British Journal of Mathematics & Computer Science
7(4): 258-265, 2015, Article no.BJMCS.2015.121
ISSN: 2231-0851

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Sharp Triangle Inequalities in Quasi-Normed Spaces

Risto Malčeski[1*]

[1]*Faculty of Informatics, FON University, Skopje, Macedonia.*

Article Information

DOI: 10.9734/BJMCS/2015/15579
Editor(s):
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Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=935&id=6&aid=8167

Original Research Article

Received: 05 December 2014
Accepted: 07 February 2015
Published: 18 February 2015

Abstract

Triangle inequality is regarded as the most prominent and important inequality in the theory of normed spaces and therefore it has been a major topic of interest treated by numerous mathematicians. However, the above inequality, as well as its generalizations are also very important in the theory of quasi-normed spaces. Therefore, that one is a subject of additional research in both quasi-normed and generalized quasi-normed space.

Keywords: Quasi-norm; generalized quasi-norm; triangle inequality.

2010 Mathematics Subject Classification: 46B20, 26D15.

1 Introduction

Quasi-norm is generalization of a norm and is defined as following

Definition 1. ([1], [2]). Let L be a real vector space. A quasi-norm is a real function $\|\cdot\|: L \to \mathbf{R}$ such that the following conditions are satisfied:

i) $\|x\| \geq 0$, for each $x \in L$ and $\|x\| = 0$ if and only if $x = 0$,
ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for each $\lambda \in \mathbf{R}$ and each $x \in L$,

*Corresponding author: risto.malceski@gmail.com;

iii) It exists a constant $C \geq 1$ such that $\| x+y \| \leq C(\| x \| + \| y \|)$, for each $x, y \in L$.

An ordered pair $(L, \| \cdot \|)$ is said to be a quasi-normed space. The smallest possible C in *iii*) is said to be a *modulus of concavity* of $\| \cdot \|$. A *quasi-Banach space* is a complete quasi-normed space.

The condition *iii*) of Definition 1 directly implies the validity of the following Lemma:

Lemma 1. ([3]). If L is a quasi-Banach space with modulus of concavity $C \geq 1$, then for each n>1 and for every $x_1, x_2, ..., x_{2n}, x_{2n+1} \in L$ the following holds true

$$\| \sum_{i=1}^{2n} x_i \| \leq C^n \sum_{i=1}^{2n} \| x_i \|, \quad \| \sum_{i=1}^{2n+1} x_i \| \leq C^{n+1} \sum_{i=1}^{2n+1} \| x_i \|. \ \blacksquare$$

Definition 2 ([1], [2]). Quasi-norm $\| \cdot \|$ is said to be $p-$ *norm*, $0 < p \leq 1$ if

$$\| x+y \|^p \leq \| x \|^p + \| y \|^p, \tag{1}$$

for each $x, y \in L$. Furthermore, a quasi-normed space is said to be a $p-$ *normed space* and a quasi-Banach space is said to be a $p-$ *Banach space*.

In quasi-normed spaces for quasi-norms and $p-$ norms the following theorem holds true.

Theorem 1. (Aoki-Rolewitz, [1], [2]). Let $(L, \| \cdot \|)$ be a quasi-normed space. Then, it exist $p, 0 < p \leq 1$ and an equivalent quasi-norm $||| \cdot |||$ with L, so that it is a $p-$ norm. ■

The above theorem enables the *p*-norms to be treated instead of quasi-norms, because in the majority of cases it has been proved as a simpler and easier way the researchers to obtain the results.

In [4] C. Park generalized the concept of quasi-normed space, i.e. gave the following definition.

Definition 3. ([4]). Let L be a real vector space. *Generalized quasi-norm* is a real function $\| \cdot \| : L \to \mathbf{R}$ such that it satisfies the following conditions:

i) $\| x \| \geq 0$, for each $x \in L$ and $\| x \| = 0$ if and only if $x = 0$,
ii) $\| \lambda x \| = | \lambda | \cdot \| x \|$, for each $\lambda \in \mathbf{R}$ and for each $x \in L$,
iii) It exists a constant $C \geq 1$ such that it satisfies the following $\| \sum_{i=1}^{\infty} x_i \| \leq \sum_{i=1}^{\infty} C \| x_i \|$, for each $x_1, x_2, x_3, ... \in L$.

An ordered pair $(L, \| \cdot \|)$ is said to be a *generalized quasi-normed space*. The smallest possible C of the condition *iii*) is said to be a *modulus of concavity* of $\| \cdot \|$. A complete generalized quasi-normed space is said to be a *generalized quasi-Banach space*.

Definition 4. ([4]). A generalized quasi-norm $\| \cdot \|$ is said to be a $p-$ *norm,* $0 < p \leq 1$ if each $x, y \in L$ satisfy the condition (1). Thus, a generalized quasi-normed space is said to be a *generalized* $p-$ *normed space,* and a generalized quasi-Banach space is said to be a *generalized* $p-$ *Banach space.*

2 Main Results

Theorem 2. Let L be a quasi-normed space with modulus of concavity $C \geq 1$. For all $x, y \in L$

$$||x+y||+||x-y||-\min\{||x-y||,||x+y||\} \leq C(||x||+||y||), \tag{2}$$

$$||x||+||y||+|\,||x||-||y||\,| \leq C(||x+y||+||x-y||). \tag{3}$$

hold true.

Proof. The condition *iii*) in Definition 1 implies that

$$\begin{aligned} &||x+y||+||x-y||-C(||x||+||y||) \leq ||x-y||, \\ &||x+y||+||x-y||-C(||x||+||-y||) \leq ||x+y||. \end{aligned}$$

Therefore the following inequality holds true

$$||x+y||+||x-y||-C(||x||+||y||) \leq \min\{||x-y||,||x+y||\}$$

The last one is equivalent to the inequality (2). Further, the condition *iii*) in Definition 1 implies that

$$\begin{aligned} &2||x|| = ||x+y+(x-y)|| \leq C(||x+y||+||x-y||), \\ &2||y|| = ||x+y-(x-y)|| \leq C(||x+y||+||x-y||), \end{aligned}$$

thus,

$$2\max\{||x||,||y||\} \leq C(||x+y||+||x-y||). \tag{4}$$

On the other side

$$||x||+||y||+|\,||x||-||y||\,| = 2\max\{||x||,||y||\}. \tag{5}$$

Finally, the equality (5) and the inequality (4) imply inequality in (3). ■

Theorem 3. Let L be a $p-$ normed space. Then for all $x_1, x_2, ..., x_n \in L$ such that $||x_1|| \geq ||x_2|| \geq ... \geq ||x_n|| > 0$ the following inequalities holds true

$$||x_1||^p \Big|\Big| \sum_{i=1}^{n} \frac{x_i}{||x_i||} \Big|\Big|^p - \sum_{i=1}^{n} (||x_1||-||x_i||)^p \leq \Big|\Big| \sum_{i=1}^{n} x_i \Big|\Big|^p, \tag{6}$$

$$\Big|\Big| \sum_{i=1}^{n} x_i \Big|\Big|^p \leq ||x_n||^p \Big|\Big| \sum_{i=1}^{n} \frac{x_i}{||x_i||} \Big|\Big|^p + \sum_{i=1}^{n} (||x_i||-||x_n||)^p, \tag{7}$$

for $0 < p \leq 1$.

Proof. The inequality (1) directly implies that

$$\begin{aligned}\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|^p &= \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|}+(\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}-\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|})\|^p\\ &\le \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|}\|^p+\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}-\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|}\|^p\\ &= \tfrac{1}{\|x_1\|^p}\|\sum_{i=1}^{n}x_i\|^p+\tfrac{1}{\|x_1\|^p}\|\sum_{i=1}^{n}\tfrac{\|x_1\|-\|x_i\|}{\|x_i\|}x_i\|^p\\ &\le \tfrac{1}{\|x_1\|^p}\|\sum_{i=1}^{n}x_i\|^p+\tfrac{1}{\|x_1\|^p}\sum_{i=1}^{n}(\|x_1\|-\|x_i\|)^p\end{aligned}$$

i.e. the inequality (6) holds true.

$$\begin{aligned}\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_n\|}\|^p &= \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}+(\sum_{i=1}^{n}\tfrac{x_i}{\|x_n\|}-\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|})\|^p\\ &\le \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|^p+\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_n\|}-\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|^p\\ &= \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|^p+\tfrac{1}{\|x_n\|^p}\|\sum_{i=1}^{n}\tfrac{\|x_i\|-\|x_n\|}{\|x_i\|}x_i\|^p\\ &\le \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|^p+\tfrac{1}{\|x_n\|^p}\sum_{i=1}^{n}(\|x_i\|-\|x_n\|)^p\end{aligned}$$

i.e. the inequality (7) holds true. ■

Theorem 4. Let L be a generalized quasi-normed space with modulus of concavity $C\ge 1$. Then for every $x_1,x_2,...,x_n\in L$ such that $\|x_1\|\ge\|x_2\|\ge...\ge\|x_n\|>0$ the following inequalities holds true:

$$C^2\sum_{i=1}^{n}\|x_i\|\le C\|\sum_{i=1}^{n}x_i\|+(C^2 n-\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|)\|x_1\|, \tag{8}$$

$$\|\sum_{i=1}^{n}x_i\|+(C^2 n-C\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\|)\|x_n\|\le C^2\sum_{i=1}^{n}\|x_i\|. \tag{9}$$

Proof. The properties of generalized quasi-norms imply the following

$$\begin{aligned}\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}\| &= \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|}+(\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}-\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|})\|\\ &\le C\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|}\|+C\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i\|}-\sum_{i=1}^{n}\tfrac{x_i}{\|x_1\|}\|\\ &= \tfrac{C}{\|x_1\|}\|\sum_{i=1}^{n}x_i\|+\tfrac{C}{\|x_1\|}\|\sum_{i=1}^{n}\tfrac{\|x_1\|-\|x_i\|}{\|x_i\|}x_i\|\\ &\le \tfrac{C}{\|x_1\|}\|\sum_{i=1}^{n}x_i\|+\tfrac{C^2}{\|x_1\|}\sum_{i=1}^{n}(\|x_1\|-\|x_i\|)\\ &= \tfrac{C}{\|x_1\|}\|\sum_{i=1}^{n}x_i\|+\tfrac{C^2}{\|x_1\|}(n\|x_1\|-\sum_{i=1}^{n}\|x_i\|),\end{aligned}$$

i.e. the inequality (8) holds true.

$$\begin{aligned}\|\sum_{i=1}^{n}\frac{x_i}{\|x_n\|}\| &= \|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}+(\sum_{i=1}^{n}\frac{x_i}{\|x_n\|}-\sum_{i=1}^{n}\frac{x_i}{\|x_i\|})\| \\ &\le C\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\|+C\|\sum_{i=1}^{n}\frac{x_i}{\|x_n\|}-\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\| \\ &= C\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\|+\frac{C}{\|x_n\|}\|\sum_{i=1}^{n}\frac{\|x_i\|-\|x_n\|}{\|x_i\|}x_i\| \\ &\le C\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\|+\frac{C^2}{\|x_n\|}\sum_{i=1}^{n}(\|x_i\|-\|x_n\|) \\ &= C\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\|+\frac{C^2}{\|x_n\|}(\sum_{i=1}^{n}\|x_i\|-n\|x_n\|),\end{aligned}$$

i.e. the inequality (9) holds true ■

Before we treat the inequalities in a quasi-normed space, which are analogies of the inequalities proven in theorems 3 and 4, we will prove that the sharp inequalities given in Lemma 1, for qausi-norm, hold true.

Lemma 2. If L is a quasi-Banach space with modulus of concavity $C\ge 1$, then for each n>1 and for each $x_1, x_2,..., x_n \in L$ the following inequality

$$\|\sum_{i=1}^{n}x_i\|\le C^{1+[\log_2(n-1)]}\sum_{i=1}^{n}\|x_i\| \tag{10}$$

is satisfied.

Proof. The above statement will be proved by the principle of mathematical induction. The properties of a quasi-norm imply the following:

$$\begin{aligned}\|x_1+x_2\| &\le C(\|x_1\|+\|x_2\|) \\ &= C^{1+[\log_2(2-1)]}(\|x_1\|+\|x_2\|)\end{aligned},$$

$$\begin{aligned}\|x_1+x_2+x_3\| &\le C(\|x_1\|+\|x_2+x_3\|) \\ &\le C\|x_1\|+C^2(\|x_2\|+\|x_3\|) \\ &\le C^2(\|x_1\|+\|x_2\|+\|x_3\|) \\ &= C^{1+[\log_2(3-1)]}(\|x_1\|+\|x_2\|+\|x_3\|)\end{aligned}$$

$$\begin{aligned}\|x_1+x_2+x_3+x_4\| &\le C(\|x_1+x_2\|+\|x_3+x_4\|) \\ &\le C^2(\|x_1\|+\|x_2\|+\|x_3\|+\|x_4\|) \\ &= C^{1+[\log_2(4-1)]}(\|x_1\|+\|x_2\|+\|x_3\|+\|x_4\|)\end{aligned}$$

The above actually means that the inequality (10) holds true for $m = 2,3,4$. Let's assume that (10) holds true for each positive integer such that it is an element of the set $\{2,3,...,2^k, 2^k+1,..., 2^{k+1}\}$. Let

$m \in \{2^{k+1}+1, 2^{k+1}+2, ..., 2^{k+2}\}$. Then it exists $p, q \in \{2^k, 2^k+1, 2^k+2, ..., 2^{k+1}\}$ such that $p+q=m$, thus

$$\begin{aligned} \| x_1 + x_2 + ... + x_{m-1} + x_m \| &\le C \| x_1 + x_2 + ... + x_p \| + C \| x_{p+1} + x_{k+2} + ... + x_m \| \\ &\le C \cdot C^{1+[\log_2(p-1)]}(\| x_1 \| + \| x_2 \| + ... + \| x_p \|) \\ &\quad + C \cdot C^{1+[\log_2(q-1)]}(\| x_{p+1} \| + \| x_{p+2} \| + ... + \| x_m \|) \\ &\le C \cdot C^{[\log_2(n-1)]}(\| x_1 \| + \| x_2 \| + ... + \| x_p \|) \\ &\quad + C \cdot C^{[\log_2(n-1)]}(\| x_{p+1} \| + \| x_{p+2} \| + ... + \| x_m \|) \\ &= C^{1+[\log_2(n-1)]}(\| x_1 \| + \| x_2 \| + ... + \| x_{m-1} \| + \| x_m \|), \end{aligned}$$

Hence, by applying the principle of mathematical induction we get that the inequality (10) holds true for each n>1 and for all $x_1, x_2, ..., x_n \in L$. ■

Theorem 5. Let L be a quasi-normed space and $n > 2$. For all elements $x_1, x_2, ..., x_n \in L$ so that $\| x_1 \| \ge \| x_2 \| \ge ... \ge \| x_n \| > 0$ the following inequalities

$$C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i \| \le C \| \sum_{i=1}^{n} x_i \| + [nC^{2+[\log_2(n-2)]} - \| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} \|] \cdot \| x_1 \|, \tag{11}$$

$$\| \sum_{i=1}^{n} x_i \| + (nC^{2+[\log_2(n-2)]} - C \| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} \|) \| x_n \| \le C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i \|, \tag{12}$$

hold true.

Proof. Let $n > 3$. The properties of quasi-norm and Lemma 2 imply the following

$$\begin{aligned} \| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} \| &= \| \sum_{i=1}^{n} \frac{x_i}{\|x_1\|} + (\sum_{i=1}^{n} \frac{x_i}{\|x_i\|} - \sum_{i=1}^{n} \frac{x_i}{\|x_1\|}) \| \\ &\le C \| \sum_{i=1}^{n} \frac{x_i}{\|x_1\|} \| + C \| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} - \sum_{i=1}^{n} \frac{x_i}{\|x_1\|} \| \\ &= \frac{C}{\|x_1\|} \| \sum_{i=1}^{n} x_i \| + \frac{C}{\|x_1\|} \| \sum_{i=2}^{n} \frac{\|x_1\| - \|x_i\|}{\|x_i\|} x_i \| \\ &\le \frac{C}{\|x_1\|} \| \sum_{i=1}^{n} x_i \| + \frac{C \cdot C^{1+[\log_2(n-2)]}}{\|x_1\|} \sum_{i=2}^{n} (\| x_1 \| - \| x_i \|) \\ &= \frac{C}{\|x_1\|} \| \sum_{i=1}^{n} x_i \| + \frac{C^{2+[\log_2(n-2)]}}{\|x_1\|} [n \| x_1 \| - \sum_{i=1}^{n} \| x_i \|], \end{aligned}$$

i.e. the following inequality, which is equivalent to the inequality (11), holds true

$$\| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} \| \le \frac{C}{\|x_1\|} \| \sum_{i=1}^{n} x_i \| + \frac{C^{2+[\log_2(n-2)]}}{\|x_1\|} [n \| x_1 \| - \sum_{i=1}^{n} \| x_i \|],$$

Let $n > 3$. The properties of quasi-norm and Lemma 2 imply

$$\begin{aligned}||\sum_{i=1}^{n}\frac{x_i}{||x_n||}|| &= ||\sum_{i=1}^{n}\frac{x_i}{||x_i||}+(\sum_{i=1}^{n}\frac{x_i}{||x_n||}-\sum_{i=1}^{n}\frac{x_i}{||x_i||})|| \\ &\leq C||\sum_{i=1}^{n}\frac{x_i}{||x_i||}||+C||\sum_{i=1}^{n}\frac{x_i}{||x_n||}-\sum_{i=1}^{n}\frac{x_i}{||x_i||}|| \\ &= C||\sum_{i=1}^{n}\frac{x_i}{||x_i||}||+\frac{C}{||x_n||}||\sum_{i=1}^{n-1}\frac{||x_i||-||x_n||}{||x_i||}x_i|| \\ &\leq C||\sum_{i=1}^{n}\frac{x_i}{||x_i||}||+\frac{C\cdot C^{1+[\log_2(n-2)]}}{||x_n||}\sum_{i=1}^{n-1}(||x_i||-||x_n||) \\ &= C||\sum_{i=1}^{n}\frac{x_i}{||x_i||}||+\frac{C^{2+[\log_2(n-2)]}}{||x_n||}(\sum_{i=1}^{n}||x_i||-n||x_n||),\end{aligned}$$

i.e. the following inequality, which is equivalent to inequality (12), holds true

$$||\sum_{i=1}^{n}\frac{x_i}{||x_n||}||\leq C||\sum_{i=1}^{n}\frac{x_i}{||x_i||}||+\frac{C^{2+[\log_2(n-2)]}}{||x_n||}(\sum_{i=1}^{n}||x_i||-n||x_n||). \blacksquare$$

Remark. a) Indeed, if $C=1$, then a quasi-norm is a norm, and the inequalities (11) and (12) are equivalent to the inequality (1.1) in Theorem 1.1 [5].

b) For $n=2$ the corresponding inequalities in Theorem 5 are proven in Theorem 2.1, [6], i.e. the following statement is proven:

If L is a quasi-norm with constant of concavity $C\geq 1$ and $x,y\in L$ are so that $||x||\geq||y||>0$, then

$$||x+y||+C(2-||\tfrac{x}{||x||}+\tfrac{y}{||y||})||y||\leq C(||x||+||y||)\leq||x+y||+(2C^2-||\tfrac{x}{||x||}+\tfrac{y}{||y||})||x||.$$

3 Conclusion

In previous considerations are proven the Sharp triangle inequalities in quasi-normed spaces, which are generalizations of results given in [7-11]. In proving of these inequalities we use the strict inequality given in Lemma 2, which can be used in proving of inequalities such as Pečarić-Rajić.

Competing Interests

Author has declared that no competing interests exist.

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