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Smooth Groups with Totally Contranormal Subgroups

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Abstract

A contranormal subgroup is a subgroup whose normal closure in the group is the whole group. Hence a subgroup H of a group G is said to be totally contranormal if $H^K = K$ for each subgroup K of G. The purpose of this article is to study the influence of contranormal (totally contranormal) subgroup on groups whose maximal subgroups are smooth groups.

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1 Introduction

All groups considered in this article will be finite. We use conventional notions and notations as in R. Schmidt [1]. In addition, n will denote the maximal length of the subgroup lattice $L(G)$ of a group G, and the set of distinct primes dividing $|G|$ will be denoted by $\pi(G)$.

A maximal chain $1 = G_0 < G_1 < G_2 < ... < G_n = G$ of subgroups of a group G is smooth if the interval $[G_{i+k}/G_i] \cong [G_k/1]$ for all $i, k \in N$ such that $i + k \leq n$. The group G is smooth if it has

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a smooth chain. Finite smooth groups have been studied by Schmidt (see [2], [3], and [4]). A group G is totally smooth if all maximal chains of subgroups of G are smooth. Totally smooth groups have been studied by Elkholy [5].

Recall that a P-group is a group lattice - isomorphic to an elementary abelian group (see [1], p. 49).

A subgroup H of a group G is said to be contranormal in G if $H^G = G$ (see [6]). It is widely known that a contranormal subgroup can be normal only if it is the whole group. But it is not true that if H is contranormal in G, then it is contranormal in every subgroup of G containing H. For example: in Dihedral group D of order 12, every non-normal subgroup H of order 2 is contranormal in D since there is no normal subgroup in D containing H. Hence we give the following:

Definition 1.1. Let G be a finite group. A subgroup H is said to be totally contranormal of G if it is contranormal in every subgroup of G containing it.

The main purpose of this article is to study the influence of a contranormal (totally contranormal) subgroup on groups whose maximal subgroups are smooth groups.

2 Preliminaries

The following lemmas will be used in the sequel.

Lemma 2.1. A group G is totally smooth if and only if one of the following holds:

(i) G is cyclic of prime power order.

(ii) G is a P- group.

(iii) G is cyclic of square free order (see [5]; Theorem 1).

Lemma 2.2. Let p and q be different primes dividing $|G|$ such that $G = PQ$ where P is an elementary abelian normal subgroup of G of order p^n $(n \in N)$ and $Q =< x >$ is a cyclic q-group. Then the following properties are equivalent:

(i) Every subgroup of Q is either irreducible on P or normalizes every subgroup of P.

(ii) One of the following holds:

(a) $G = P \times Q$ or x induces a power automorphism in P.

(b) $q \mid p - 1$, $|P| = p^q$, and x induces an automorphism of order q^{k+1} in P where k is the largest integer such that $q^k \mid p - 1$.

(c) $n \geq 2$, $q^m \mid p^n - 1$, q does not divide $p^r - 1$ where $(1 \leq r < n)$, and x induces an automorphism of order q^m in P $(m \in N)$ (see [7]; Lemma 3.1).

3 Main Results

We begin with the following result:

Theorem 3.1. Suppose that p and q are distinct primes in $\pi(G)$ such that $|G| = p^\alpha q^\beta$; where α and β are non-zero positive integers and let $n \geq 3$. If the maximal subgroups of G are totally smooth and G has a contranormal subgroup of prime order, then one of the following holds:

(i) G is a nonabelian P -group.

(ii) $G/P \cong Q$ is a Sylow q-subgroup of G of order q and P is an elementary abelian minimal normal Sylow p-subgroup of G.

(iii) $G = PQ$, where P is an elementary abelian normal Sylow $p-$subgroup of order p^q, Q is cyclic of order q^2 which operates irreducibly on P and $q|p-1$.

(iv) $n = 3$ and $|G| = p^2q$, where p and q are distinct primes in $\pi(G)$.

Proof. Suppose that P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G. Since G is solvable, G has a minimal normal subgroup N, say, which is elementary abelian. We handel the cases:

Case 1. N is a p-group and p is the largest prime in $\pi(G)$.

Hence $N \leq P$. If H is a contranormal subgroup of G, we have the two subcases:

Subcase a. $H \leq P$.

If $N = P$, P is an elementary abelian minimal normal subgroup of G. If $|P| = p$, it follows that $H = P$ is normal in G and since H is contranormal in G, it follows that $|G| = p$ which contradicts our choice of G. Thus $H < P$.

If $|Q| = q$, Q operates irreducibly on P and (ii) holds. So assume that $|Q| > q$. By hypothesis, Q is totally smooth and by Lemma 2.1, it is elementary abelian or cyclic. If $Q \lhd G$, $HQ < G$ which is totally smooth by hypothesis. Since $p > q$, we get by Lemma 2.1 that HQ is cyclic of square free order. Hence $|Q| = q$, a contradiction. Thus Q is not normal in G.

If Q^* is a maximal subgroup of Q, it follows by hypothesis that PQ^* is totally smooth. Since $|P| > p$, Lemma 2.1 shows that PQ^* would be nonabelian P-group and $|Q^*| = q$. Hence $H \lhd PQ^*$. If Q is elementary abelian, $H \lhd G$ since Q^* is any maximal subgroup of Q. Therefore $G = H$; a contradiction. Thus Q is cyclic of order q^2 operates irreducibly on P and $Q^* = \Phi(Q)$ normalizes every subgroup of P. Since $q|p-1$, it follows by Lemma 2.2 that $|P| = p^q$ and (iii) holds.

Now assume that $N < P$. Hence $NQ < G$ and by hypothesis, it is totally smooth which implies by Lemma 2.1 that $|Q| = q$. It follows that $P \lhd G$. Since H is a contranormal subgroup of G of order p, P would be elementary abelian by Lemma 2.1. We claim that $|N| = p$.

Suppose for a contradiction that $|N| > p$. It is clear by Lemma 2.1 that NQ would be nonabelian P-group since NQ is totally smooth and $|N| > p$. If L is a p-subgroup of N of order p, $L \lhd NQ$. As P is elementary abelian, we get $L \lhd G$, a contradiction. Thus $|N| = p$.

If $n = 3$, then $|G| = p^2q$ and (iv) holds. So assume that $n \geq 4$. Hence $|P| \geq p^3$. By Maschke's Theorem, P is completely reducible under Q and so N has a complement V, say, in P which is normal in G. Hence VQ is a proper subgroup of G. Once again, by hypothesis and Lemma 2.1, VQ is a nonabelian P -group or cyclic. Since $n \geq 4$, $|V| > p$ and hence VQ would be nonabelian P-group. Therefore, all subgroups of V are normal in G.

Let K be a subgroup of order p with $K \neq N$. Clearly, $U = NK < P$ as $|P| \geq p^3$. By Dedekind's rule, $U = NV \cap U$. Since N and $V \cap U$ are Q-invariant, we get UQ is a totally smooth subgroup of G and so UQ is a nonabelian P-group of order p^2q. Then $K \lhd UQ$ and hence $K \lhd UQ$. Therefore, every subgroup of P is normal in G and hence $H \lhd G$; a contradiction.

Subcase b. $H \leq Q$.

Assume first that $H = Q$. Then $|Q| = q$ and hence $P \lhd G$. If $N = P$ is a minimal normal subgroup of G, Then Q is of order q which operates irreducibly on P and (ii) holds and we are done. Thus $N < P$.

If $n = 3$, $|G| = p^2q$ and we are done. So assume that $n \geq 4$. By hypothesis, P is elementary abelian or cyclic. If P is cyclic and since N is a minimal normal subgroup of G, we get $|N| = p$ and every subgroup of P is normal in G. If $P_1 < P$ such that $|P_1| > p$, it follows that the chain $[P_1Q/1]$ is not smooth which contradicts our assumption. Thus P is elementary abelian. Once again, by Maschkes Theorem, P is completely reducible under Q and N has a complement K. Similarly, we can prove that every subgroup of P is normal in G and Q does not centralize any subgroup of P. Then G is a nonabelian P-group and (i) holds. Therefore, $H < Q$.

If $|P| = p$ and since $n \geq 3$, we have $|Q| \geq q^2$. Then PQ^* is a maximal subgroup of G, where Q^* is a maximal subgroup of Q. By hypothesis and Lemma 2.1, PQ^* is a nonabelian P-group or cyclic of square free order which implies that $|Q^*| = q$ since p is the largest prime. Then $n = 3$ and $|G| = pq^2$ and (iv) holds. So $|P| > p$.

If $N < P$, NQ is a proper subgroup of G and hence it is totally smooth. Then by Lemma 2.1, $|Q| = q$, a contradiction.

Case 2. N is a q-group and q is the smallest prime in $\pi(G)$. Hence $N \leq Q$.

Suppose first that $N = Q$. Then Q is a minimal normal subgroup of G and so Q is elementary abelian.

If $H < P$, $HQ < G$. By hypothesis, HQ is totally smooth. Hence Lemma 2.1 shows that HQ must be cyclic since $p > q$ and $Q \lhd G$. Hence $H \lhd HQ$. Since $H \lhd P$, $H \lhd G$ which implies that $H = G$ which contradicts our hypothesis. Thus $H = P$ is contranormal in G. Therefore $G/Q \cong P$ and (ii) holds. So $|P| > p$ and H is a subgroup of Q. Since $Q \lhd G$, H would be a proper subgroup of Q. Let P_1 be a maximal subgroup of P. Since $P_1Q < G$ and $p > q$, it follows by Lemma 2.1 that P_1Q is cyclic of square free order. Then $H = Q$, a contradiction.

Now suppose that N is a proper subgroup of Q. If Q is cyclic, $P \lhd G$. Since $NP < G$, we have by Lemma 2.1 that NP would be cyclic of square free order which implies that $|P| = p$. Let Q_1 be a maximal subgroup of Q. Since $PQ_1 < G$ and $p > q$, we have by Lemma 2.1 that PQ_1 is a nonabelian P-group or cyclic of square free order which implies that $|Q_1| = q$ and hence $|Q| = q^2$. Then $|G| = pq^2$ and we are done. So assume that Q is elementary abelian. By hypothesis, G/N is totally smooth which by applying Lemma 2.1, G/N is a nonabelian P-group or cyclic of square free order. This implies that $PN/N \lhd G/N$ and hence $P \lhd G$. Once again, $|G| = pq^2$ and we are done. This completes the proof.

we are now ready to prove:

Theorem 3.2. Suppose that p and q are distinct primes in $\pi(G)$ such that $|G| = p^\alpha q^\beta$; where α and β are non-zero positive integers and let $n \geq 3$. If the maximal subgroups of G are totally smooth and G has a totally contranormal subgroup of prime order, then one of the following holds:

(i) G is a nonabelian P-group of order $p^{n-1}q$.

(ii) $G = PQ$, where P is an elementary abelian minimal normal Sylow p-subgroup and Q is a totally contranormal Sylow q-subgroup of order $q \neq p$.

(iii) $n = 3$ and $|G| = p^2q$, where p and q are distinct primes in $\pi(G)$.

Proof. Suppose that P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G. Since G is solvable, G has a minimal normal subgroup N, say, which is elementary abelian. Let H be a totally contranormal subgroup of G.

Assume first that N is a p-group and p is the largest prime in $\pi(G)$. Hence $N \leq P$. If $N = P$, P is an elementary abelian minimal normal subgroup of G. If $n = 3$, $G = PQ$ where P is a normal Sylow p-subgroup of G of order p^2 and Q is a totally contranormal Sylow q-subgroup of G. Then (iii) holds and we are done. So let $n \geq 4$.

Assume that $H \leq P$. Since P is totally smooth, Lemma 2.1 shows that $H \lhd P$. As H is totally contranormal in G, it follows that $H = P$ is a minimal normal subgroup of G. Then $H = G$ and G would be of order p, a contradiction. Thus $H = Q$ as H is totally contranormal in G. Since P is a minimal normal subgroup of G, Q operates irreducibly on P and (ii) holds and we are done.

Now assume that $N < P$. Hence NQ is a totally smooth proper subgroup of G which implies that $|Q| = q$. Then $P \lhd G$ and since $n \geq 4$, $|P| > p^2$. If P is cyclic and since every maximal subgroup of G is totally smooth, we have $|P| = p^2$, a contradiction. This implies that P is elementary abelian and since H is totally contranormal in G, it follows that $H = Q$. By applying Maschkes Theorem, P is completely reducible under Q and N has a complement M and so we can prove that every subgroup of P is normal in G and Q does not centralize every subgroup of P. Then G is a nonabelian P-group and (i) holds.

To complete the proof, we have to consider that $N \leq Q$.

If $N < Q$ and since H is totally contranormal in G, it follows that $H = P$ and PN is a totally smooth proper subgroup of G and hence $P \lhd PN$, a contradiction as P is totally contranormal in G. Thus $N = Q$ is a minimal normal Sylow q-subgroup of G and $H \leq P$.

If $|P| > p$, $P_1Q < G$ where P_1 is a maximal subgroup of P containing H. Then by Lemma 2.1, P_1Q is cyclic of square free order. Hence $H = P_1 \lhd P_1Q$, a contradiction. Thus $N = Q$ is a minimal normal Sylow q-subgroup of G and (ii) holds.

Now we are consider the case when a group G is divisible by at least three different primes.

Theorem 3.3. Suppose that $n \geq 3$ and let $p_1, p_2, ..., p_r$ be different primes in $\pi(G)$ $(r \geq 3)$. Suppose further that all maximal subgroups of a group G are totally smooth. If G has a contranormal subgroup of prime order, then $n = 3$, and $|G| = p_1p_2p_3$.

Proof. Suppose that M is a maximal subgroup of G.By hypothesis, M is totally smooth. It follows by Lemma 2.1 that M is a nonabelian P-group or cyclic of square free order since $|\pi(G)| \geq 3$. In particular, M is supersolvable. Therefore, all maximal subgroups of G are supersolvable. Then by Huppert's Theorem, G is solvable. Let H be a contranormal subgroup of G of prime order such that $H \leq M$.

If M would be cyclic, we get $H \lhd M$ and hence H is normal in every maximal subgroup of G containing it. Then $H \lhd G$. Since H is contranormal in G, $H = G$ is of order p; a contradiction. Thus M is a nonabelian P -group for each maximal subgroup M of G. Then $|\pi(G)| = 3$.

Let $p_i \in \pi(G)$; $i = 1, 2, 3$. Suppose, for a contradiction, that $p_1^e||G|$ $(e \geq 2)$ where $p_1 > p_j$; $j = 2, 3$. Since G is solvable, G has a Sylow basis $\{P_1, P_2, P_3\}$. Consequently P_1P_j is a totally smooth proper subgroup of G. Since $p_1 > p_j$, we get by Lemma 2.1 that P_1P_j is a nonabelian P-group. Then $|P_j| = p_j$ $(j = 2, 3)$ and P_1 has a proper normal subgroup L of G of order p_1. Hence LP_2P_3 is a proper subgroup of G. Since P_j does not centralize L, it follows that $[LP_2P_3/1]$ is not smooth which contradicts our hypothesis. Thus $|P_i| = p_i$ for $i = 1, 2, 3$. Let p_3 be the smallest prime in $\pi(G)$. Then $n = 3$ and $|G| = p_1p_2p_3$ and G has a totally contranormal subgroup of order p_3.

Finally, we prove the following result:

Theorem 3.4. Suppose that $n \geq 3$ and let $p_1, p_2, ..., p_r$ be different primes in $\pi(G)$ $(r \geq 3)$. Suppose further that all maximal subgroups of a group G are totally smooth. If G has a totally contranormal subgroup of prime order, then $|\ G\ |= p_1p_2p_3$ and the Sylow p_3-subgroup is a totally contranormal subgroup of G, where p_3 is the smallest prime in $\pi(G)$.

Proof. Since every maximal subgroup of G is totally smooth, it follows that all maximal subgroups of G are supersolvabe and by Hupper's Theorem, the solvability of G holds.

Let H be a totally contranormal subgroup of G of order q. Then $|\pi(G)| = 3$, every subgroup containing H is a nonabelian P-group and H is a Sylow q-subgroup of G where q is the smallest prime in $\pi(G)$.

If $p^2 \mid |G|$ for some prime $p \in \pi(G)$, G has a normal subgroup L of order p. Then there exists a proper subgroup K of G containing LH such that $|\pi(K)| = 3$. By hypothesis and Lemma 2.1, K would be cyclic of square free order, a contradiction since H is totally contranormal in G. Thus G is of square free order and this completes the proof.

4 Conclusion

Let G be a finite group. A contranormal subgroup is a subgroup whose normal closure in G is the whole group G. Hence a subgroup H of G is said to be totally contranormal if $H^K = K$ for each subgroup K of G. We have studied the structure of G which has a contranormal (totally contranormal) subgroup of prime order under the assumption that its maximal subgroups are totally smooth.

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Competing Interests

The author declares that no competing interests exist.

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