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Generalized Olivier Numbers

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper, we introduce and investigate the generalized Olivier sequences and we deal with, in detail, two special cases, namely, Olivier and Olivier-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. We also provide various matrices and identities associated with these sequences. Furthermore, we show that there are close relations between Olivier, Olivier-Lucas and adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers. Moreover, we give some identities and matrices related with these sequences.

Keywords: Olivier numbers, Olivier-Lucas numbers; Pell-Padovan numbers; Pell-Perrin numbers; third order Fibonacci-Pell numbers; third order Lucas-Pell numbers numbers; Fibonacci numbers; Lucas numbers.

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1 Introduction

Adjusted Pell-Padovan sequence $\{M_n\}_{n\geq 0}$, third order Lucas-Pell sequence $\{B_n\}_{n\geq 0}$ (OEIS: A099925, [1]), third order Fibonacci-Pell sequence $\{G_n\}_{n\geq 0}$ (OEIS: A008346, [1]), Pell-Perrin sequence $\{C_n\}_{n\geq 0}$, Pell-Padovan sequence $\{R_n\}_{n\geq 0}$ (OEIS: A066983, [1]), are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}
M_{n+3} &= 2M_{n+1}+M_n, \quad M_0=0, M_1=1, M_2=0, &(1.1)\\
B_{n+3} &= 2B_{n+1}+B_n, \quad B_0=3, B_1=0, B_2=4 &(1.2)\\
G_{n+3} &= 2G_{n+1}+G_n, \quad G_0=1, G_1=0, G_2=2, &(1.3)\\
C_{n+3} &= 2C_{n+1}+C_n, \quad C_0=3, C_1=0, C_2=2, &(1.4)\\
R_{n+3} &= 2R_{n+1}+R_n, \quad R_0=1, R_1=1, R_2=1. &(1.5)
\end{aligned}$$

The sequences $\{M_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, $\{G_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
M_{-n} &= -2M_{-(n-1)}+M_{-(n-3)},\\
B_{-n} &= -2B_{-(n-1)}+B_{-(n-3)},\\
G_{-n} &= -2G_{-(n-1)}+G_{-(n-3)},\\
C_{-n} &= -2C_{-(n-1)}+C_{-(n-3)},\\
R_{-n} &= -2R_{-(n-1)}+R_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1)-(1.5) hold for all integer n. For more details on the generalized Pell-Padovan numbers and its special cases, see Soykan [2].

Now, we define two sequences related to Adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell , Pell-Perrin, Pell-Padovan numbers. Olivier and Olivier-Lucas numbers are defined as

$$O_n = 2O_{n-2}+O_{n-3}+1, \quad \text{with } O_0=0, O_1=1, O_2=1, \quad n\geq 3,$$

and

$$K_n = 2K_{n-2}+K_{n-3}-2, \quad \text{with } K_0=4, K_1=1, K_2=5\ , \quad n\geq 3,$$

respectively.

The first few values of Olivier and Olivier-Lucas numbers are

$$0, 1, 1, 3, 4, 8, 12, 21, 33, 55, 88, 144, 232, 377, ...$$

and

$$4, 1, 5, 4, 9, 11, 20, 29, 49, 76, 125, 199, 324, 521, ...$$

respectively.

The sequences $\{O_n\}$ and $\{K_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned}
O_n &= O_{n-1}+2O_{n-2}-O_{n-3}-O_{n-4}, \quad O_0=0, O_1=1, O_2=1, O_3=3, \quad n\geq 4,\\
K_n &= K_{n-1}+2K_{n-2}-K_{n-3}-K_{n-4}, \quad K_0=4, K_1=1, K_2=5, K_3=4, \quad n\geq 4,
\end{aligned}$$

There are close relations between Olivier, Olivier-Lucas and Adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers. For example, they satisfy the following interrelations:

$$\begin{aligned}
2O_n &= M_{n+2}+M_{n+1}+M_n-1,\\
K_n &= -2M_{n+2}+3M_{n+1}+4M_n+1,\\
10O_n &= -7B_{n+2}+9B_{n+1}+11B_n-5,\\
K_n &= B_n+1,\\
2O_n &= G_{n+2}+G_{n+1}-G_n-1,\\
K_n &= 4G_{n+2}-2G_{n+1}-5G_n+1,\\
22O_n &= 13C_{n+2}-3C_{n+1}-5C_n-11,\\
11K_n &= -12C_{n+2}+18C_{n+1}+19C_n+11,\\
2O_n &= R_{n+2}-1,\\
2K_n &= -3R_{n+2}+4R_{n+1}+5R_n+2,
\end{aligned}$$

and

$$\begin{aligned}
M_n &= -O_{n+2} + 3O_n + 1, \\
5M_n &= -9K_{n+2} + 8K_{n+1} + 12K_n - 11, \\
B_n &= -6O_{n+2} + 5O_{n+1} + 9O_n + 4, \\
2B_n &= K_{n+3} - 2K_{n+1} + K_n, \\
G_n &= O_{n+1} - O_n, \\
5G_n &= 8K_{n+2} - 6K_{n+1} - 9K_n + 7, \\
C_n &= -12O_{n+2} + 7O_{n+1} + 21O_n + 8, \\
5C_n &= -24K_{n+2} + 18K_{n+1} + 37K_n - 31, \\
R_n &= -4O_{n+2} + 2O_{n+1} + 8O_n + 3, \\
5R_n &= -13K_{n+2} + 11K_{n+1} + 19K_n - 17.
\end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Olivier, Olivier-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1.6}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [3,4,5,6,7,8,9,**??**,11]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (1.6) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{1.7}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= r, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\
\alpha\beta\gamma\delta &= -u.
\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers n, Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{1.8}$$

where

$$\begin{aligned} p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0, \\ p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0, \\ p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0, \\ p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.7) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.7) for non-negative integers is valid for all integers n (see [**??**]).

Next, we consider two special cases of the generalized (r, s, t, u) sequence $\{W_n\}$ which we call them (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas sequences. (r, s, t, u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and (r, s, t, u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, \qquad (1.9) \\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, \\ H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, \qquad (1.10) \\ H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.9) and (1.10) hold for all integers n.
For all integers n, (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas numbers (using initial conditions in (1.9) or (1.10)) can be expressed using Binet's formulas as in the following corollary.

Corollary 1.2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas numbers are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

and

$$H_n = \alpha^n+\beta^n+\gamma^n+\delta^n,$$

respectively.

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1.1, respectively. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 1.3. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized (r, s, t, u) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0+(W_1-rW_0)z+(W_2-rW_1-sW_0)z^2+(W_3-rW_2-sW_1-tW_0)z^3}{1-rz-sz^2-tz^3-uz^4}. \qquad (1.11)$$

Proof. For a proof, see Soykan [7, Lemma 1]. □

The following theorem presents Simson's formula of generalized (r,s,t,u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

Theorem 1.4 (Simson's Formula of Generalized (r,s,t,u) Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{1.12}$$

Proof. (1.12) is given in Soykan [13]. □

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.5. *For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r,s,t,u)-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [14, Theorem 1.]. □

Using Theorem 1.5, we have the following corollary, see Soykan [14, Corollary 4].

Corollary 1.6. *For $n \in \mathbb{Z}$, we have*

(a) $2(-u)^{n+4}G_{-n} = -(3ru^2+t^3-3stu)^2G_n^3 - (2su-t^2)^2G_{n+3}^2G_n - (-rt^2-tu+2rsu)^2G_{n+2}^2G_n - (-st^2+2s^2u+4u^2+rtu)^2G_{n+1}^2G_n + 2(3ru^2+t^3-3stu)((-2su+t^2)G_{n+3} + (-rt^2-tu+2rsu)G_{n+2} + (-st^2+2s^2u+4u^2+rtu)G_{n+1})G_n^2 + 2(2su-t^2)(-rt^2-tu+2rsu)G_{n+3}G_{n+2}G_n + 2(2su-t^2)(-st^2+2s^2u+4u^2+rtu)G_{n+3}G_{n+1}G_n - 2(-st^2+2s^2u+4u^2+rtu)(-rt^2-tu+2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su+t^2)G_{2n+3}G_n + u^2(-rt^2-tu+2rsu)G_{2n+2}G_n + u^2(-st^2+2s^2u+4u^2+rtu)G_{2n+1}G_n - 2u^2(2su-t^2)G_{2n}G_{n+3} + 2u^2(-rt^2-tu+2rsu)G_{2n}G_{n+2} + 2u^2(-st^2+2s^2u+4u^2+rtu)G_{2n}G_{n+1} - 3u^2(3ru^2+t^3-3stu)G_{2n}G_n.$

(b) $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 1.5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), \tag{1.13}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right), \tag{1.14}$$

respectively.

If we define the square matrix A of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 1.7. *For all integers m, n, we have*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b) $U_1 A^n = A^n U_1$.

(c) $U_{n+m} = U_n B_m = B_m U_n$.

Proof. For the proof, see Soykan [7, Theorem 19]. □

Theorem 1.8. *For all integers m, n, we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \quad (1.15)$$

Proof. For the proof, see Soykan [7, Theorem 20]. □

In the next sections, we present new results.

2 Generalized Olivier Sequence

In this paper, we consider the case $r = 1, s = 2, t = -1, u = -1$. A generalized Olivier sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = W_{n-1} + 2W_{n-2} - W_{n-3} - W_{n-4} \quad (2.1)$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + 2W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - 2z^2 + z + 1 = (z^3 - 2z - 1)(z - 1) = (z^2 - z - 1)(z + 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned}\alpha &= \frac{1+\sqrt{5}}{2},\\ \beta &= \frac{1-\sqrt{5}}{2},\\ \gamma &= -1,\\ \delta &= 1.\end{aligned}$$

Note that

$$\begin{aligned}\alpha+\beta+\gamma+\delta &= 1,\\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -2,\\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= -1,\\ \alpha\beta\gamma\delta &= 1.\end{aligned}$$

Note also that

$$\begin{aligned}\alpha+\beta+\gamma &= 0,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= -2,\\ \alpha\beta\gamma &= 1.\end{aligned}$$

The first few generalized Olivier numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Olivier numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$2W_1-W_0+W_2-W_3$
2	W_2	$3W_0-W_1-2W_2+W_3$
3	W_3	$4W_1-4W_0+4W_2-3W_3$
4	$2W_2-W_1-W_0+W_3$	$8W_0-4W_1-7W_2+4W_3$
5	$W_2-2W_1-W_0+3W_3$	$9W_1-12W_0+12W_2-8W_3$
6	$4W_2-4W_1-3W_0+4W_3$	$21W_0-12W_1-20W_2+12W_3$
7	$4W_2-7W_1-4W_0+8W_3$	$22W_1-33W_0+33W_2-21W_3$
8	$9W_2-12W_1-8W_0+12W_3$	$55W_0-33W_1-54W_2+33W_3$
9	$12W_2-20W_1-12W_0+21W_3$	$56W_1-88W_0+88W_2-55W_3$
10	$22W_2-33W_1-21W_0+33W_3$	$144W_0-88W_1-143W_2+88W_3$
11	$33W_2-54W_1-33W_0+55W_3$	$145W_1-232W_0+232W_2-144W_3$
12	$56W_2-88W_1-55W_0+88W_3$	$377W_0-232W_1-376W_2+232W_3$
13	$88W_2-143W_1-88W_0+144W_3$	$378W_1-609W_0+609W_2-377W_3$

Note that the sequences $\{O_n\}$ and $\{K_n\}$ which are defined in the section Introduction, are the special cases of the generalized Olivier sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$ in this section as well. Olivier sequence $\{O_n\}_{n\geq 0}$ and Olivier-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}O_n &= O_{n-1}+2O_{n-2}-O_{n-3}-O_{n-4}, \quad O_0=0, O_1=1, O_2=1, O_3=3, \quad n\geq 4,\\ K_n &= K_{n-1}+2K_{n-2}-K_{n-3}-K_{n-4}, \quad K_0=4, K_1=1, K_2=5, K_3=4, \quad n\geq 4,\end{aligned}$$

The sequences $\{O_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} O_{-n} &= -O_{-(n-1)} + 2O_{-(n-2)} + O_{-(n-3)} - O_{-(n-4)} \\ K_{-n} &= -K_{-(n-1)} + 2K_{-(n-2)} + K_{-(n-3)} - K_{-(n-4)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Next, we present the first few values of the Olivier and Olivier-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
O_n	0	1	1	3	4	8	12	21	33	55	88	144	232	377
O_{-n}	0	0	0	-1	1	-3	4	-8	12	-21	33	-55	88	-144
K_n	4	1	5	4	9	11	20	29	49	76	125	199	324	521
K_{-n}	4	-1	5	-4	9	-11	20	-29	49	-76	125	-199	324	-521

Theorem 1.1 can be used to obtain the Binet formula of generalized Olivier numbers. Using these (the above) roots and the recurrence relation, Binet's formula of generalized Olivier numbers can be given as follows:

Theorem 2.1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of generalized Olivier numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(1-\alpha)W_2 + (-\alpha^2+1)W_1 - W_0)\alpha^n}{4\alpha^2 - \alpha - 3} \\ &\quad + \frac{(\beta W_3 - \beta(1-\beta)W_2 + (-\beta^2+1)W_1 - W_0)\beta^n}{4\beta^2 - \beta - 3} \\ &\quad + \frac{(-W_3 + 2W_2 - W_0)(-1)^n}{2} - \frac{W_3 - 2W_1 - W_0}{2}. \end{aligned}$$

Olivier and Olivier-Lucas numbers can be expressed using Binet's formulas as follows:

Corollary 2.2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formulas of Olivier and Olivier-Lucas numbers are*

$$\begin{aligned} O_n &= \frac{(2\alpha+1)\alpha^n}{4\alpha^2 - \alpha - 3} + \frac{(2\beta+1)\beta^n}{4\beta^2 - \beta - 3} - \frac{1}{2}\gamma^n - \frac{1}{2} \\ &= \frac{1}{10}\left((5-\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^n + (5+\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n - 5(-1)^n - 5\right) \end{aligned}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n + 1 = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n + (-1)^n + 1$$

respectively.

Note that for all integers n, adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1} \\ &= (\frac{1}{2}-\frac{1}{10}\sqrt{5})\alpha^n+(\frac{1}{2}+\frac{1}{10}\sqrt{5})\beta^n-\gamma^n, \\ B_n &= \alpha^n+\beta^n+\gamma^n, \\ G_n &= \frac{1}{\sqrt{5}}\alpha^n-\frac{1}{\sqrt{5}}\beta^n+\gamma^n, \\ C_n &= (2-\frac{3}{\sqrt{5}})\alpha^n+(2+\frac{3}{\sqrt{5}})\beta^n-\gamma^n, \\ R_n &= (1-\frac{1}{\sqrt{5}})\alpha^n+(1+\frac{1}{\sqrt{5}})\beta^n-\gamma^n, \end{aligned}$$

respectively, see Soykan [2] for more details. So, by using Binet's formulas of Olivier, Olivier-Lucas and adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell , Pell-Perrin, Pell-Padovan numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

Lemma 2.3. *For all integers n, the following equalities (identities) are true:*

(a)

- $M_{n+1} = O_{n+1} - O_n$.
- $M_n = O_{n+3} - O_{n+2} - 2O_{n+1} + 2O_n$.
- $2O_{n+4} = 7M_{n+2} + 9M_{n+1} + 3M_n - 1$.
- $2O_n = M_{n+2} + M_{n+1} + M_n - 1$.
- $M_n = -O_{n+2} + 3O_n + 1$.

(b)

- $10M_{n+3} = -3K_{n+3} - 2K_{n+1} + 14K_{n+2} - 9K_n$.
- $10M_n = 11K_{n+3} - 18K_{n+2} - 6K_{n+1} + 13K_n$.
- $K_{n+4} = 3M_{n+2} + 8M_{n+1} + 4M_n + 1$.
- $K_n = -2M_{n+2} + 3M_{n+1} + 4M_n + 1$.
- $5M_n = -9K_{n+2} + 8K_{n+1} + 12K_n - 11$.
- $8M_n + 9M_{n+1} + 5 = 3K_n + 2K_{n+1}$.

(c)

- $B_{n+3} = 4O_{n+2} - O_{n+1} - 3O_n$.
- $B_n = 4O_{n+3} - 6O_{n+2} - 3O_{n+1} + 5O_n$.
- $10O_{n+4} = 3B_{n+2} + 19B_{n+1} + 11B_n - 5$.
- $10O_n = -7B_{n+2} + 9B_{n+1} + 11B_n - 5$.
- $B_n = -6O_{n+2} + 5O_{n+1} + 9O_n + 4$.
- $5B_n + 6B_{n+1} - 8 = 9O_n + 7O_{n+1}$.

(d)

- $2B_{n+3} = 3K_{n+3} - 2K_{n+1} - K_n$.
- $2B_n = K_{n+3} - 2K_{n+1} + K_n$.
- $K_{n+4} = 2B_{n+2} + B_{n+1} + 1$.
- $K_n = B_n + 1$.
- $B_n = K_n - 1$.

(e)

- $G_{n+3} = 2O_{n+2} - O_{n+1} - O_n$.
- $G_n = O_{n+1} - O_n$.
- $2O_{n+4} = 3G_{n+2} + 7G_{n+1} + 3G_n - 1$.
- $2O_n = G_{n+2} + G_{n+1} - G_n - 1$.

(f)

- $10G_{n+3} = 11K_{n+3} - 8K_{n+2} - 6K_{n+1} + 3K_n$.
- $10G_n = -7K_{n+3} + 16K_{n+2} + 2K_{n+1} - 11K_n$.
- $K_{n+4} = 4G_{n+2} + 3G_{n+1} + 1$.
- $K_n = 4G_{n+2} - 2G_{n+1} - 5G_n + 1$.
- $5G_n = 8K_{n+2} - 6K_{n+1} - 9K_n + 7$.
- $3G_n + 4G_{n+1} + 3 = K_n + 2K_{n+1}$.

(g)

- $C_{n+3} = 2O_{n+2} + O_{n+1} - 3O_n$.
- $C_n = 8O_{n+3} - 12O_{n+2} - 9O_{n+1} + 13O_n$.
- $22O_{n+4} = 39C_{n+2} + 35C_{n+1} + 7C_n - 11$.
- $22O_n = 13C_{n+2} - 3C_{n+1} - 5C_n - 11$.
- $C_n = -12O_{n+2} + 7O_{n+1} + 21O_n + 8$.
- $7C_n + 12C_{n+1} - 8 = 3O_n + 13O_{n+1}$.

(h)

- $10C_{n+3} = -3K_{n+3} + 24K_{n+2} - 2K_{n+1} - 19K_n$.
- $10C_n = 31K_{n+3} - 48K_{n+2} - 26K_{n+1} + 43K_n$.
- $11K_{n+4} = 8C_{n+2} + 43C_{n+1} + 24C_n + 11$.
- $11K_n = -12C_{n+2} + 18C_{n+1} + 19C_n + 11$.
- $5C_n = -24K_{n+2} + 18K_{n+1} + 37K_n - 31$.
- $3C_n + 4C_{n+1} + 5 = 3K_n + 2K_{n+1}$.

(i)

- $R_{n+3} = O_{n+3} - O_n$.
- $R_n = 3O_{n+3} - 4O_{n+2} - 4O_{n+1} + 5O_n$.

- $2O_{n+4} = 4R_{n+2} + 4R_{n+1} + R_n - 1.$
- $2O_n = R_{n+2} - 1.$
- $R_n = -4O_{n+2} + 2O_{n+1} + 8O_n + 3.$
- $2O_{n+1} = 2R_{n+1} + R_n - 1.$

(j)

- $10R_{n+3} = -K_{n+3} + 18K_{n+2} - 4K_{n+1} - 13K_n.$
- $10R_n = 17K_{n+3} - 26K_{n+2} - 12K_{n+1} + 21K_n.$
- $2K_{n+4} = 2R_{n+2} + 9R_{n+1} + 5R_n + 2.$
- $2K_n = -3R_{n+2} + 4R_{n+1} + 5R_n + 2.$
- $5R_n = -13K_{n+2} + 11K_{n+1} + 19K_n - 17.$
- $11R_n + 13R_{n+1} + 14 = 2(4K_n + 3K_{n+1}).$

Proof. We only prove $M_{n+1} = O_{n+1} - O_n$ by using Binet's formulas of M_n and O_n as the others can be proved similarly. By using Binet's formulas, we get

$$\begin{aligned} O_{n+1} - O_n &= \frac{(2\alpha+1)\alpha^{n+1}}{4\alpha^2-\alpha-3} - \frac{(2\alpha+1)\alpha^n}{4\alpha^2-\alpha-3} + \frac{(2\beta+1)\beta^{n+1}}{4\beta^2-\beta-3} - \frac{(2\beta+1)\beta^n}{4\beta^2-\beta-3} - \frac{1}{2}\gamma^{n+1} + \frac{1}{2}\gamma^n \\ &= \frac{2\alpha+1}{4\alpha+3}\alpha^n + \frac{2\beta+1}{4\beta+3}\beta^n - \frac{1}{2}(\gamma-1)\gamma^n \\ &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+2} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+2} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+2} \\ &= M_{n+1} \end{aligned}$$

where

$$\begin{aligned} \frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} &= \frac{2\alpha+1}{4\alpha+3}, \\ \frac{\beta^2}{(\beta-\alpha)(\beta-\gamma)} &= \frac{2\beta+1}{4\beta+3}, \\ \frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)} &= -\frac{1}{2}(\gamma-1). \quad \square \end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 2.4. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Olivier sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z + (W_2 - W_1 - 2W_0)z^2 + (W_3 - W_2 - 2W_1 + W_0)z^3}{1 - z - 2z^2 + z^3 + z^4}.$$

Proof. Take $r = 1, s = 2, t = -1, u = -1$ in Lemma 1.3. □

The previous lemma gives the following results as particular examples.

Corollary 2.5. *Generating functions of Olivier and Olivier-Lucas numbers are*

$$\sum_{n=0}^{\infty} O_n z^n = \frac{z}{1-z-2z^2+z^3+z^4},$$
$$\sum_{n=0}^{\infty} K_n z^n = \frac{4-3z-4z^2+z^3}{1-z-2z^2+z^3+z^4},$$

respectively.

3 Simson Formulas

Now, we present Simson's formula of generalized Olivier numbers.

Theorem 3.1 (Simson's Formula of Generalized Olivier Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (W_3-2W_2+W_0)(W_3-2W_1-W_0)(W_3^2-W_2^2+W_1^2-W_0^2-W_2W_3-$$
$$2W_1W_3+W_1W_2+W_0W_3+2W_0W_2-W_0W_1).$$

Proof. Take $r=1, s=2, t=-1, u=-1$ in Theorem 1.4. □
The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers n, the Simson's formulas of Olivier and Olivier-Lucas numbers are given as*

$$\begin{vmatrix} O_{n+3} & O_{n+2} & O_{n+1} & O_n \\ O_{n+2} & O_{n+1} & O_n & O_{n-1} \\ O_{n+1} & O_n & O_{n-1} & O_{n-2} \\ O_n & O_{n-1} & O_{n-2} & O_{n-3} \end{vmatrix} = 1,$$
$$\begin{vmatrix} K_{n+3} & K_{n+2} & K_{n+1} & K_n \\ K_{n+2} & K_{n+1} & K_n & K_{n-1} \\ K_{n+1} & K_n & K_{n-1} & K_{n-2} \\ K_n & K_{n-1} & K_{n-2} & K_{n-3} \end{vmatrix} = 20,$$

respectively.

4 Some Identities

In this section, we obtain some identities of Olivier and Olivier-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{O_n\}$.

Lemma 4.1. *The following equalities are true:*

(a) $W_n = (8W_0-4W_1-7W_2+4W_3)O_{n+5}+(8W_1-12W_0+11W_2-7W_3)O_{n+4}+(3W_1-9W_0+8W_2-4W_3)O_{n+3}+(12W_0-9W_1-12W_2+8W_3)O_{n+2}.$

(b) $W_n = (4W_1-4W_0+4W_2-3W_3)O_{n+4}+(7W_0-5W_1-6W_2+4W_3)O_{n+3}+(4W_0-5W_1-5W_2+4W_3)O_{n+2}+(4W_1-8W_0+7W_2-4W_3)O_{n+1}.$

(c) $W_n = (3W_0-W_1-2W_2+W_3)O_{n+3}+(3W_1-4W_0+3W_2-2W_3)O_{n+2}+(3W_2-4W_0-W_3)O_{n+1}+(4W_0-4W_1-4W_2+3W_3)O_n.$

(d) $W_n = (2W_1-W_0+W_2-W_3)O_{n+2}+(2W_0-2W_1-W_2+W_3)O_{n+1}+(W_0-3W_1-2W_2+2W_3)O_n+(W_1-3W_0+2W_2-W_3)O_{n-1}.$

(e) $W_n = W_0O_{n+1} + (W_1 - W_0)O_n + (W_2 - W_1 - 2W_0)O_{n-1} + (W_0 - 2W_1 - W_2 + W_3)O_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times O_{n+5} + b \times O_{n+4} + c \times O_{n+3} + d \times O_{n+2}$$

and solving the system of equations

$$\begin{array}{rcl} W_0 & = & a \times O_5 + b \times O_4 + c \times O_3 + d \times O_2 \\ W_1 & = & a \times O_6 + b \times O_5 + c \times O_4 + d \times O_3 \\ W_2 & = & a \times O_7 + b \times O_6 + c \times O_5 + d \times O_4 \\ W_3 & = & a \times O_8 + b \times O_7 + c \times O_6 + d \times O_5 \end{array}$$

we find that $a = 8W_0 - 4W_1 - 7W_2 + 4W_3, b = 8W_1 - 12W_0 + 11W_2 - 7W_3, c = 3W_1 - 9W_0 + 8W_2 - 4W_3, d = 12W_0 - 9W_1 - 12W_2 + 8W_3$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{K_n\}$.

Lemma 4.2. *The following equalities are true:*

(a) $10W_n = (22W_0 - 19W_1 - 27W_2 + 19W_3)K_{n+5} - (41W_0 - 22W_1 - 46W_2 + 27W_3)K_{n+4} - (17W_0 - 24W_1 - 22W_2 + 19W_3)K_{n+3} + (41W_0 - 17W_1 - 41W_2 + 22W_3)K_{n+2}$.

(b) $10W_n = -(19W_0 - 3W_1 - 19W_2 + 8W_3)K_{n+4} + (27W_0 - 14W_1 - 32W_2 + 19W_3)K_{n+3} + (19W_0 + 2W_1 - 14W_2 + 3W_3)K_{n+2} - (22W_0 - 19W_1 - 27W_2 + 19W_3)K_{n+1}$.

(c) $10W_n = (8W_0 - 11W_1 - 13W_2 + 11W_3)K_{n+3} - (19W_0 - 8W_1 - 24W_2 + 13W_3)K_{n+2} - (3W_0 - 16W_1 - 8W_2 + 11W_3)K_{n+1} + (19W_0 - 3W_1 - 19W_2 + 8W_3)K_n$.

(d) $10W_n = -(11W_0 + 3W_1 - 11W_2 + 2W_3)K_{n+2} + (13W_0 - 6W_1 - 18W_2 + 11W_3)K_{n+1} + (11W_0 + 8W_1 - 6W_2 - 3W_3)K_n - (8W_0 - 11W_1 - 13W_2 + 11W_3)K_{n-1}$.

(e) $10W_n = (2W_0 - 9W_1 - 7W_2 + 9W_3)K_{n+1} - (11W_0 - 2W_1 - 16W_2 + 7W_3)K_n + (3W_0 + 14W_1 + 2W_2 - 9W_3)K_{n-1} + (11W_0 + 3W_1 - 11W_2 + 2W_3)K_{n-2}$.

Now, we give a few basic relations between $\{O_n\}$ and $\{K_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{array}{rcl} 10O_n & = & 11K_{n+5} - 13K_{n+4} - 11K_{n+3} + 8K_{n+2}, \\ 10O_n & = & -2K_{n+4} + 11K_{n+3} - 3K_{n+2} - 11K_{n+1}, \\ 10O_n & = & 9K_{n+3} - 7K_{n+2} - 9K_{n+1} + 2K_n, \\ 10O_n & = & 2K_{n+2} + 9K_{n+1} - 7K_n - 9K_{n-1}, \\ 10O_n & = & 11K_{n+1} - 3K_n - 11K_{n-1} - 2K_{n-2}, \end{array}$$

and

$$\begin{array}{rcl} K_n & = & 9O_{n+5} - 13O_{n+4} - 9O_{n+3} + 11O_{n+2}, \\ K_n & = & -4O_{n+4} + 9O_{n+3} + 2O_{n+2} - 9O_{n+1}, \\ K_n & = & 5O_{n+3} - 6O_{n+2} - 5O_{n+1} + 4O_n, \\ K_n & = & -O_{n+2} + 5O_{n+1} - O_n - 5O_{n-1}, \\ K_n & = & 4O_{n+1} - 3O_n - 4O_{n-1} + O_{n-2}. \end{array}$$

5 Relations Between Special Numbers

In this section, we present identities on Olivier, Olivier-Lucas numbers and adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers. We know from Lemma 2.3 that

$$\begin{aligned} 22O_n &= 13C_{n+2} - 3C_{n+1} - 5C_n - 11, \\ 2K_n &= -3R_{n+2} + 4R_{n+1} + 5R_n + 2. \end{aligned}$$

Note also that from Lemma Lemma 4.1 and Lemma 4.2, we have the formulas of W_n as

$$\begin{aligned} W_n &= (3W_0 - W_1 - 2W_2 + W_3)O_{n+3} + (3W_1 - 4W_0 + 3W_2 - 2W_3)O_{n+2} \\ &\quad +(3W_2 - 4W_0 - W_3)O_{n+1} + (4W_0 - 4W_1 - 4W_2 + 3W_3)O_n, \\ 10W_n &= (8W_0 - 11W_1 - 13W_2 + 11W_3)K_{n+3} - (19W_0 - 8W_1 - 24W_2 + 13W_3)K_{n+2} \\ &\quad -(3W_0 - 16W_1 - 8W_2 + 11W_3)K_{n+1} + (19W_0 - 3W_1 - 19W_2 + 8W_3)K_n. \end{aligned}$$

Using the above identities, we obtain relation of generalized Olivier numbers in the following forms (in terms of Pell-Perrin and Pell-Padovan numbers):

Lemma 5.1. *For all integers n, we have the following identities:*

(a) $22W_n = (7W_3 - 12W_2 + 4W_1 + W_0)C_{n+2} - (5W_3 - 18W_2 + 6W_1 + 7W_0)C_{n+1} - (W_3 - 8W_2 + 10W_1 - 3W_0)C_n - 11W_3 + 22W_1 + 11W_0$.

(b) $2W_n = (W_1 - W_0)R_{n+2} + (W_2 - W_1)R_{n+1} + (W_3 - W_2 - 2W_1 + 2W_0)R_n - W_3 + 2W_1 + W_0$.

6 On the Recurrence Properties of Generalized Olivier Sequence

Taking $r = 1, s = 2, t = -1, u = -1$ in Theorem 1.5, we obtain the following Proposition.

Proposition 6.1. *For $n \in \mathbb{Z}$, generalized Olivier numbers (the case $r = 1, s = 2, t = -1, u = -1$) have the following identity:*

$$W_{-n} = \frac{1}{6}(-6W_{3n} + 6K_nW_{2n} - 3K_n^2W_n + 3K_{2n}W_n + W_0K_n^3 + 2W_0K_{3n} - 3W_0K_nK_{2n}).$$

From the above Proposition 6.1 (or by taking $G_n = O_n$ and $H_n = K_n$ in (1.13) and (1.14) respectively), we have the following corollary which gives the connection between the special cases of generalized Olivier sequence at the positive index and the negative index: for Olivier and Olivier-Lucas numbers: take $W_n = O_n$ with $O_0 = 0, O_1 = 1, O_2 = 1, O_3 = 3$ and take $W_n = K_n$ with $K_0 = 4, K_1 = 1, K_2 = 5, K_3 = 4$, respectively. Note that in this case $H_n = K_n$.

Corollary 6.2. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Olivier sequence:*

$$O_{-n} = \frac{1}{6}(-6O_{3n} + 6K_nO_{2n} - 3K_n^2O_n + 3K_{2n}O_n).$$

(b) *Olivier-Lucas sequence:*

$$K_{-n} = \frac{1}{6}(K_n^3 + 2K_{3n} - 3K_{2n}K_n).$$

We can also present the formulas of O_{-n} and K_{-n} in the following forms.

Corollary 6.3. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) $O_{-n} = \frac{1}{6}(-6O_{3n} + 6(5O_{n+3} - 6O_{n+2} - 5O_{n+1} + 4O_n)O_{2n} - 3(5O_{n+3} - 6O_{n+2} - 5O_{n+1} + 4O_n)^2 O_n + 3(5O_{2n+3} - 6O_{2n+2} - 5O_{2n+1} + 4O_{2n})O_n)$.

(b)

- **(i)** $2O_{-n} = -4M_n^2 - 4M_{n-1}^2 - 4M_{n-2}^2 + (2M_{n+2} - 3M_{n+1} - 3M_{n-1} + 2M_{n-2})M_n + (2M_{n+1} - 3M_{n-2})M_{n-1} + M_{2n} + M_{2n-2} + M_{2n-4} - 1$.
- **(ii)** $K_{-n} = -16M_n^2 - 12M_{n-1}^2 + 8M_{n-2}^2 - (-8M_{n+2} + 12M_{n+1} + 9M_{n-1} + 4M_{n-2})M_n + 6(M_{n+1} + M_{n-2})M_{n-1} + 4M_{2n} + 3M_{2n-2} - 2M_{2n-4} + 1$.

(c)

- **(i)** $20O_{-n} = 11B_n^2 + 9B_{n-1}^2 - 7B_{n-2}^2 - 11B_{2n} - 9B_{2n-2} + 7B_{2n-4} - 10$.
- **(ii)** $2K_{-n} = B_n^2 - B_{2n} + 2$.

(d)

- **(i)** $4O_{-n} = -16G_{n+2}^2 + 12G_{n+1}^2 - 15G_n^2 + 39G_{n-1}^2 + 35G_{n-2}^2 + 8(6G_{n+2} - 5G_{n+1} + G_{n-1} - 6G_{n-2})G_n - 24(2G_{n+1} - G_{n-2})G_{n-1} + 16G_{n+1}G_{n+2} + 4G_{2n+2} - 2G_{2n+1} - 11G_{2n} + 2G_{2n-1} + 3G_{2n-2} + 2G_{2n-3} + 7G_{2n-4} - 2$.
- **(ii)** $2K_{-n} = -80G_{n+2}^2 - 52G_{n+1}^2 - 119G_n^2 - 54G_{n-1}^2 + 140G_{n-2}^2 + 8(30G_{n+2} - 11G_{n+1} - 14G_{n-1} - 24G_{n-2})G_n + 96(G_{n+1} + G_{n-2})G_{n-1} + 80G_{n+1}G_{n+2} + 20G_{2n+2} - 10G_{2n+1} - 27G_{2n} - 4G_{2n-1} - 30G_{2n-2} + 8G_{2n-3} + 28G_{2n-4} + 2$.

(e)

- **(i)** $5324O_{-n} = -2160C_{n+2}^2 - 6156C_{n+1}^2 - 625C_n^2 + 10\,641C_{n-1}^2 + 8645C_{n-2}^2 - 24(-230C_{n+2} + 183C_{n+1} + 909C_{n-1} + 598C_{n-2})C_n + 1656(2C_{n+1} + 13C_{n-2})C_{n-1} + 6480C_{n+1}C_{n+2} - 1980C_{2n+2} + 2970C_{2n+1} + 737C_{2n} + 1782C_{2n-1} + 6303C_{2n-2} - 7722C_{2n-3} - 5005C_{2n-4} - 2662$.
- **(ii)** $2662K_{-n} = 8208C_{n+2}^2 + 26244C_{n+1}^2 + 24947C_n^2 + 306C_{n-1}^2 - 7980C_{n-2}^2 + 24(-874C_{n+2} + 339C_{n+1} + 1890C_{n-1} + 552C_{n-2})C_n - 19872(C_{n+1} + C_{n-2})C_{n-1} - 24624C_{n+1}C_{n+2} + 7524C_{2n+2} - 11\,286C_{2n+1} - 187C_{2n} - 10\,692C_{2n-1} - 11\,682C_{2n-2} + 7128C_{2n-3} + 4620C_{2n-4} + 2662$.

(f)

- **(i)** $16O_{-n} = 9R_n^2 + 16R_{n-1}^2 + 5R_{n-2}^2 - 6(4R_{n-1} + 3R_{n-2})R_n + 24R_{n-1}R_{n-2} + 6R_{2n-2} - 8R_{2n-3} - 2R_{2n-4} - 8$.
- **(ii)** $16K_{-n} = 45R_{n+2}^2 + 116R_{n+1}^2 + 62R_n^2 - 28R_{n-1}^2 - 15R_{n-2}^2 + 6(-15R_{n+2} + 4R_{n+1} + 28R_{n-1} + 9R_{n-2})R_n - 72(R_{n+1} + R_{n-2})R_{n-1} - 120R_{n+1}R_{n+2} + 30R_{2n+2} - 40R_{2n+1} + 14R_{2n} - 32R_{2n-1} - 26R_{2n-2} + 24R_{2n-3} + 6R_{2n-4} + 16$.

Proof. We use the identities, see Soykan [15],

$$\begin{aligned} M_{-n} &= -4M_n^2 + M_{2n} + 2M_{n+2}M_n - 3M_{n+1}M_n, \\ B_{-n} &= \frac{1}{2}(B_n^2 - B_{2n}), \end{aligned}$$

and

$G_{-n} = \frac{1}{2}(16G_{n+2}^2 + 4G_{n+1}^2 + 35G_n^2 - 4G_{2n+2} + 2G_{2n+1} + 7G_{2n} - 16G_{n+2}G_{n+1} - 48G_{n+2}G_n + 24G_{n+1}G_n)$,

$C_{-n} = \frac{1}{242}(432C_{n+2}^2 + 972C_{n+1}^2 + 665C_n^2 + 396C_{2n+2} - 594C_{2n+1} - 385C_{2n} - 1296C_{n+2}C_{n+1} - 1104C_{n+2}C_n + 1656C_{n+1}C_n)$,

$R_{-n} = \frac{1}{8}(16R_{n+1}^2 + 9R_{n+2}^2 + 5R_n^2 + 6R_{2n+2} - 8R_{2n+1} - 2R_{2n} - 24R_{n+2}R_{n+1} - 18R_{n+2}R_n + 24R_{n+1}R_n)$.

We also use the identities

$$\begin{aligned}
2O_n &= M_{n+2}+M_{n+1}+M_n-1,\\
K_n &= -2M_{n+2}+3M_{n+1}+4M_n+1,\\
10O_n &= -7B_{n+2}+9B_{n+1}+11B_n-5,\\
K_n &= B_n+1,\\
2O_n &= G_{n+2}+G_{n+1}-G_n-1,\\
K_n &= 4G_{n+2}-2G_{n+1}-5G_n+1,\\
22O_n &= 13C_{n+2}-3C_{n+1}-5C_n-11,\\
11K_n &= -12C_{n+2}+18C_{n+1}+19C_n+11,\\
2O_n &= R_{n+2}-1,\\
2K_n &= -3R_{n+2}+4R_{n+1}+5R_n+2.
\end{aligned}$$

(a) By using the identity $K_n = 5O_{n+3} - 6O_{n+2} - 5O_{n+1} + 4O_n$ and Corollary 6.2, (or by using Corollary 1.6 (a)), we obtain (a).

(b) Since $2O_n = M_{n+2} + M_{n+1} + M_n - 1$, $K_n = -2M_{n+2} + 3M_{n+1} + 4M_n + 1$, and $M_{-n} = -4M_n^2 + M_{2n} + 2M_{n+2}M_n - 3M_{n+1}M_n$, we get (b)

(c) Since $10O_n = -7B_{n+2} + 9B_{n+1} + 11B_n - 5$, $K_n = B_n + 1$ and $B_{-n} = \frac{1}{2}(B_n^2 - B_{2n})$, we obtain (c).

(d) Since $2O_n = G_{n+2} + G_{n+1} - G_n - 1$, $K_n = 4G_{n+2} - 2G_{n+1} - 5G_n + 1$ and $G_{-n} = \frac{1}{2}(16G_{n+2}^2 + 4G_{n+1}^2 + 35G_n^2 - 4G_{2n+2} + 2G_{2n+1} + 7G_{2n} - 16G_{n+2}G_{n+1} - 48G_{n+2}G_n + 24G_{n+1}G_n)$, we get (d).

(e) Since $22O_n = 13C_{n+2}-3C_{n+1}-5C_n-11$, $11K_n = -12C_{n+2}+18C_{n+1}+19C_n+11$ and $C_{-n} = \frac{1}{242}(432C_{n+2}^2 + 972C_{n+1}^2 + 665C_n^2 + 396C_{2n+2} - 594C_{2n+1} - 385C_{2n} - 1296C_{n+2}C_{n+1} - 1104C_{n+2}C_n + 1656C_{n+1}C_n)$, we obtain (e).

(f) Since $2O_n = R_{n+2}-1$, $2K_n = -3R_{n+2}+4R_{n+1}+5R_n+2$ and $R_{-n} = \frac{1}{8}(16R_{n+1}^2+9R_{n+2}^2+5R_n^2+6R_{2n+2} - 8R_{2n+1} - 2R_{2n} - 24R_{n+2}R_{n+1} - 18R_{n+2}R_n + 24R_{n+1}R_n)$, we get (f). □

7 Sum Formulas

The following Corollary gives sum formulas of Pell-Padovan numbers.

Corollary 7.1. *For $n \geq 0$, Pell-Padovan numbers have the following property:*

(a) $\sum_{k=0}^{n} R_k = \frac{1}{2}\left(R_{n+3} + R_{n+2} - R_{n+1} - 1\right).$

(b) $\sum_{k=0}^{n} R_{2k} = R_{2n+1} - n.$

(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{2}\left(R_{2n+3} + R_{2n+2} - R_{2n+1} + 2n - 1\right).$

Proof. It is given in Soykan [2]. □

The following Corollary presents sum formulas of Olivier and Olivier-Lucas numbers.

Corollary 7.2. *For $n \geq 0$, Olivier and Olivier-Lucas numbers have the following properties (in terms of Pell-Padovan numbers):*

(a)

(i) $\sum_{k=0}^{n} O_k = \frac{1}{4}(3R_{n+2} + 3R_{n+1} + R_n - 2n - 7).$

(ii) $\sum_{k=0}^{n} O_{2k} = \frac{1}{2}(R_{2n+2} + R_{2n+1} - 2(n+1))$.

(iii) $\sum_{k=0}^{n} O_{2k+1} = \frac{1}{4}(R_{2n+2} + 5R_{2n+1} + 3R_{2n} - 5)$.

(b)

(i) $\sum_{k=0}^{n} K_k = \frac{1}{2}(4R_{n+1} + 3R_n + 2n + 1)$.

(ii) $\sum_{k=0}^{n} K_{2k} = \frac{1}{2}(-R_{2n+2} + 4R_{2n+1} + 2R_{2n} + 4n + 3)$.

(iii) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{2}(5R_{2n+2} - R_{2n+1} - 2R_{2n})$.

Proof. The proof follows from Corollary 7.1 and the identities

$$\begin{aligned} 2O_n &= R_{n+2} - 1, \\ 2K_n &= -3R_{n+2} + 4R_{n+1} + 5R_n + 2. \end{aligned} \qquad \square$$

8 Matrices and Identities Related With Generalized Olivier Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} O_{n+1} & 2O_n - O_{n-1} - O_{n-2} & -O_n - O_{n-1} & -O_n \\ O_n & 2O_{n-1} - O_{n-2} - O_{n-3} & -O_{n-1} - O_{n-2} & -O_{n-1} \\ O_{n-1} & 2O_{n-2} - O_{n-3} - O_{n-4} & -O_{n-2} - O_{n-3} & -O_{n-2} \\ O_{n-2} & 2O_{n-3} - O_{n-4} - O_{n-5} & -O_{n-3} - O_{n-4} & -O_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & 2W_n - W_{n-1} - W_{n-2} & -W_n - W_{n-1} & -W_n \\ W_n & 2W_{n-1} - W_{n-2} - W_{n-3} & -W_{n-1} - W_{n-2} & -W_{n-1} \\ W_{n-1} & 2W_{n-2} - W_{n-3} - W_{n-4} & -W_{n-2} - W_{n-3} & -W_{n-2} \\ W_{n-2} & 2W_{n-3} - W_{n-4} - W_{n-5} & -W_{n-3} - W_{n-4} & -W_{n-3} \end{pmatrix}.$$

then we get the following Theorem.

Theorem 8.1. *For all integers m, n, we have*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} O_{n+1} & 2O_n - O_{n-1} - O_{n-2} & -O_n - O_{n-1} & -O_n \\ O_n & 2O_{n-1} - O_{n-2} - O_{n-3} & -O_{n-1} - O_{n-2} & -O_{n-1} \\ O_{n-1} & 2O_{n-2} - O_{n-3} - O_{n-4} & -O_{n-2} - O_{n-3} & -O_{n-2} \\ O_{n-2} & 2O_{n-3} - O_{n-4} - O_{n-5} & -O_{n-3} - O_{n-4} & -O_{n-3} \end{pmatrix}.$$

(b) $U_1 A^n = A^n U_1$.

(c) $U_{n+m} = U_n B_m = B_m U_n$.

Proof. Take $r = 1, s = 2, t = -1, u = -1$ in Theorem 1.7. □

Using the above last Theorem and the identity

$$2O_n = R_{n+2} - 1,$$

we obtain the following identity for Pell-Padovan numbers.

Corollary 8.2. *For all integers n, we have the following formula for Pell-Padovan numbers:*

$$A^n = \frac{1}{2}\begin{pmatrix} R_{n+3}-1 & R_{n+4}-R_{n+3} & -R_{n+2}-R_{n+1}+2 & -R_{n+2}+1 \\ R_{n+2}-1 & R_{n+3}-R_{n+2} & -R_{n+1}-R_n+2 & -R_{n+1}+1 \\ R_{n+1}-1 & R_{n+2}-R_{n+1} & -R_n-R_{n-1}+2 & -R_n+1 \\ R_n-1 & R_{n+1}-R_n & -R_{n-1}-R_{n-2}+2 & -R_{n-1}+1 \end{pmatrix}.$$

Next, we present an identity for W_{n+m}.

Theorem 8.3. *For all integers m, n, we have*

$$W_{n+m} = W_n O_{m+1} + W_{n-1}(2O_m - O_{m-1} - O_{m-2}) + W_{n-2}(-O_m - O_{m-1}) - W_{n-3}O_m.$$

Proof. Take $r = 1, s = 2, t = -1, u = -1$ in Theorem 1.8.. □

As particular cases of the above theorem, we give identities for O_{n+m} and K_{n+m}.

Corollary 8.4. *For all integers m, n, we have*

$$\begin{aligned} O_{n+m} &= O_n O_{m+1} + O_{n-1}(2O_m - O_{m-1} - O_{m-2}) + O_{n-2}(-O_m - O_{m-1}) - O_{n-3}O_m, \\ K_{n+m} &= K_n O_{m+1} + K_{n-1}(2O_m - O_{m-1} - O_{m-2}) + K_{n-2}(-O_m - O_{m-1}) - K_{n-3}O_m. \end{aligned}$$

9 Conclusions

Numerous studies of number sequences have been published in the literature, and these studies have been applied to a wide range of fields including physics, engineering, architecture, nature, and the arts. The most well-known second order recurrence sequences are those based on integer numbers, including the Fibonacci, Lucas, Pell, and Jacobsthal sequences. In Leonardo de Pisa's book Liber Abaci from 1202, which is where he first presented the rabbit breeding conundrum, the Fibonacci numbers are likely most well-known for their appearance. The Fibonacci and Lucas sequences are sources of many nice and interesting identities. For example, in [16], authors study on the solutions of the connection problems between Fermat and generalized Fibonacci polynomials. For rich applications of second order sequences in science and nature, one can see the citations in [17].

As a fourth order sequence, we introduce the generalized Olivier sequence (and it's two special cases, namely, Olivier and Olivier-Lucas sequences) and we present Binet's formulas, generating functions, Simson formulas, the sum formulas, some identities, recurrence properties and matrices for these sequences.

We have shown that there are close relations between Olivier, Olivier-Lucas numbers (which are fourth order linear recurences) and special third order linear recurences (numbers), namely adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers.

Linear recurrence relations (sequences) have many applications. We now present one of them. The ratio of two consecutive Padovan numbers converges to the plastic ratio, α_P (which is given in (9.1) below), which have many applications to such as architecture, see [18]. Padovan numbers is defined by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, \qquad P_0 = 1, P_1 = 1, P_2 = 1.$$

The characteristic equation associated with Padovan sequence is $x^3 - x - 1 = 0$ with roots α, β and γ in which

$$\alpha = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \simeq 1.32471795724 \tag{9.1}$$

is called plastic number (or plastic ratio or plastic constant or silver number) and

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \alpha.$$

The plastic number is used in art and architecture. Richard Padovan studied on plastic number in Architecture and Mathematics in [19, 20].

We now present some other applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [21] and [22], respectively.
- For the application of Tribonacci numbers to special matrices, see [23].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [24] and [25], respectively.
- For the application of Pell-Padovan numbers to groups, see [26].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [27].
- For the application of Gaussian Tribonacci numbers to various graphs, see [28].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [29].
- For the application of Narayan numbers to finite groups see [30].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [31].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [32, 33].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [34].
- For the application of generalized Tribonacci numbers to Sedenions, see [35].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [36].
- For the application of generalized Tribonacci numbers to circulant matrix, see [37].

Next, we list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [38].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [39].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [40].
- For the application of generalized Tetranacci numbers to binomial transform, see [41].

Competing Interests

Author has declared that no competing interests exist.

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