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Probability Risk Model of Claim Amount Affected by a Threshold Value

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Author's contribution

This work was carried out in collaboration between both authors.Both authors read and approved the final manuscript.

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Abstract

In this paper, we consider a new risk model of claim amount affected by a threshold value. The comparision between the claim interval and the threshold will affect the distribution of claims. The hypothesis of the model is presented and then we derive the roots of the Lundberg equation, and the expected discounted penalty function and its Laplace Transform. Besides, the Gerber-Shiu penalty function and some other functions are given when the initial surplus is zero and when they satifie some defective renewal equations. Some explicit expressions about the ruin probability are obtained too.

Keywords: Threshold value; Gerber-Shiu penalty function; Lundberg equation; ruin probability.

1 Introduction

In this area, the classial risk model and the renewal risk model have been extensively studied. And they both assumed that the interarrival times and the claim amounts are independent. However, this assumptions is inappropriate in the real world. To solve this problem, some papers started to study the dependent risk models. For example, [1], [2], [3]. M. Boudreault et al. [4] studied the dependence structure among the interclaim time and the claim size. Several renewal risk models with different interclaim times have been studied by many authors, see Cheng D, Yu C. [5] and Li J.[6] H.Cossette et al.[7] and Stathis et al. [8] considered an extension to the renewal process with dependence structure through a copula function. Zhang and Liu [9] considered a discrete-time

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dependent risk model with impulsive dividend payments. And some other distributions are also applied to the risk model, see Guan and Hu [10] and Xu and Wang [11].

Here, we talk about other case in which the distribution of every claim size is controlled by a threshold value.

The paper is organsized as follows. We introduce the new risk model and give some basic assumptions in section 2. And in section 3, we analyse the generalised Lundberg equation and the number of roots . The Laplace Transform (LT) of the Gerber-Shiu expected discount penalty function is given in section 4. And then we analyse the Gerber-Shiu penalty function when u=0 in Section 5. In section 6, the defective renewal functions are given to sovle the expressions for the Gerber-Shiu penalty function. In the last section 7, explicit expressions and numerical examples are given.

2 The New Model

In this section, the new surplus process $\{U(t), t \geq 0\}$ defined as follows:

$$U(t) = u + qt - \sum_{i=1}^{N(t)} X_i,$$

where $u = U(0) \geq 0$ is the initial surplus and $q(q > 0)$ is the premium rate. The claim number process $\{N(t), t \geq 0\}$ is a homogeneous Possion process. $\{W_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed(i.i.d.) interclaim times and the claim arrival times is $T_j, j \in N^+$ which $T_j = W_1 + \ldots + W_j$, and the random variable (r.v.) W_i has an exponentially distribution with expectation $1/k, k > 0$. The p.d.f. gives:

$$K_W(t) = ke^{-kt}.t \geq 0$$

The random variable(r.v.) X_i represents the size of the ith claim. We assume that $M_i, i = 1, 2, \ldots$ a sequence of i.i.d. non-negative random variables distributed as M with Erlang(2) distribution with expectation $2/l, l > 0$ with p.d.f. given by:

$$x(t) = l^2 te^{-lt}, t \geq 0.$$

Then the claim sizes are determined as follows: If T_i is smaller than M_i, then the claim size X_i has density function $f_1(x)$, otherwise its density function is $f_2(x)$. Here $M_i, i = 1, 2, \ldots$ are independent of T_i and X_i. From above notations, we get that:

$$P(M \leq T) = 1 - e^{-lt} - \frac{1}{l}x(t)$$

$$P(M > T) = e^{-lt} + \frac{1}{l}x(t)$$

We ask $\rho = inf_{t\geq 0}\{t, U_t < 0\}$ to be the ruin time which $\rho = \infty$ if $X_t \geq 0$. The deficit at ruin is denoted by $|U_\rho|$ and $U_{\rho-}$ is the surplus just prior to ruin. The Gerber-Shiu discounted penalty function $m_\theta(u)$ is defined as

$$m_\theta(u) = E[e^{-\theta\rho} w(U_{\rho-}, |U_\rho|) 1_{\rho<\infty} | U_0 = u],$$

where $\theta > 0, w : R^+ \times R^+ \to R^+$ is the penalty function. And also $m_\theta(u)$ a defective renewal equation in section 6. Especially, the infinite-time ruin probability is $\psi(u) = Pr(\rho < \infty)$. To ensure the ruin does not occur, the premium rate c needs to satisfy

$$E[qW_j - X_j] > 0, j = 1, 2, \ldots \tag{1}$$

which providing a positive safety loading.

3 The Lundberg Equation

We will derive a Lundberg's generalised equation in this section. We consider the discrete-time process embedded in the continuous-time surplus process $\{U(t); t \geq 0\}$. Firstly, we define the discrete-time process by

$$U_k = u + \sum_{i=1}^{k} (qW_i - X_i), k = 1, 2, \dots$$

to be the surplus immediately after the kth claim, where $U_0 = u$. In this new risk model, the process $\{e^{-\theta \sum_{i=1}^{k} W_i + sU_k}, \quad k = 0, 1, 2, \dots\}$ for $s > 0$ is a martingale if and only if

$$E[e^{-\theta W} e^{s(qW-X)}] = E[e^{(qs-\theta)W} e^{-sX}] = 1, \tag{2}$$

which is called the *generalised Lundberg equation*. From the definition in scetion 2, the left side of Equation (2) is

$$\begin{aligned} E[e^{-\theta W} e^{s(qW-X)}] &= \int_0^\infty \int_0^\infty e^{-(\theta-qs)t} K(t) P(M > T) f_1(x) e^{-sx} dxdt \\ &+ \int_0^\infty \int_0^\infty e^{-(\theta-qs)t} K(t) P(M \leq T) f_2(x) e^{-sx} dxdt \\ &= \int_0^\infty \int_0^\infty e^{-(\theta-qs)t} k e^{-kt} \left[e^{-lt} + \frac{1}{l} l^2 t e^{-lt} \right] f_1(x) e^{-sx} dxdt \\ &+ \int_0^\infty \int_0^\infty e^{-(\theta-qs)t} k e^{-kt} \left[1 - e^{-lt} - \frac{1}{l} l^2 t e^{-lt} \right] f_2(x) e^{-sx} dxdt \\ &= \frac{k}{q} \left[\frac{(k+l+\theta-qs+l)\hat{f}_1(s)}{q(\frac{k+l+\theta}{q}-s)^2} + \frac{l^2 \hat{f}_2(s)}{q^2(\frac{k+l+\theta}{q}-s)^2(\frac{k+\theta}{q}-s)} \right]. \end{aligned} \tag{3}$$

Then, from (2) we have

$$\frac{k}{q} \frac{\left(\frac{k+\theta}{q}-s\right)\left[\frac{1}{l}(k+l+\theta-qs)+1\right]\hat{f}_1(s) + \frac{l}{q}\hat{f}_2(s)}{\frac{q}{l}\left(\frac{k+\theta}{q}-s\right)\left(\frac{k+l+\theta}{q}-s\right)^2} = 1. \tag{4}$$

We will use Rouche's theorem in the following proposition.
PROPOSITION 1. For $\theta > 0$, the equation (4) has exactly 3 roots, say $\beta_1(\theta)$, $\beta_2(\theta)$, $\beta_3(\theta)$, with $Re(\beta_i(\theta)) > 0$, $i = 1, 2, 3$.

Proof. Rewrite the equation (4) as

$$\begin{aligned} &k(k+\theta-qs)(k+2l+\theta-qs)\hat{f}_1(s) + kl^2\hat{f}_2(s) \\ &= (k+\theta-qs)(k+l+\theta-qs)^2, \end{aligned} \tag{5}$$

it can be seen from the right side of above equation, it has exactly 3 roots with positive real parts. We denote $D_r = \{s \in D : |s| = r, Re(s) \geq 0, r > 0\}$ is the contour, which containing the imaginary axis running from -ir to ir and a semicircle with radius r running clockwise from -ir to ir.

(1) For $Re(s) > 0$, since $|k + l - qs| \to \infty, \quad |\theta + k + l - qs| \to \infty$ as $r \to \infty$, then

$$\begin{aligned}
&\Bigg| \left[\frac{k}{(k+l+\theta-qs)} + \frac{kl}{(k+l+\theta-qs)^2} \right] \hat{f}_1(s) \\
&+ \left[\frac{k}{(k+\theta-qs)} - \frac{k}{(k+l+\theta-qs)} - \frac{kl}{(k+l+\theta-qs)^2} \right] \hat{f}_2(s) \Bigg| \\
&\leq \left| \frac{k}{(k+l+\theta-qs)} + \frac{kl}{(k+l+\theta-qs)^2} \right| |\hat{f}_1(s)| \\
&+ \left| \frac{k}{(k+\theta-qs)} - \frac{k}{(k+l+\theta-qs)} - \frac{kl}{(k+l+\theta-qs)^2} \right| |\hat{f}_2(s)| \to 0
\end{aligned}$$

on C. For $r \to \infty$, we have

$$\begin{aligned}
&\Bigg| \left[\frac{k}{(k+l+\theta-qs)} + \frac{kl}{(k+l+\theta-qs)^2} \right] \hat{f}_1(s) \\
&+ \left[\frac{k}{(k+\theta-qs)} - \frac{k}{(k+l+\theta-qs)} - \frac{kl}{(k+l+\theta-qs)^2} \right] \hat{f}_2(s) \Bigg| < 1. \qquad (6)
\end{aligned}$$

on C.

(2) For $Re(s) = 0$, we give the similar discussion to Cossette et al. [12], let

$$\hat{d}_\theta(s) = \frac{k}{k+\theta-qs} - \frac{k}{k+l+\theta-qs} - \frac{kl}{(k+l+\theta-qs)^2}$$

then we have

$$\begin{aligned}
|\hat{d}_\theta(s)| &= \left| \frac{k}{k+\theta-qs} - \frac{k}{k+l+\theta-qs} - \frac{kl}{(k+l+\theta-qs)^2} \right| \\
&= k \left| \frac{l^2}{(k+\theta-qs)(k+l+\theta-qs)^2} \right| \\
&\leq k \left| \frac{l^2}{(k+\theta)(k+l+\theta)} \right| = |\hat{d}_\theta(0)|
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg| \left[\frac{k}{(k+l+\theta-qs)} + \frac{kl}{(k+l+\theta-qs)^2} \right] \hat{f}_1(s) \\
&+ \left[\frac{k}{(k+\theta-qs)} - \frac{k}{(k+l+\theta-qs)} - \frac{kl}{(k+l+\theta-qs)^2} \right] \hat{f}_2(s) \Bigg| \\
&= \left| \left(\frac{k}{k+l+\theta-qs} + \frac{kl}{(k+l+\theta-qs)^2} \right) \hat{f}_1(s) + \hat{f}_2(s)\hat{d}_\theta(s) \right| \\
&\leq \left| \frac{k}{k+l+\theta-qs} \right| + \left| \frac{kl}{(k+l+\theta-qs)^2} \right| + |\hat{d}_\theta(s)| \\
&\leq \left| \frac{k}{k+l+\theta} \right| + \left| \frac{kl}{(k+l+\theta)^2} \right| + |\hat{d}_\theta(0)|. \qquad (7)
\end{aligned}$$

For $\theta > 0$, it has $\hat{d}_\theta(0) > 0$. Indeed,

$$\hat{d}_\theta(0) = \frac{k}{k+\theta} - \frac{k}{k+l+\theta} - \frac{kl}{(k+l+\theta)^2} = \frac{kl^2}{(k+l+\theta)^2(k+l)} > 0.$$

Therefore, for s on the imaginary axis and for $\theta > 0$, Equation (7) becomes

$$\begin{aligned}
&\Bigg| \left[\frac{k}{(k+l+\theta-qs)} + \frac{kl}{(k+l+\theta-qs)^2}\right]\hat{f}_1(s) \\
&+ \left[\frac{k}{(k+\theta-qs)} - \frac{k}{(k+l+\theta-qs)} - \frac{kl}{(k+l+\theta-qs)^2}\right]\hat{f}_2(s)\Bigg| \\
&\leq \left|\frac{k}{k+l+\theta}\right| + \left|\frac{kl}{(k+l+\theta)^2}\right| + |\hat{d}_\theta(0)| \\
&\leq \frac{(k+l+\theta)^2-\theta^2-2l\theta-k\theta}{(k+l+\theta)^2} < 1.
\end{aligned}$$

Above all, we proved

$$\begin{aligned}
&|k(k+\theta-qs)(k+2l+\theta-qs)\hat{f}_1(s) + kl^2\hat{f}_2(s)| \\
&< |(k+\theta-qs)(k+l+\theta-qs)^2|
\end{aligned}$$

in two case. Then by Rouche's theorem, it derives that Equation(5) and the equation $(k+\theta-qs)(k+l+\theta-qs)^2$ inside C_r have the same number of roots. That is, Equation (4) has exactly 3 roots.
For simiplicity, we rewrite β_j as $\beta_j(\theta), j = 1, 2, 3.$ in following part.

REMARK. Since

$$\begin{aligned}
&\Bigg| \left[\frac{k}{(k+l+\theta-qs)} + \frac{kl}{(k+l+\theta-qs)^2}\right]\hat{f}_1(s) \\
&+ \left[\frac{k}{(k+\theta-qs)} - \frac{k}{(k+l+\theta-qs)} - \frac{kl}{(k+l+\theta-qs)^2}\right]\hat{f}_2(s)\Bigg| \\
&= \left|\frac{k}{k+l} + \frac{kl}{(k+l)^2} + \left[1 - \frac{k}{k+l} - \frac{kl}{k+l}\right]\right| = 1
\end{aligned}$$

for $s = 0$ and $\theta = 0$, the conditions to Rouche's theorem are no longer true. But the case is important to evaluate ruin probablity. So we apply the Klimenok [13] to derive the number of roots to the generalized Lundberg's equation with $\theta = 0$.

PROPOSITION 2. For $\theta = 0$, Lundberg's generalised Equation(4) has exaclty 2 roots, say $\beta_1(0)$, $\beta_2(0)$, with $Re(\beta_i) > 0$ and one root equals zero.

Proof. Define $D_k = s : |w| = 1$ and let $w = 1 - \frac{s}{r}$. In terms of s, the contour D_k is a circle with origin at r and radius r. Using the same arguments as Proposition 1, Equation(7) also holds on D (excluding s=0 or equivalently w=1) for $\theta = 0$. Besides, the functions $k(k+\theta-qs)(k+2l+\theta-qs)\hat{f}_1(s) + kl^2\hat{f}_2(s)$ and $(k+\theta-qs)(k+l+\theta-qs)^2$ are continuous on D. We need prove that

$$\begin{aligned}
&\frac{d}{dw}\Bigg\{1 - \left[\frac{k}{(k+l+\theta-qr(1-w))} + \frac{kl}{(k+l+\theta-qr(1-w))^2}\right]\hat{f}_1(r-rw) \\
&- \left[\frac{k}{(k+\theta-qr(1-w))} - \frac{k}{(k+l+\theta-qr(1-w))}\right. \\
&\left. - \frac{kl}{(k+l+\theta-qr(1-w))^2}\right]\hat{f}_2(r-rw)\Bigg\}\Bigg|_{w=1} > 0.
\end{aligned}$$

The left side of this relation equals

$$\frac{d}{dw}\left\{1 - E\left[e^{(r-rw)(qW-X)}\right]\right\}\Big|_{w=1} = rE\left[qW - X\right]$$

where $E\left[qW - X\right] > 0$ given the solvability condition in equation (1).

Based on theorem 1 of Klimenok (2001), we conclude the number of roots of Equation (5) is equal to 2 inside D. Moreover, a trival root to equation (4) equals zero.

4 The Laplace Transform of $m_\theta(u)$

Here we want to derive the Laplace Transform of $m_\theta(u)$. For $u \geq 0$ and setting $y = u + qt$, we have

$$\begin{aligned} m_\theta(u) &= E[e^{-\theta\rho} w(U_{\rho-}, |U_\rho|) 1_{\rho<\infty} | U_0 = u] \\ &= \frac{k}{q}\int_u^\infty e^{-(\theta+k+l)(\frac{y-u}{q})}(\zeta_1(y) - \zeta_2(y))dy \\ &+ \frac{k}{q}\frac{1}{l}\int_u^\infty e^{-(\theta+k)(\frac{y-u}{q})} f\left(\frac{y-u}{q}\right)(\zeta_1(y) - \zeta_2(y))dy \\ &+ \frac{k}{q}\int_u^\infty e^{-(\theta+k)(\frac{y-u}{q})}\zeta_2(y)dy, \end{aligned}$$

where

$$\begin{aligned} \zeta_{1,\theta}(u) &= \int_0^u m_\theta(u-x) f_1(x)dx + \sigma_1(u), \quad \sigma_1(u) = \int_u^\infty f_1(x)dx, \\ \zeta_{2,\theta}(u) &= \int_0^u m_\theta(u-x) f_2(x)dx + \sigma_2(u), \quad \sigma_2(u) = \int_u^\infty f_2(x)dx. \end{aligned}$$

Then we obtain

$$\begin{aligned} \frac{q}{k}\hat{m}_\theta(s) &= \int_0^\infty e^{-su}\frac{q}{k} m_\theta(u)du \\ &= \int_0^\infty e^{-\frac{k+l+\theta}{q}y}\left(\zeta_1(y) - \zeta_2(y)\right)\int_0^y e^{-\left(s-\frac{k+l+\theta}{q}u\right)} dudy \\ &+ \frac{l}{q}\int_0^\infty e^{-\frac{k+l+\theta}{q}y}\left(\zeta_1(y) - \zeta_2(y)\right)\int_0^y e^{-\left(s-\frac{k+l+\theta}{q}u\right)}(y-u)dudy \\ &+ \int_0^\infty e^{-\frac{k+\theta}{q}y}\zeta_2(y)\int_0^y e^{-\left(s-\frac{k+\theta}{q}u\right)} dudy. \end{aligned} \tag{8}$$

It can be easily proved that for $a > 0$

$$\begin{aligned} \int_0^y e^{-au}du &= -\frac{e^{-ay}}{a} \\ \int_0^y (y-u)e^{-au}du &= \frac{y}{a} - \frac{1}{a^2} + \frac{e^{-ay}}{a^2} \end{aligned} \tag{9}$$

Therefore, using Equation (9), Equation (8) can be written as

$$\begin{aligned} \frac{q}{k}\hat{m}_\theta(s) &= \frac{1}{(\frac{k+l+\theta}{q} - s)}\left(\hat{\zeta}_1(s) - \hat{\zeta}_2(s)\right) + \frac{1}{\left(\frac{k+\theta}{q} - s\right)}\hat{\zeta}_2(s) \\ &+ \frac{l}{q}\frac{1}{\left(s - \frac{k+l+\theta}{q}\right)^2}\left(\hat{\zeta}_1(s) - \hat{\zeta}_2(s)\right) + \hat{E}_\theta(s). \end{aligned} \tag{10}$$

where

$$\hat{\zeta}_{i,\theta}(s) = \int_0^\infty e^{-su}\zeta_{i,\theta}(u)du \quad i = 1, 2.$$

and

$$\hat{E}_\theta(s) = \int_0^\infty ye^{-(\theta+k+l)\frac{y}{q}}(\zeta_1(y) - \zeta_2(y))\frac{1}{\left(s - \frac{k+\theta+l}{q}\right)}dy - \int_0^\infty e^{-(\theta+k+l)\frac{y}{q}}(\zeta_1(y) - \zeta_2(y))\frac{1}{(s - \frac{\theta+k+l}{q})^2}dy.$$

Let $\hat{\sigma}_i(s) = \int_0^\infty e^{-su}\sigma_i(u)du, \quad i = 1, 2$, Equation (10) becomes

$$\hat{m}_\theta(s)\left[\frac{q}{k} - \frac{\hat{f}_1(s) - \hat{f}_2(s)}{\left(\frac{k+l+\theta}{q} - s\right)} - \frac{\hat{f}_2(s)}{\left(\frac{k+\theta}{q} - s\right)} - \frac{\hat{f}_1(s) - \hat{f}_2(s)}{\frac{q}{l}\left(\frac{k+l+\theta}{q} - s\right)^2}\right] = \frac{\hat{\sigma}_1(s) - \hat{\sigma}_2(s)}{\left(\frac{k+l+\theta}{q} - s\right)} + \frac{\hat{\sigma}_2(s)}{\left(\frac{k+\theta}{q} - s\right)} + \frac{\hat{\sigma}_1(s) - \hat{\sigma}_2(s)}{\frac{q}{l}\left(\frac{k+l+\theta}{qc} - s\right)^2} + \hat{E}_\theta(s). \tag{11}$$

Now using Equation (11), we have the following theorem about $\hat{m}_\theta(s)$.

THEOREM 1. In this new risk process, the expression for $\hat{m}_\theta(s)$ is

$$\hat{m}_\theta(s) = \frac{\hat{l}_{1,\theta}(s) + \hat{l}_{2,\theta}(s)}{\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s)}, \tag{12}$$

where

$$\hat{p}_{1,\theta}(s) = \frac{q}{k}\frac{q}{l}\left(\frac{\theta+k+l}{q} - s\right)^2\left(\frac{\theta+k}{q} - s\right), \tag{13}$$

$$\hat{p}_{2,\theta}(s) = \left(\frac{\theta+k}{q} - s\right)\left[\frac{q}{l}\left(\frac{\theta+k+l}{q} - s\right) + 1\right]\hat{f}_1(s) + \frac{l}{q}\hat{f}_2(s), \tag{14}$$

$$\hat{l}_{1,\theta}(s) = \left(\frac{\theta+k}{q} - s\right)\left[\frac{q}{l}\left(\frac{\theta+k+l}{q} - s\right) + 1\right]\hat{\sigma}_1(s) + \frac{l}{q}\hat{\sigma}_2(s), \tag{15}$$

$$\hat{l}_{2,\theta}(s) = -\sum_{j=1}^{3}\hat{l}_{1,\theta}(\beta_j)\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j - \beta_k}.$$

Proof. Multiplying Equation (11) by $\frac{q}{l}\left(\frac{\theta+k+l}{q} - s\right)^2\left(\frac{\theta+k}{q} - s\right)$ and solving the resulting equation, then we get the equation (12), with

$$\begin{aligned}
\hat{l}_{2,\theta}(s) &= \left(\frac{k+\theta}{q} - s\right)\left(s - \frac{k+\theta+l}{q}\right)^2\hat{E}_\theta(s) \\
&= \left(\frac{k+\theta}{q} - s\right)\left(s - \frac{k+\theta+l}{q}\right)^2\left[\int_0^\infty ye^{-(\theta+k+l)\frac{y}{q}}(\zeta_1(y) - \zeta_2(y))\frac{1}{\left(s - \frac{k+\theta+l}{q}\right)}dy\right. \\
&\left.- \int_0^\infty e^{-(\theta+k+l)\frac{y}{q}}(\zeta_1(y) - \zeta_2(y))\frac{1}{(s - \frac{\theta+k+l}{q})^2}dy\right] \\
&= \left(\frac{k+\theta}{q} - s\right)\left(\frac{k+\theta+l}{q} - s\right)\hat{\mu}_1\left(\frac{k+l+\theta}{q}\right) + \left(\frac{k+\theta}{q} - s\right)\hat{\mu}_0\left(\frac{k+l+\theta}{q}\right),
\end{aligned}$$

where

$$\hat{\mu}_j\left(\frac{\theta+k+l}{q}\right)=\int_0^\infty e^{-(\theta+k+l)y/q}(\zeta_1(y)-\zeta_2(y))y^j dy \quad (j=0,1).$$

The equation (4) can be written as $\hat{p}_{1,\theta}(s)-\hat{p}_{2,\theta}(s)=0$, meaning that $\beta_i's, i=1,\ldots,3$ are the roots of the fraction's denominator in Equation (12). In the previous discussion, we let $\hat{m}_\theta(s)$ be analyzed at $Re(s)\geq 0$. That is, $\beta_i's, i=1,\ldots,3$ are also roots of the Equation (12), and then $\hat{l}_{2,\theta}(\beta_i)=-\hat{l}_{1,\theta}(\beta_i), i=1,\ldots,3$.

By using Lagrange interpolation formula at 3 roots β_1,β_2,β_3, we have

$$\hat{l}_{2,\theta}(s)=\sum_{j=1}^{3}\hat{l}_{2,\theta}(\beta_j)\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j-\beta_k}=-\sum_{j=1}^{3}\hat{l}_{1,\theta}(\beta_j)\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j-\beta_k},$$

and then the proof is end.

5 Analysis of the Function when u=0

In this part, we look at the $m_\theta(0)$, $m_\rho(0)$ and $\psi(0)$ when u=0.

THEOREM 2. When u=0, the expression for $m_\theta(0)$ is

$$m_\theta(0)=\sum_{j=1}^{3}\frac{\hat{l}_{1,\theta}(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k-\beta_j)}.$$

Proof. We assume that the roots of Lundberg's equation β_1,β_2,β_3 are all distinct. According to the initial value theorem, we easily get

$$\begin{aligned}
m_\theta(0)&=\lim_{s\to\infty}s\hat{m}(s)=\lim_{s\to\infty}s\frac{\hat{l}_{1,\theta}(s)+\hat{l}_{2,\theta}(s)}{\hat{h}_{1,\theta}(s)-\hat{h}_{2,\theta}(s)}\\
&=\lim_{s\to\infty}s\frac{\hat{l}_{1,\theta}(s)-\sum_{j=1}^{3}\hat{l}_{1,\theta}(\beta_j)\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j-\beta_k}}{\hat{h}_{1,\theta}(s)-\hat{h}_{2,\theta}(s)}\\
&=\lim_{s\to\infty}\frac{\frac{\hat{l}_1(s)}{s^3}-\frac{1}{s^3}\sum_{j=1}^{3}\hat{l}_{1,\theta}(\beta_j)\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j-\beta_k}}{\frac{\hat{h}_1(s)}{s^4}-\frac{\hat{h}_2(s)}{s^4}}\\
&=\lim_{s\to\infty}\frac{-\frac{1}{s^2}\sum_{j=1}^{3}\hat{l}_{1,\theta}(\beta_j)\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j-\beta_k}}{(-1)^3}\\
&=\sum_{j=1}^{3}\frac{\hat{l}_{1,\theta}(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k-\beta_j)}.
\end{aligned}\tag{16}$$

THEOREM 3. When u=0, the expression of the Laplace Transform of the time ruin $m_\rho(0)$ is:

$$m_\rho(0)=1-\frac{(\theta+k+l)^2\theta}{qkl\prod_{i=1}^{3}\beta_i}.$$

Proof. Let

$$d_{1,\theta}(s)=\left(\frac{k+\theta}{q}-s\right)\left[\frac{q}{l}\left(\frac{k+l+\theta}{q}-s\right)+1\right],\tag{17}$$

$$d_{2,\theta}(s)=\frac{l}{q}.\tag{18}$$

And from Equation (15) we obtain

$$\hat{l}_{1,\theta}(s) = d_{1,\theta}(s)\hat{\sigma}_1(s) + d_{2,\theta}(s)\hat{\sigma}_2(s). \tag{19}$$

then $m_\theta(0)$ becomes

$$m_\theta(0) = \sum_{j=1}^{3} \frac{d_{1,\theta}(\beta_j)\hat{\sigma}_1(\beta_j) + d_{2,\theta}(\beta_j)\hat{\sigma}_2(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k - \beta_j)} = \sum_{i=1}^{2}\sum_{j=1}^{3} d_{i,j}\hat{\sigma}_i(\beta_j), \tag{20}$$

which

$$d_{i,j} = \frac{d_{i,\theta}(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k - \beta_j)}. \quad i = 1,2 \quad j = 1,2,3 \tag{21}$$

Since

$$\hat{\sigma}_i(s) = \int_0^\infty e^{-sx}\sigma_i(x)dx = \int_0^\infty \int_0^\infty e^{-sx} w(x,y) f_i(x+y)dydx,$$

then we get that

$$m_\theta(0) = \int_0^\infty \int_0^\infty w(x,y)\left[f_1(x+y)\sum_{j=1}^{3} d_{1,j}e^{-\beta_j x} + f_2(x+y)\sum_{j=1}^{3} d_{2,j}e^{-\beta_j x}\right]dydx. \tag{22}$$

Define $h(x, y, t|0)$ be the joint defective density of the time of ruin (t), the surplus prior to ruin (x), and the deficit at ruin (y) where U(0)=0, and $h_\theta(x, y|0)$ be the discounted (nondiscounted if $\theta \to 0$) p.f.d. of the surplus just before ruin and ruin deficit. The relationship between the two is

$$h_\theta(x,y|0) = \int_0^\infty e^{-\theta t} h(x,y,t|0)dt.$$

For u=0, and according to Willmot et al. [14], it obtains

$$m_\theta(0) = \int_0^\infty \int_0^\infty \int_0^\infty w(x,y)e^{-\theta t}h(x,y,t|0)dtdydx = \int_0^\infty \int_0^\infty w(x,y)h_\theta(x,y|0)dydx,$$

which combined with Equation (22) yields

$$h_\theta(x,y|0) = f_1(x+y)\sum_{j=1}^{3} d_{1,j}e^{-\beta_j x} + f_2(x+y)\sum_{j=1}^{3} d_{2,j}e^{-\beta_j x}. \tag{23}$$

We also let $h_{1,\theta}(x|0) = \int_0^\infty f_\theta(x,y|0)dy$ as well as $h_{2,\theta}(y|0) = \int_0^\infty f_\theta(x,y|0)dx$. Since

$$\int_0^\infty f_i(x+y)dy = \bar{F}_i(x), i = 1,2$$

Then from Equation (23) we obtain

$$h_{1,\theta}(x|0) = \int_0^\infty h_\theta(x,y|0)dy = \bar{F}_1(x)\sum_{j=1}^{3} d_{1,j}e^{-\beta_j x} + \bar{F}_2(x)\sum_{j=1}^{3} d_{2,j}e^{-\beta_j x},$$

and according to Li and Garrido [15], we have

$$\begin{aligned} h_{2,\theta}(y|0) &= \int_0^\infty h_\theta(x,y|0)dx \\ &= \int_0^\infty \left(f_1(x+y)\sum_{j=1}^{3} d_{1,j}e^{-\beta_j x} + f_2(x+y)\sum_{j=1}^{3} d_{2,j}e^{-\beta_j x}\right)dx \\ &= \sum_{j=1}^{3} d_{1,j}T_{\beta_j}f_1(y) + \sum_{j=1}^{3} d_{2,j}T_{\beta_j}f_2(y). \end{aligned} \tag{24}$$

The Laplace transform of $h_{2,\theta}(y|0)$ is that

$$\begin{aligned}\hat{h}_{2,\theta}(s) &= \int_0^\infty e^{-sy} h_{2,\theta}(y|0)dy = T_s h_{2,\theta}(0|0) \\ &= \sum_{j=1}^{3} d_{1,j} T_s T_{\beta_j} f_1(0) + \sum_{j=1}^{3} d_{2,j} T_s T_{\beta_j} f_2(0) \\ &= \sum_{j=1}^{3} \frac{d_{1,\theta}\hat{f}_1(\beta_j) + d_{2,j}\hat{f}_2(\beta_j)}{s-\beta_j} - \hat{f}_1(s)\sum_{j=1}^{3}\frac{d_{1,j}}{s-\beta_j} - \hat{f}_2(s)\sum_{j=1}^{3}\frac{d_{2,j}}{s-\beta_j}. \end{aligned} \tag{25}$$

Using Equation (22), (17) and (18), it gets that $\hat{o}_{2,\theta}(s) = d_{1,\theta}(s)\hat{f}_1(s) + d_{2,\theta}(s)\hat{f}_2(s)$, then for $j = 1, \ldots, 3$, we have

$$\begin{aligned} d_{1,j}\hat{f}_1(\beta_j) + d_{2,j}\hat{f}_2(\beta_j) &= \frac{d_{1,\theta}(\beta_j)\hat{f}_1(\beta_j) + d_{2,\theta}(\beta_j)\hat{f}_2(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k-\beta_j)} = \frac{\hat{p}_{2,\theta}(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k-\beta_j)} \\ &= \frac{\hat{p}_{1,\theta}(\beta_j)}{\prod_{k=1,k\neq j}^{3}(\beta_k-\beta_j)}. \end{aligned}$$

Then using Equation (22) and (25), we have

$$\begin{aligned}\hat{h}_{2,\theta}(s) &= \sum_{j=1}^{3}\frac{(\theta+k+l-q\beta_j)^2(\theta+k-q\beta_j)}{qkl(s-\beta_j)\prod_{k=1,k\neq j}^{3}(\beta_k-\beta_j)} - \hat{f}_1(s)\sum_{j=1}^{3}\frac{d_{1,j}}{s-\beta_j} \\ &\quad - \hat{f}_2(s)\sum_{j=1}^{3}\frac{d_{2,j}}{s-\beta_j}. \end{aligned} \tag{26}$$

According to Li and Garrido [16] of the Equation(17)and (18), Equation (26) rewrites as

$$\begin{aligned}\hat{f}_{2,\theta}(s) &= 1 - \frac{(\theta+k+l-qs)^2(\theta+k-qs)}{qkl\prod_{i=1}^{3}(\beta_i-s)} \\ &\quad + \hat{f}_1(s)\frac{(k+\theta-qs)(k+\theta+l-qs+l)}{ql\prod_{i=1}^{3}(\beta_i-s)} + \hat{f}_2(s)\frac{l}{q\prod_{i=1}^{3}(\beta_i-s)} \\ &= 1 - \frac{1}{\prod_{i=1}^{3}(\beta_i-s)}\left\{\frac{q^2}{kl}\left(\frac{\theta+k}{q}-s\right)\left(\frac{k+l+\theta}{q}-s\right)^2\right. \\ &\quad \left. - \left(\frac{k+\theta}{q}-s\right)\left[\left(\frac{k+l+\theta}{q}-s\right)\frac{q}{l}+1\right]\hat{f}_1(s) - \frac{l}{q}\hat{f}_2(s)\right\} \\ &= 1 - \frac{\hat{p}_{1,\theta}(s)-\hat{p}_{2,\theta}(s)}{\prod_{i=1}^{3}(\beta_i-s)}. \end{aligned}$$

Let $w(x,y) = 1$, when $U(0) = 0$, since $\hat{f}_1(0) = 1, \hat{f}_2(0) = 1$, then we have

$$\begin{aligned} m_\rho(0) &= E\left[I(\rho<\infty)e^{-\theta\rho}|U(0)=0\right] = \int_0^\infty\int_0^\infty h_\theta(u,y|0)dydu \\ &= \int_0^\infty h_{2,\theta}(y|0)dy = \lim_{s\to 0}\hat{h}_{2,\theta}(s) = 1 - \frac{\hat{p}_{1,\theta}(0)-\hat{p}_{2,\theta}(0)}{\beta_1\beta_2\beta_3} \\ &= 1 - \frac{(\theta+k+l)^2\theta}{qkl\prod_{i=1}^{3}\beta_i}. \end{aligned} \tag{27}$$

Due to $\theta > 0$, we can derive that $m_\rho(0) < 1$.

THEOREM 4. When U(0)=0, the ruin probability $\psi(0)$ is

$$\psi(0) = 1 - \frac{(k+l)^2}{qkl\beta_1'(0)\beta^*(0)}.$$

Proof. We get that

$$\begin{aligned}\psi(0) &= \lim_{\theta\to 0^+} E\left[I(\rho<\infty)e^{-\theta\rho}|U(0)=u\right]\\ &= 1 - \lim_{\theta\to 0^+}\frac{(\theta+k+l)^2\theta}{qkl\prod_{i=1}^{3}\beta_i}\\ &= 1 - \frac{(k+l)^2}{qkl\beta_1'(0)\beta^*(0)},\end{aligned} \tag{28}$$

where $\beta^*(0) = \prod_{i=2}^{3}\beta_i(0)$ and $\beta_1^{'}(0) = \frac{d}{d\theta}\beta_1(\theta)|_\theta \to 0^+$ and when U(0)=0. Using the fact that $\beta_1(\theta)$ is a root of Lundberg equation, we have $\hat{p}_1(\beta_1(\theta)) = \hat{p}_2(\beta_1(\theta))$. By differentiating with respect to θ and then letting $\theta \to 0^+$, we obtain

$$(k+l)^2(1-c\beta_1^{'}(0)) = -k^2(k+2l)\mu_1\beta_1^{'}(0) - kl^2\mu_2\beta_1^{'}(0) \tag{29}$$

where $\hat{f}_1^{'}(0) = -\mu_1, \hat{f}_2^{'}(0) = -\mu_2$. From Equation (29) and the Equation (1), we know that

$$\beta_1^{'}(0) = \frac{(k+l)^2}{q(k+l)^2 - k^2(k+2l)\mu_1 - kl^2\mu_2} = \frac{E(W)}{qE(W)-E(X)} \tag{30}$$

which is always positive. Thus, using Equation (28) and (30), we have that

$$\psi(0) = 1 - \frac{(k+l)^2}{qkl\beta_1'(0)\beta^*(0)} = 1 - \frac{[qE(W)-E(X)]}{ckl\beta^*(0)} < 1. \tag{31}$$

6 Expressions for the Gerber-Shiu Penalty Function

THEOREM 5. An another expression to Gerber-Shiu penalty function is,

$$m_\theta(u) = \int_0^u m_\theta(u-y)\zeta_\theta(y)dy + B_\theta(u), \quad u \geq 0. \tag{32}$$

where

$$\begin{aligned}\zeta_\theta(y) &= H_{2,\theta}(y|0) = T_{\beta_1}T_{\beta_2}T_{\beta_3}p_{2,\theta}(u),\\ B_\theta(u) &= T_{\beta_1}T_{\beta_2}T_{\beta_3}l_{1,\theta}(u).\end{aligned} \tag{33}$$

Proof. Since $\int_0^\infty h_{2,\theta}(y|0)dy = m_\rho(0) < 1$, Equation (32) is a defective renewal equation. Using Lagrange interpolating formula, we derive that

$$\hat{p}_{1,\theta}(s) = \hat{p}_{1,\theta}(0)\prod_{k=1}^{3}\frac{s-\beta_k}{(-\beta_k)} + s\sum_{j=1}^{3}\frac{\hat{p}_{1,\theta}(\beta_j)}{\beta_j}\prod_{k=1,k\neq j}^{3}\frac{s-\beta_k}{\beta_j-\beta_k}.$$

Following a similar discussion as in the references, the relation mentioned before implies

$$\begin{aligned}\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s) = \pi_3(s)\Bigg[&\frac{\hat{p}_{1,\theta}(0)}{\pi_3(0)} - \sum_{j=1}^{3}\frac{\hat{p}_{2,\theta}(\beta_j)}{(-\beta_j)\pi_3'(\beta_j)}\\ &+ \sum_{j=1}^{3}\frac{\hat{p}_{2,\theta}(\beta_j)}{(s-\beta_j)\pi_3'(\beta_j)} - \frac{\hat{p}_{2,\theta}(s)}{\pi_3(s)}\Bigg],\end{aligned} \tag{34}$$

where $\pi_3(s) = \prod_{i=1}^{3}(s-\beta_j)$. Since $\hat{p}_{2,\theta}(\beta_j) = \hat{p}_{1,\theta}(\beta_j)$, $j = 1, 2, 3$, for s=0, we obtain

$$\begin{aligned}\frac{\hat{p}_{1,\theta}(0)}{\pi(0)} + \sum_{j=1}^{3} \frac{\hat{p}_{2,\theta}(\beta_j)}{\beta_j \pi'(\beta_j)} &= \frac{\frac{q^2}{kl}\left(\frac{\theta+k+l}{q}\right)^2\left(\frac{\theta+k}{q}\right)}{\prod_{i=1}^{3}(-\beta_i)} + \sum_{j=1}^{3} \frac{\frac{q^2}{kl}\left(\frac{\theta+k}{q}-\beta_j\right)\left(\frac{k+\theta+l}{q}-\beta_j\right)^2}{\beta_j \prod_{k=1,k\neq j}^{3}(\beta_j-\beta_k)}\\ &= \frac{(\theta+k+l)^2(\theta+k)}{qkl\prod_{i=1}^{3}(-\beta_j)} + (-1)^3\left[1-\frac{(\theta+k+l)^2(\theta+k)}{qkl\prod_{i=1}^{3}(\beta_j)}\right]\\ &= -1.\end{aligned}$$

Then Equation (34) becomes

$$\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s) = (-1)^3\pi_3(s)\left[1 - T_sT_{\beta_1}T_{\beta_2}T_{\beta_3}h_{2,\theta}(0)\right]. \tag{35}$$

Furthermore, from (35) we get that

$$\begin{aligned}\hat{h}_{2,\theta}(s) &= 1 - \frac{\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s)}{\prod_{i=1}^{3}(\beta_i - s)}\\ &= 1 - \frac{(-1)^3\pi_3(s)\left[1 - T_sT_{\beta_1}T_{\beta_2}T_{\beta_3}p_{2,\theta}(0)\right]}{(-1)^3\pi_3(s)}\\ &= T_sT_{\beta_1}T_{\beta_2}T_{\beta_3}p_{2,\theta}(0).\end{aligned} \tag{36}$$

Since $\hat{h}_{2,\theta}(s) = \int_0^\infty e^{-su}h_{2,\theta}(y|0)du$, then

$$h_{2,\theta}(y|0) = T_{\beta_1}T_{\beta_2}T_{\beta_3}p_{2,\theta}(u).$$

Using the Dickson-Hipp operator, we have

$$\begin{aligned}B_\theta(u) &= \int_0^\infty\int_u^\infty w(s,t)\left[f_1(s+t)\sum_{j=1}^{3} d_{1,j}e^{-\beta_j(s-u)} + f_2(s+t)\sum_{j=1}^{3} d_{2,j}e^{-\beta_j(s-u)}\right]dsdt\\ &= \sum_{j=1}^{3} d_{1,j}\int_u^\infty e^{-\beta_j(s-u)}\sigma_1(s)ds + \sum_{j=1}^{3} d_{2,j}\int_u^\infty e^{-\beta_j(s-u)}\sigma_2(s)ds\\ &= \sum_{i=1}^{2}\sum_{j=1}^{3} d_{ij}T_{\beta_j}\sigma_i(u).\end{aligned} \tag{37}$$

From Equation (37), we obtain that the LT of the $B_\theta(u)$,

$$\begin{aligned}\hat{B}_\theta(s) &= \int_0^\infty e^{-su}B_\theta(u)du = T_sB_\theta(0) = \sum_{i=1}^{2}\sum_{j=1}^{3} d_{ij}T_sT_{\beta_j}\sigma_i(0)\\ &= \sum_{j=1}^{3}\frac{d_{1,\theta}\hat{\sigma}_1(\beta_j) + d_{2,\theta}\hat{\sigma}_2(\beta_j)}{(s-\beta_j)} - \hat{\sigma}_1(s)\sum_{j=1}^{3}\frac{d_{1,j}}{s-\beta_j} - \hat{\sigma}_2(s)\sum_{j=1}^{3}\frac{d_{2,j}}{s-\beta_j}\\ &= (-1)^3\left[\frac{\hat{l}_{1,\theta}(s)}{\pi(s)} - \sum_{j=1}^{3}\frac{\hat{l}_{1,\theta}(\beta_j)}{(s-\beta_j)\pi'(\beta_j)}\right]\\ &= T_sT_{\beta_1}T_{\beta_2}T_{\beta_3}l_{1,\theta}(0),\end{aligned}$$

Thus, by inverting Equation (37), we can get another expression for $G_\theta(u)$,

$$G_\theta(u) = T_{\beta_1}T_{\beta_2}T_{\beta_3}l_{1,\theta}(u).$$

THEOREM 6. The defective renewal equation of $m_\rho(u)$ is:

$$m_\rho(u) = \int_0^u m_\rho(u-y)\zeta_\theta(y)dy + \int_u^\infty \zeta_\theta(y)dy, \quad u \geq 0. \tag{38}$$

7 Numerical Illustration

In this section, we give some examples. If T_i is smaller than M_i, then the following claim size X_i has density function $f_1(x)$, otherwise its density function is $f_2(x)$. They are both exponential distribution with parameter k_1, k_2, that is, $f_1(x) = k_1 e^{-k_1 x}, f_2(x) = k_2 e^{-k_1 x}$, and $\hat{f}_1(s) = \frac{k_1}{k_1+s}, \hat{f}_2(s) = \frac{k_2}{k_2+s}$. We get an explicit expression for taking Laplace Transform of the first equation in Theorem 6,

$$\hat{m}_\rho(s) = \frac{m_\rho(0) - \hat{\zeta}_{2,\theta}(s)}{s\left[1 - \hat{\zeta}_{2,\theta}(s)\right]} = \frac{1 - \hat{\zeta}_{2,\theta}(s) - [1 - m_\rho(0)]}{s\left[1 - \hat{\zeta}_{2,\theta}(s)\right]}. \tag{39}$$

From Equation (35) and (36) we have

$$\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s) = \left[1 - \hat{\zeta}_{2,\theta}(s)\right] \prod_{i=1}^{3} (\beta_i - s),$$

and thus Equation (39) becomes

$$\hat{m}_\rho(s) = \frac{\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s) - [1 - m_\rho(0)] \prod_{i=1}^{3} (\beta_i - s)}{s\left[\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s)\right]}. \tag{40}$$

From Equation (21),(22) we easily have

$$\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s) = \frac{Q_{3,\theta}(s)}{qkl(k_1+s)(k_2+s)}, \tag{41}$$

where

$$\begin{aligned} Q_{3,\theta}(s) = {} & (k_1+s)(k_2+s)(k+\theta-qs)(\theta+k+l-qs)^2 - kl^2 k_2(k_1+s) \\ & - k_1(k_2+s)(k+\theta-qs)\left[k(k+\theta+l-qs)+kl\right]. \end{aligned}$$

Since $Q_{3,\theta}(s)$ is a polynomial of degree 3 and then we have that $Q_{3,\theta}(s) = 0$ has 3 roots in the complex plane. Since $\hat{p}_{1,\theta}(s) - \hat{p}_{2,\theta}(s) = 0$ is the Lundberg equation, that is equation $Q_{3,\theta}(s) = 0$ has 3 roots β_1, β_2, β_3 and two roots say $-M_i = -M_i(\theta)$, with $Re(M_i) > 0, i = 1, 2$. Thus, we rewrite $Q_{3,\theta}(s)$ as

$$Q_{3,\theta}(s) = qkl(s+M_1)(s+M_2) \prod_{i=1}^{3} (\beta_i - s). \tag{42}$$

So, from Equation (42)and (41), Equation (39) yields

$$\hat{m}_\rho(s) = \frac{\prod_{j=1}^{2} (s+M_j) - [1 - m_\rho(0)]\,(k_1+s)(k_2+s)}{s \prod_{j=1}^{2} (s+M_j)}. \tag{43}$$

Since $\hat{m}_\rho(s) < \infty$ for $s \geq 0$, that is

$$1 - m_\rho(0) = \frac{M_1 M_2}{k_1 k_2}$$

and then Equation (43) becomes

$$\hat{m}_\rho(s) = \frac{\left(1 - \frac{M_1 M_2}{k_1 k_2}\right) s + M_1 + M_2 - \frac{M_1 M_2 (k_1+k_2)}{k_1 k_2}}{(s+M_1)(s+M_2)}.$$

We assume R_1, R_2 are different and use partial fractions,

$$\hat{m}_\rho(s) = \sum_{j=1}^{2} \frac{\xi_{i,\theta}}{s+M_j},$$

where

$$\xi_{1,\theta} = \frac{M_2}{M_2 - M_1}\left(1 - \frac{M_1(k_1+k_2)}{k_1k_2} + \frac{M_1^2}{k_1k_2}\right)$$

$$\xi_{2,\theta} = \frac{M_1}{M_2 - M_1}\left(1 - \frac{M_2(k_1+k_2)}{k_1k_2} + \frac{M_2^2}{k_1k_2}\right)$$

Inverting $\hat{m}_\rho(s)$ gives that

$$m_\rho(u) = \xi_{1,\theta}e^{-M_1u} + \xi_{2,\theta}e^{-M_2u}, u \geq 0 \tag{44}$$

We also can get ruin probability $\psi(u)$ by letting $\theta \to 0$.

7.1 When $\theta = 0$

Let $k_1 = 2$, $k_2 = 4$, $c = 1.5$, $k = 2$,
with $l = 1$,

$$\psi(u) = -0.00054031652280488e^{-3.9900610193824644u} + 0.6425490681568e^{-0.7155993125651344u},$$

with $l = 3$,

$$\psi(u) = -0.0041727320835225e^{-3.938341980843664u} + 0.57252489270819e^{-0.8598590518898737u},$$

with $l = 5$,

$$\psi(u) = -0.00997224149241e^{-3.8733037753948447u} + 0.5182533288617954e^{-0.9744124540765244u},$$

with $l = 10$,

$$\psi(u) = -0.027705461665553e^{-3.7169851159041327u} + 0.43132361981897e^{-1.1643219472701969u}.$$

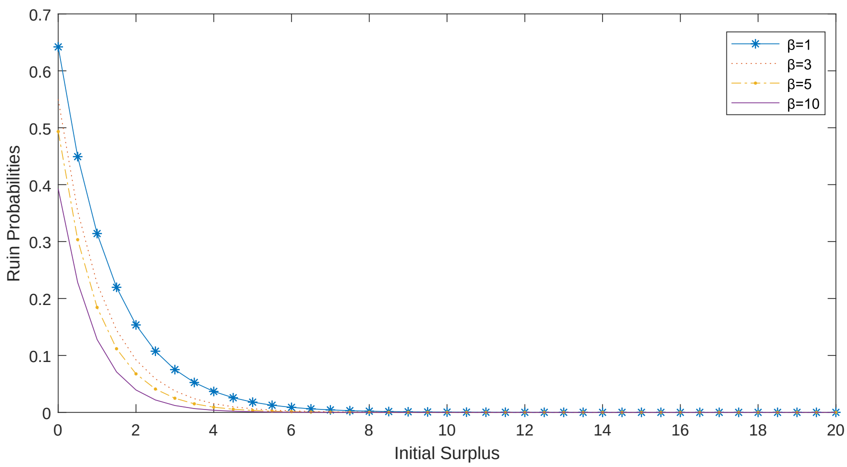

Fig. 1. Ruin probabilities when $\theta = 0$

From Fig. 1 we can see that the parameter l has an important impact on ruin probabilities $\psi(u)$.

7.2 When $\theta = 1$

Furthermore, we can analyze the expressions for the $m_\theta(u)$ by using $\theta = 1$. Let $k_1 = 2$, $k_2 = 4$, $c = 1.5$, $k = 2$,
with $l = 1$,

$$m_\rho(u) = -0.000753258169851e^{-3.992695543278276u} + 0.4158061959884069e^{-1.16901584810085u},$$

with $l = 3$,

$$m_\rho(u) = -0.005125813528442236e^{-3.9533678066596094u} + 0.3884081526472388e^{-1.2272392825230218u},$$

with $l = 5$,

$$m_\rho(u) = -0.011401900767716385e^{-3.902448624640329u} + 0.3612870826739349e^{-1.2859844203405293u},$$

with $l = 10$,

$$m_\rho(u) = -0.02929701439042134e^{-3.7763937788329516u} + 0.30918222107489424e^{-1.401380900990933u}.$$

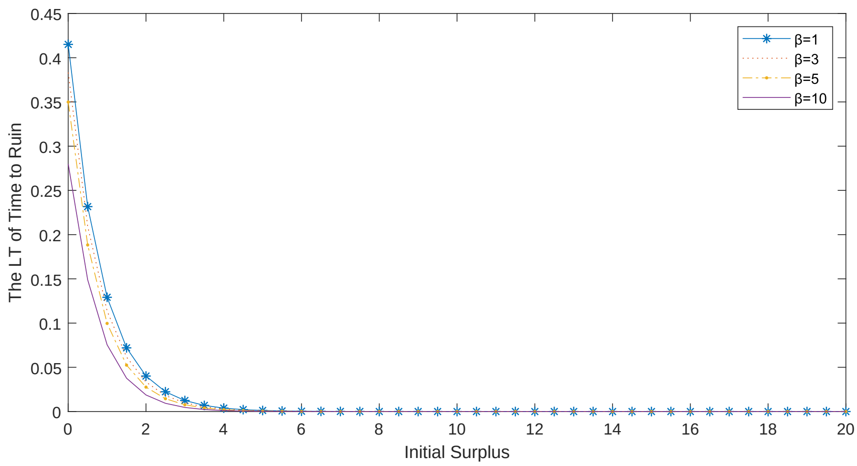

Fig. 2. Ruin probabilities when $\theta = 1$

From Fig. 2 we can see that the parameter l has an important impact on the values of the $m_\rho(u)$.

8 Conclusion

In this paper, we have considered a new risk model of claim amount affected by a threshold value. We derived that the generalised Lundberg equation has three roots and the Laplace Transform of the expected discounted penalty function. Besides, we analyzed the function when the initial surplus is zero. And also, we gave the expressions for the penalty function and some defective renewal equations. Some explicit expressions about the ruin probability are given to show that as the dependence parameter l is higher, the ruin probability and the value of the LT of time to ruin are both lower.

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Competing Interests

Authors have declared that no competing interests exist.

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