $x, y, z \in X$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots$ (Symmetry in all three variables),

(G5) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$. (Rectangle inequality).

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space, G-cone metric space.

Definition 2.3 (See [10]). Let X be a G-cone metric space, and x_n be a sequence in X. We say that x_n is

- Cauchy sequence if for every $c \in E$ with $0 \prec\prec c$, there exists N such that $G(x_n, x_m, x_\ell) \prec\prec c$, for all $n, m, \ell > N$.
- convergent sequence if for every $c \in E$ with $0 \prec\prec c$, there exists N such that for all $n, m > N$, $G(x_n, x_m, x) \prec\prec c$ for some fixed $x \in X$. Here, x is called the limit of the sequence x_n, and is denoted by $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

A $G-$cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

To illustrate this new concept we give the following example.

Example 2.1. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be a cone in E, and $X \subseteq \mathbb{R}$. Define the mapping $G : X \times X \times X \to E$ by

$$G(x, y, z) = \left(\frac{1}{3}(|x-y| + |y-z| + |x-z|), \frac{2}{3}(|x-y| + |y-z| + |x-z|)\right), \text{ for each } x, y, z \in X.$$

Then G is a generalized cone metric on X and X is a $G-$cone metric space.
For more examples of G-cone metric spaces, and some convergence properties of sequences in G-cone metric spaces, one could refer to the paper by Beg *et. al.* [10].

3 Main Results

We prove here, some fixed point theorems in G-cone metric spaces introduced in Section 2. We should note that we would not need to use the normality property of cone to obtain the results.

Theorem 3.1. *Let X be a complete G-cone metric space, and let $T : X \to X$ be a mapping satisfying, for each $x, y \in X$,*

$$\begin{aligned} G(Tx, Ty, Ty) \quad & \preceq \alpha(x, y, y)G(x, y, y) \\ & \quad + \beta(x, y, y)[G(x, Tx, Tx) + 2G(y, Ty, Ty)] \\ & \quad + \gamma(x, y, y)[G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)] \\ & \quad + \delta(x, y, y)[G(y, y, Tx) + G(y, y, Ty) + G(x, x, Ty)] \\ & \quad + \sigma(x, y, y)[G(x, x, Tx) + 2G(y, y, Ty)], \end{aligned} \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0, 1)$ such that

$$\lambda := \sup\left\{\frac{\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+5\delta(x,y,y)+8\sigma(x,y,y)}{1-(2\beta(x,y,y)+2\gamma(x,y,y)+4\delta(x,y,y)+4\sigma(x,y,y))} : x, y \in X\right\} < 1. \tag{3.2}$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$.

Proof. Let $x_0 \in X$ be an arbitrary initial guess, and let the sequence $\{x_n\}$ be defined by $x_n = T^n x_0$, or equivalently, $x_n = Tx_{n-1}$, $n \geq 1$. Then, from (3.1), using (G1) from Definition 2.2, we get

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) & \\ & \preceq \alpha G(x_{n-1}, x_n, x_n) \\ & \quad + \beta[G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})] \\ & \quad + \gamma[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})] \\ & \quad + \delta[G(x_n, x_n, x_{n+1}) + G(x_{n-1}, x_{n-1}, x_{n+1})] \\ & \quad + \sigma[G(x_{n-1}, x_{n-1}, x_n) + 2G(x_n, x_n, x_{n+1})], \end{aligned} \tag{3.3}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are evaluated at (x_{n-1}, x_n, x_n). By rectangle inequality, (G5) in Definition 2.2, we have

$$\begin{cases} G(x_{n-1}, x_{n+1}, x_{n+1}) \preceq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n-1}, x_{n+1}) \preceq 2G(x_{n-1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_n, x_{n+1}) \preceq G(x_{n-1}, x_{n-1}, x_{n+1}) + G(x_{n-1}, x_n, x_n), \\ G(x_{n-1}, x_{n-1}, x_n) \preceq 2G(x_{n-1}, x_n, x_n), \end{cases} \tag{3.4}$$

and together with equation (3.3), we get

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \quad & \preceq \alpha G(x_{n-1}, x_n, x_n) \\ & \quad + \beta[G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})] \\ & \quad + \gamma[G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})] \\ & \quad + \delta[5G(x_{n-1}, x_n, x_n) + 4G(x_n, x_{n+1}, x_{n+1})] \\ & \quad + \sigma[8G(x_{n-1}, x_n, x_n) + 4G(x_n, x_{n+1}, x_{n+1})], \end{aligned}$$

or

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \quad &\preceq (\alpha+\beta+\gamma+5\delta+8\sigma)G(x_{n-1}, x_n, x_n) \\ &\quad +(2\beta+2\gamma+4\delta+4\sigma)G(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

which implies

$$G(x_n, x_{n+1}, x_{n+1}) \preceq \frac{\alpha+\beta+\gamma+5\delta+8\sigma}{1-(2\beta+2\gamma+4\delta+4\sigma)} G(x_{n-1}, x_n, x_n) \preceq \lambda G(x_{n-1}, x_n, x_n),$$

where, λ is given in (3.2), and hence, we have

$$G(x_n, x_{n+1}, x_{n+1}) \preceq \lambda^n G(x_0, x_1, x_1). \tag{3.5}$$

Now, for all $n, m \in \mathbb{N}$, with $n < m$, we have, using (G5) from Definition 2.2,

$$G(x_n, x_m, x_m) \preceq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m).$$

Using equation (3.5), implies

$$\begin{aligned} &G(x_n, x_m, x_m) \\ &\qquad \preceq (G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)) \\ &\qquad \preceq (\lambda^n G(x_0, x_1, x_1) + \lambda^{n+1} G(x_0, x_1, x_1) + \cdots + \lambda^{m-1} G(x_0, x_1, x_1)) \\ &\qquad \preceq \tfrac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1). \end{aligned} \tag{3.6}$$

We want next to show that x_n is a Cauchy sequence. Given c such that $0 \prec\prec c$, we choose $\hat{\delta} > 0$ such that $c + N_{\hat{\delta}}(0) \subseteq P$, where $N_{\hat{\delta}}(0) = \{y \in E : \|y\| < \hat{\delta}\}$. We also choose a natural number N_0 such that $\frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1) \in N_{\hat{\delta}}(0)$, for all $m \geq N_0$. Then, $\frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1) \prec\prec c$, for all $m \geq N_0$. Therefore, we have $G(x_n, x_m, x_m) \prec\prec c$, for all $m > n$. Hence, x_n is a Cauchy sequence. By completeness, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.7}$$

To show that u is indeed a fixed point for T, we again use (3.1) to get

$$\begin{aligned} G(x_n, Tu, Tu) & \\ &\preceq \alpha G(x_{n-1}, x_n, x_n) \\ &\quad +\beta[G(x_{n-1}, x_n, x_n) + 2G(u, Tu, Tu)] \\ &\quad +\gamma[G(x_{n-1}, Tu, Tu) + G(u, Tu, Tu) + G(u, x_n, x_n)] \\ &\quad +\delta[G(u, u, x_n) + G(u, u, Tu) + G(x_{n-1}, x_{n-1}, Tu)] \\ &\quad +\sigma[G(x_{n-1}, x_{n-1}, x_n) + 2G(u, u, Tu)], \end{aligned}$$

where α, β, γ, δ, and σ are now evaluated at (x_{n-1}, u, u). Taking the limit as $n \to \infty$, leads to

$$G(u, Tu, Tu) \preceq 2(\beta+\gamma)G(u, Tu, Tu) + 2(\delta+\sigma)G(u, u, Tu),$$

and using the rectangle inequality to derive the fact that

$$G(u, u, Tu) \preceq 2G(Tu, Tu, u),$$

we get

$$G(u, Tu, Tu) \preceq 2(\beta+\gamma+2(\delta+\sigma))G(u, Tu, Tu). \tag{3.8}$$

Now, from (3.2) we know that

$$\sup\left\{ \tfrac{\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+5\delta(x,y,y)+8\sigma(x,y,y)}{1-(2\beta(x,y,y)+2\gamma(x,y,y)+4\delta(x,y,y)+4\sigma(x,y,y))} : x, y \in X \right\} < 1,$$

and hence,

$$\tfrac{\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+5\delta(x,y,y)+8\sigma(x,y,y)}{1-(2\beta(x,y,y)+2\gamma(x,y,y)+4\delta(x,y,y)+4\sigma(x,y,y))} < 1, \forall x, y \in X.$$

Consequently,

$$\alpha(x,y,y)+3\beta(x,y,y)+3\gamma(x,y,y)+9\delta(x,y,y)+12\sigma(x,y,y)<1, \forall x,y\in X, \tag{3.9}$$

which implies that $2(\beta+\gamma+2(\delta+\sigma))<1$. Therefore, we have $G(u,Tu,Tu)=0$, and hence, $Tu=u$.

For uniqueness, suppose that $u\neq v=Tv$. Then, from (3.1) after simplifying, we get

$$G(u,v,v)\preceq \alpha G(u,v,v)+\gamma[G(u,v,v)+G(v,u,u)]+\delta[G(v,v,u)+G(u,u,v)],$$

where α, γ, and δ are evaluated at (u,v,v). Using again the rectangle inequality, we have

$$G(v,u,u)\preceq 2G(u,v,v).$$

Therefore, after simplifying, we get

$$G(u,v,v)\preceq (\alpha+3(\gamma+\delta))G(u,v,v).$$

Form (3.9), we have $\alpha+3(\gamma+\delta)<1$, and hence, we have $G(u,v,v)=0$, which implies $u=v$.

Finally, since $x_0\in X$ was arbitrary, then from (3.7) we conclude that $T^n x\to u$ as $n\to\infty$ for all $x\in X$. □

Corollary 3.2. *Let X be a complete G-cone metric space, and let $T:X\to X$ be a mapping satisfying, for each $x,y\in X$,*

$$G(Tx,Ty,Ty)\preceq \alpha(x,y,y)G(x,y,y), \tag{3.10}$$

where, $0\leq\lambda:=\sup\{\alpha(x,y,y):x,y\in X\}<1$. Then T has a unique fixed point, say $u\in X$. Moreover, $T^n x\to u$ as $n\to\infty$ for all $x\in X$.

Proof. If one takes $\beta=\gamma=\delta=\sigma=0$ in Theorem 3.1, then $\lambda=\sup\{\alpha(x,y,y):x,y\in X\}<1$, and the proof is straightforward from the proof of Theorem 3.1. □

Remark 3.1. As one can see from Corollary 3.2, other selections or variations from the condition in (3.1) are valid similarly, i.e., one could take $\alpha=\gamma=\delta=\sigma=0$, $\alpha=\beta=\gamma=0$, $\alpha=\beta=\delta=0$, or $\alpha=\beta=0$, ..., and so on, and a similar proof would follows, because any map satisfies the new condition using one of the above would definitely, satisfies the original condition (3.1) in Theorem 3.1.

Corollary 3.3. *Let X be a complete G-cone metric space, and let $T:X\to X$ be a mapping satisfying, for each $x,y\in X$,*

$$\begin{aligned} G(Tx,Ty,Ty) \preceq\ & \alpha(x,y,y)G(x,y,y)\\ &+\beta(x,y,y)[G(x,Tx,Tx)+G(y,Ty,Ty)]\\ &+\gamma(x,y,y)[G(x,Ty,Ty)+G(y,Ty,Ty)+G(y,Tx,Tx)]\\ &+\delta(x,y,y)[G(y,y,Tx)+G(y,y,Ty)+G(x,x,Ty)]\\ &+\sigma(x,y,y)[G(x,x,Tx)+2G(y,y,Ty)], \end{aligned} \tag{3.11}$$

where $\alpha,\beta,\gamma,\delta,\sigma$ are some functions from $X\times X\times X$ into $[0,1)$ such that

$$\lambda:=\sup\left\{\frac{\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+5\delta(x,y,y)+8\sigma(x,y,y)}{1-(2\beta(x,y,y)+2\gamma(x,y,y)+4\delta(x,y,y)+4\sigma(x,y,y))}:x,y\in X\right\}<1.$$

Then T has a unique fixed point, say $u\in X$. Moreover, $T^n x\to u$ as $n\to\infty$ for all $x\in X$.

Proof. From Theorem 3.1, one can see that every map satisfies (3.11) would definitely, satisfy (3.1), and hence, the proof is straightforward from proof of Theorem 3.1. □

Remark 3.2. In condition (3.11), we remove the coefficient 2 of $G(y, Ty, Ty)$ which was in condition (3.1). Therefore, following the proof of Theorem 3.1, one would realize that λ in Corollary 3.3, could be replaced by

$$\tilde{\lambda} := \sup\left\{ \frac{\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+5\delta(x,y,y)+8\sigma(x,y,y)}{1-(\beta(x,y,y)+2\gamma(x,y,y)+4\delta(x,y,y)+4\sigma(x,y,y))} : x, y \in X \right\} < 1,$$

where we have β instead of 2β in the denominator, and the proof follows straightforwardly. Similarly, one could also remove the coefficient 2 of $G(y, y, Ty)$ in condition (3.1) and similar Corollary as Corollary 3.3, and similar arguments as those for Corollary 3.3 follow straightforwardly.

Corollary 3.4. *Let X be a complete G-cone metric space, and let $T : X \to X$ be a mapping satisfying for some $m \in \mathbb{N}$, for each $x, y \in X$,*

$$\begin{aligned} G(T^m x, T^m y, T^m y) \preceq\ & \alpha(x,y,y)G(x,y,y) \\ & +\beta(x,y,y)[G(x,T^m x,T^m x)+2G(y,T^m y,T^m y)] \\ & +\gamma(x,y,y)[G(x,T^m y,T^m y)+G(y,T^m y,T^m y)+G(y,T^m x,T^m x)] \\ & +\delta(x,y,y)[G(y,y,T^m x)+G(y,y,T^m y)+G(x,x,T^m y)] \\ & +\sigma(x,y,y)[G(x,x,T^m x)+2G(y,y,T^m y)], \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0, 1)$ such that

$$\lambda := \sup\left\{ \frac{\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+5\delta(x,y,y)+8\sigma(x,y,y)}{1-(2\beta(x,y,y)+2\gamma(x,y,y)+4\delta(x,y,y)+4\sigma(x,y,y))} : x, y \in X \right\} < 1.$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$.

Proof. From Theorem 3.1, T^m has a unique fixed point, say u, i.e., $T^m u = u$, and since $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$, we have Tu as another fixed point for T^m, and by uniqueness, $Tu = u$. The rest of the proof follows similarly. □

Theorem 3.5. *Let X be a complete G-cone metric space, and let $T : X \to X$ be a mapping satisfying, for each $x, y, z \in X$,*

$$\begin{aligned} G(Tx,Ty,Tz) \preceq\ & \alpha(x,y,z)G(x,y,z) \\ & +\beta(x,y,z)[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)] \\ & +\gamma(x,y,z)[G(x,Ty,Ty)+G(y,Tz,Tz)+G(z,Tx,Tx)] \\ & +\delta(x,y,z)[G(y,y,Tx)+G(z,z,Ty)+G(x,x,Tz)] \\ & +\sigma(x,y,z)[G(x,x,Tx)+G(y,y,Ty)+G(z,z,Tz)], \end{aligned} \tag{3.12}$$

where $\alpha, \beta, \gamma, \delta, \sigma$ are some functions from $X \times X \times X$ into $[0, 1)$ such that

$$\lambda := \sup\left\{ \frac{\alpha(x,y,z)+\beta(x,y,z)+\gamma(x,y,z)+5\delta(x,y,z)+8\sigma(x,y,z)}{1-(2\beta(x,y,z)+2\gamma(x,y,z)+4\delta(x,y,z)+4\sigma(x,y,z))} : x, y, z \in X \right\} < 1.$$

Then T has a unique fixed point, say $u \in X$. Moreover, $T^n x \to u$ as $n \to \infty$ for all $x \in X$.

Proof. If we Take $z = y$ in (3.12), then we get the condition (3.1) in Theorem 3.1, and therefore, the proof follows straightforwardly from the proof of Theorem 3.1. □

It should be noted that previous Corollaries following Theorem 3.1 also follow Theorem 3.5 straightforwardly.

Remark 3.3. If $\alpha, \beta, \gamma, \delta, \sigma$ are only constants in $[0, 1)$ instead of functions from $X \times X \times X$ into $[0, 1)$, then (3.2) in Theorem 3.1 will be replaced by

$$0 \leq \alpha + 3\beta + 3\gamma + 9\delta + 12\sigma < 1,$$

which clearly implies

$$\lambda := \frac{\alpha + \beta + \gamma + 5\delta + 8\sigma}{1 - (2\beta + 2\gamma + 4\delta + 4\sigma)} < 1,$$

and the proof follows straightforwardly. Similarly, for the other theorem and other corollaries.

We give in what follows an example to validate our results.

Example 3.1. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be a cone in E, and $X = [-1, 1] \subset \mathbb{R}$. Define the mapping $G : X \times X \times X \to E$, as given in Example 2.1, by

$$G(x, y, z) = \left(\frac{1}{3}(|x-y| + |y-z| + |x-z|), \frac{2}{3}(|x-y| + |y-z| + |x-z|)\right), \forall x, y, z \in X.$$

Let $T : X \to X$ be given by

$$T(x) = \begin{cases} -\frac{1}{6}xe^{-\frac{1}{|x|}}, & x \in [-1, 0) \cup (0, 1], \\ 0, & x = 0. \end{cases}$$

Then, for all $x, y, z \in X$, we have

$$G(Tx,Ty,Tz) = \left(\tfrac{1}{3}(|Tx - Ty| + |Ty - Tz| + |Tx - Tz|), \tfrac{2}{3}(|Tx - Ty| + |Ty - Tz| + |Tx - Tz|)\right).$$

Now, we consider each coordinate separately. So, for the first coordinate, we have

$$\begin{aligned} \frac{1}{3}\left(|Tx - Ty| + |Ty - Tz| + |Tx - Tz|\right) = & \\ & \tfrac{1}{3}\left(\left|-\tfrac{1}{6}xe^{-\frac{1}{|x|}} + \tfrac{1}{6}ye^{-\frac{1}{|y|}}\right| + \left|-\tfrac{1}{6}xe^{-\frac{1}{|x|}} + \tfrac{1}{6}ze^{-\frac{1}{|z|}}\right| \right. \\ & \left. + \left|-\tfrac{1}{6}ye^{-\frac{1}{|y|}} + \tfrac{1}{6}ze^{-\frac{1}{|z|}}\right|\right) \\ \leq & \quad \tfrac{1}{3}\left(\tfrac{1}{6}|x| + \tfrac{1}{6}|y| + \tfrac{1}{6}|x| + \tfrac{1}{6}|z| + \tfrac{1}{6}|y| + \tfrac{1}{6}|z|\right) \end{aligned}$$

$$\begin{aligned} \leq \quad & \frac{1}{3}\left(\frac{1}{6}\left|x + \frac{1}{6}xe^{-\frac{1}{|x|}}\right| + \frac{1}{6}\left|y + \frac{1}{6}ye^{-\frac{1}{|y|}}\right|\right. \\ & + \frac{1}{6}\left|x + \frac{1}{6}xe^{-\frac{1}{|x|}}\right| + \frac{1}{6}\left|z + \frac{1}{6}ze^{-\frac{1}{|z|}}\right| \\ & \left. + \frac{1}{6}\left|y + \frac{1}{6}ye^{-\frac{1}{|y|}}\right| + \frac{1}{6}\left|z + \frac{1}{6}ze^{-\frac{1}{|z|}}\right|\right) \\ = \quad & \frac{1}{3}\left(\frac{1}{6}|Tx - x| + \frac{1}{6}|Ty - y| + \frac{1}{6}|Tx - x| + \frac{1}{6}|Tz - z|\right. \\ & \left. + \frac{1}{6}|Ty - y| + \frac{1}{6}|Tz - z|\right) \\ = \quad & \frac{1}{3}\left(\frac{1}{6}(|Tx - x| + |Tx - x|) + \frac{1}{6}(|Ty - y| + |Ty - y|)\right. \\ & \left. + \frac{1}{6}(|Tz - z| + |Tz - z|)\right). \end{aligned}$$

Similar arguments hold for the other coordinate, and we get

$$\begin{aligned} \frac{2}{3}(|Tx - Ty| + |Ty - Tz| + |Tx - Tz|) \leq & \\ & \tfrac{2}{3}\left(\tfrac{1}{6}(|Tx - x| + |Tx - x|) + \tfrac{1}{6}(|Ty - y| + |Ty - y|)\right. \\ & \left. + \tfrac{1}{6}(|Tz - z| + |Tz - z|)\right). \end{aligned}$$

If we let
$$w = (|Tx - x| + |Tx - x|) + (|Ty - y| + |Ty - y|) + (|Tz - z| + |Tz - z|),$$
then, since
$$\frac{1}{6}\left(\frac{1}{3}w, \frac{2}{3}w\right) - G(Tx, Ty, Tz) \in P,$$
we have
$$G(Tx, Ty, Tz) \preceq \frac{1}{6}\left(\frac{1}{3}w, \frac{2}{3}w\right),$$
which implies
$$G(Tx, Ty, Tz) \preceq \frac{1}{6}(G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)).$$
If we take $\beta = \frac{1}{6} \in [0, 1)$, and $\alpha = \gamma = \delta = \sigma = 0$ in Theorem 3.5, then the contraction condition is satisfied with $\lambda = 3\beta < 1$, and T has a unique fixed point in $[-1, 1]$ which is clearly $u = 0$.

4 conclusion

New fixed point results in G-cone metric spaces for mappings satisfying certain contractions with variable coefficients have been proved. These results have been proved with no use of the normality property of cone and are generalizations of previous results in the literature.

Acknowledgment

This work was supported by the council of scientific research (Dean of Graduate Studies and Scientific Research-Yarmouk University-Jordan).

Competing Interests

The authors declare that no competing interests exist.

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